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STRENGTHENED CAUCHY-SCHWARZ AND HÖLDER INEQUALITIES

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ABSTRACT. We present some identities related to the Cauchy-Schwarz inequality in complex inner product spaces. A new proof of the basic result on the subject of strengthened Cauchy-Schwarz inequalities is derived using these identities. Also, an analogous version of this result is given for strengthened Hölder inequalities.

Key words and phrases: Strengthened Cauchy-Schwarz, Strengthened Hölder inequality.

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1. INTRODUCTION

In [1], the parallelogram identity in a *real* inner product space, is rewritten in Cauchy-Schwarz form (with the deviation from equality given as a function of the angular distance between vectors) thereby providing another proof of the Cauchy-Schwarz inequality in the real case. The first section of this note complements this result by presenting related identities for complex inner product spaces, and thus a proof of the Cauchy-Schwarz inequality in the complex case.

Of course, using angular distances is equivalent to using angles. An advantage of the angular distance is that it makes sense in arbitrary normed spaces, in addition to being simpler than the notion of an angle. And in some cases it may also be easier to compute. Angular distances are used in Section 2 to give a proof of the basic theorem in the subject of strengthened Cauchy-Schwarz inequalities (Theorem 3.1 below). We also point out that the result is valid not just for vector subspaces, but also for cones. Strengthened Cauchy-Schwarz inequalities are fundamental in the proofs of convergence of iterative, finite element methods in numerical analysis, cf. for instance [8]. They have also been considered in the context of wavelets, cf. for example [4], [5], [6].

Finally, Section 4 presents a variant, for cones and in the Hölder case when $1 < p < \infty$, of the basic theorem on strengthened Cauchy-Schwarz inequalities, cf. Theorem 4.1.

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2. IDENTITIES RELATED TO THE CAUCHY-SCHWARZ INEQUALITY IN COMPLEX INNER PRODUCT SPACES

It is noted in [1] that in a *real* inner product space, the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.1}$$

provides the following stability version of the Cauchy-Schwarz inequality, valid for non-zero vectors x and y:

$$(x,y) = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right). \tag{2.2}$$

Basically, this identity says that the size of (x,y) is determined by the angular distance $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$ between x and y. In particular, $(x,y) \le \|x\|\|y\|$, with equality precisely when the angular distance is zero. In this section we present some complex variants of this identity, involving (x,y) and $|(x,y)|$; as a byproduct, the Cauchy-Schwarz inequality in the complex case is obtained. Since different conventions appear in the literature, we point out that in this paper (x,y) is taken to be linear in the first argument and conjugate linear in the second.

We systematically replace in the proofs nonzero vectors x and y by unit vectors $u = x/\|x\|$ and $v = y/\|y\|$.

Theorem 2.1. *For all nonzero vectors x and y in a complex inner product space, we have*

$$\operatorname{Re}(x,y) = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right) \tag{2.3}$$

and

$$\operatorname{Im}(x,y) = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{iy}{\|y\|}\right\|^2\right). \tag{2.4}$$

Proof. Let $\|u\| = \|v\| = 1$. From (2.1) we obtain

$$\begin{aligned} 4 - \|u-v\|^2 &= \|u+v\|^2 \\ &= 2 + (u,v) + (v,u) \\ &= 2 + (u,v) + \overline{(u,v)} = 2 + 2\operatorname{Re}(u,v). \end{aligned}$$

Thus, $\operatorname{Re}(u,v) = 1 - \frac{1}{2}\|u-v\|^2$. The same argument, applied to $\|u+iv\|^2$, yields $\operatorname{Im}(u,v) = 1 - \frac{1}{2}\|u-iv\|^2$. $\square$

Writing $(x,y) = \operatorname{Re}(x,y) + i\operatorname{Im}(x,y)$ we obtain the following:

Corollary 2.2. *For all nonzero vectors x and y in a complex inner product space, we have*

$$(x,y) = \|x\|\|y\| \left(\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right) + i\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{iy}{\|y\|}\right\|^2\right)\right). \tag{2.5}$$

Thus,

$$|(x,y)| = \|x\|\|y\| \sqrt{\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right)^2 + \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{iy}{\|y\|}\right\|^2\right)^2}. \tag{2.6}$$

Next we find some shorter expressions for $|(x, y)|$. Let $\operatorname{Arg} z$ denote the principal argument of $z \in \mathbb{C}$, $z \neq 0$. That is, $0 \leq \operatorname{Arg} z < 2\pi$, and in polar coordinates, $z = e^{i \operatorname{Arg} z} r$. We choose the principal argument for definiteness; any other argument will do equally well.

Theorem 2.3. *Let x and y be nonzero vectors in a complex inner product space. Then, for every $\alpha \in \mathbb{R}$ we have*

$$\|x\| \|y\| \left(1 - \frac{1}{2} \left\| \frac{e^{i\alpha} x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right) \leq |(x, y)| = \|x\| \|y\| \left(1 - \frac{1}{2} \left\| \frac{e^{-i \operatorname{Arg}(x,y)} x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right). \tag{2.7}$$

Proof. By a normalization, it is enough to consider unit vectors u and v. Let α be an arbitrary real number, and set $t = \operatorname{Arg}(u, v)$, so $(u, v) = e^{it} r$ in polar form. Using (2.3) we obtain

$$\begin{aligned} 1 - \frac{1}{2} \left\| e^{i\alpha} u - v \right\|^2 &= \operatorname{Re}(e^{i\alpha} u, v) \leq |(e^{i\alpha} u, v)| \\ &= |(u, v)| = r = (e^{-it} u, v) \\ &= \operatorname{Re}(e^{-it} u, v) = 1 - \frac{1}{2} \left\| e^{-it} u - v \right\|^2. \end{aligned}$$

□

The preceding result can be regarded as a variational expression for $|(x, y)|$, since it shows that this quantity can be obtained by maximizing the left hand side of (2.7) over α, or, in other words, by minimizing $\left\| \frac{e^{i\alpha} x}{\|x\|} - \frac{y}{\|y\|} \right\|$ over α.

Corollary 2.4 (Cauchy-Schwarz inequality)**.** *For all vectors x and y in a complex inner product space, we have $|(x, y)| \leq \|x\| \|y\|$, with equality if and only if the vectors are linearly dependent.*

Proof. Of course, if one of the vectors x, y is zero, the result is trivial, so suppose otherwise and normalize, writing $u = x/\|x\|$ and $v = y/\|y\|$. From (2.7) we obtain, first, $|(u, v)| \leq 1$, second, $e^{-i \operatorname{Arg}(u,v)} u = v$ if $|(u, v)| = 1$, so equality implies linear dependency, and third, $|(u, v)| = 1$ if $e^{i\alpha} u = v$ for some $\alpha \in \mathbb{R}$, so linear dependency implies equality. □

3. STRENGTHENED CAUCHY-SCHWARZ INEQUALITIES

Such inequalities, of the form $|(x, y)| \leq \gamma \|x\| \|y\|$ for some fixed $\gamma \in [0, 1)$, are fundamental in the proofs of convergence of iterative, finite element methods in numerical analysis. The basic result in the subject is the following theorem (see Theorem 2.1 and Remark 2.3 of [8]).

Theorem 3.1. *Let H be a Hilbert space, let $F \subset H$ be a closed subspace, and let $V \subset H$ be a finite dimensional subspace. If $F \cap V = \{0\}$, then there exists a constant $\gamma = \gamma(V, F) \in [0, 1)$ such that for every $x \in V$ and every $y \in F$,*

$$|(x, y)| \leq \gamma \|x\| \|y\|.$$

There are, at least, two natural notions of angles between subspaces. To see this, consider a pair of distinct 2 dimensional subspaces V and W in $\mathbb{R}^3$. They intersect in a line L, so we may consider that they are parallel in the direction of the subspace L, and thus the angle between them is zero. This is the notion of angle relevant to the subject of strengthened Cauchy-Schwarz inequalities.

Alternatively, we may disregard the common subspace L, and (in this particular example) determine the angle between subspaces by choosing the minimal angle between their unit normals. Note however that the two notions of angle suggested by the preceding example coincide when the intersection of subspaces is $\{0\}$ (cf. [7] for more information on angles between subspaces).

From the perspective of angles, or equivalently, angular distances, what Theorem 3.1 states is the intuitively plausible assertion that the angular distance between V and F is strictly positive provided that F is closed, V is finite dimensional, and $F \cap V = \{0\}$. Finite dimensionality of one of the subspaces is crucial, though. It is known that if both V and F are infinite dimensional, the angular distance between them can be zero, even if both subspaces are closed.

Define the angular distance between V and F as

$$\kappa(V, F) := \inf\{\|v - w\| : v \in V, w \in F, \text{ and } \|v\| = \|w\| = 1\}. \tag{3.1}$$

The proof (by contradiction) of Theorem 3.1 presented in [8] is not difficult, but deals only with the case where both V and F are finite dimensional. And it is certainly not as simple as the following

Proof. If either $V = \{0\}$ or $F = \{0\}$ there is nothing to show, so assume otherwise. Let $S(V)$ be the unit sphere of the finite dimensional subspace V, and let $v \in S$. Denote by $f(v)$ the distance from v to the unit sphere $S(F)$ of F. Then $f(v) > 0$ since F is closed and $v \notin F$. Thus, f achieves a minimum value $\kappa > 0$ over the compact set $S(V)$. By the right hand side of formula (2.7), for every $x \in V \setminus \{0\}$ and every $y \in F \setminus \{0\}$ we have $|(x, y)| \leq (1 - \kappa^2/2)\, \|x\|\, \|y\|$. □

In concrete applications of the strengthened Cauchy-Schwarz inequality, a good deal of effort goes into estimating the size of $\gamma = \cos\theta$, where θ is the angle between subspaces appearing in the discretization schemes. Since we also have $\gamma = 1 - \kappa^2/2$, this equality can provide an alternative way of estimating γ, via the angular distance κ rather than the angle.

Next we state a natural extension of Theorem 3.1, to which the same proof applies (so we will not repeat it). Consider two nonzero vectors u, v in a real inner product space E, and let S be the unit circumference in the plane spanned by these vectors. The angle between them is just the length of the smallest arc of S determined by $u/\|u\|$ and $v/\|v\|$. So to speak about angles, or angular distances, we only need to be able to multiply nonzero vectors x by positive scalars $\lambda = 1/\|x\|$. This suggests that the natural setting for Theorem 3.1 is that of cones, rather than vector subspaces. Recall that C is a *cone* in a vector space over a field containing the real numbers if for every $x \in C$ and every $\lambda > 0$ we have $\lambda x \in C$. In particular, every vector subspace is a cone. If C_1 and C_2 are cones in a Hilbert space, the angular distance between them can be defined exactly as before:

$$\kappa(C_1, C_2) := \inf\{\|v - w\| : v \in C_1, w \in C_2, \text{ and } \|v\| = \|w\| = 1\}. \tag{3.2}$$

Theorem 3.2. *Let H be a Hilbert space with unit sphere $S(H)$, and let $C_1, C_2 \subset H$ be (topologically) closed cones, such that $C_1 \cap S(H)$ is a norm compact set. If $C_1 \cap C_2 = \{0\}$, then there exists a constant $\gamma = \gamma(C_1, C_2) \in [0, 1)$ such that for every $x \in C_1$ and every $y \in C_2$,*

$$|(x, y)| \leq \gamma\, \|x\|\, \|y\|.$$

Example 3.1. Let $H = \mathbb{R}^2$, $C_1 = \{(x, y) \in \mathbb{R}^2 : x = -y\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$, that is, C_1 is the one dimensional subspace with slope -1 and C_2 is the union of the first and third quadrants. Here we can explicitly see that $\gamma(C_1, C_2) = \cos(\pi/4) = 1/\sqrt{2}$. However, if C_2 is extended to a vector space V, then the condition $C_1 \cap V = \{0\}$ no longer holds and $\gamma(C_1, V) = 1$. So stating the result in terms of cones rather than vector subspaces does cover new, nontrivial cases.

4. A STRENGTHENED HÖLDER INEQUALITY

For $1 < p < \infty$, it is possible to give an $L^p - L^q$ version of the strengthened Cauchy-Schwarz inequality. Here $q := p/(p-1)$ denotes the conjugate exponent of p. We want to find suitable conditions on $C_1 \subset L^p$ and $C_2 \subset L^q$ so that there exists a constant $\gamma = \gamma(C_1, C_2) \in [0, 1)$ with $\|fg\|_1 \leq \gamma \|f\|_p \|g\|_q$ for every $f \in C_1$ and every $g \in C_2$. An obvious difference between the Hölder and the Cauchy-Schwarz cases is that in the pairing $(f, g) := \int f\overline{g}$, the functions f and g belong to different spaces (unless $p = q = 2$). This means that the hypothesis $C_1 \cap C_2 = \{0\}$ needs to be modified. A second obvious difference is that Hölder's inequality actually deals with $|f|$ and $|g|$ rather than with f and g. So when finding angular distances we will also deal with $|f|$ and $|g|$. Note that $f \in C_i$ does not necessarily imply that $|f| \in C_i$ (consider, for instance, the second quadrant in $\mathbb{R}^2$).

We make standard nontriviality assumptions on measure spaces $(X, \mathcal{A}, \mu)$: X contains at least one point and the (positive) measure μ is not identically zero. We write L^p rather than $L^p(X, \mathcal{A}, \mu)$.

To compare cones in different L^p spaces, we map them into L^2 via the Mazur map. Let us write $\operatorname{sign} z = e^{i\theta}$ when $z = re^{i\theta} \neq 0$, and $\operatorname{sign} 0 = 1$ (so $|\operatorname{sign} z| = 1$ always). The Mazur map $\psi_{r,s} : L^r \to L^s$ is defined first on the unit sphere $S(L^r)$ by $\psi_{r,s}(f) := |f|^{r/s} \operatorname{sign} f$, and then extended to the rest of L^r by homogeneity (cf. [3, pp. 197–199] for additional information on the Mazur map). More precisely,

$$\psi_{r,s}(f) := \|f\|_r \psi_{r,s}(f/\|f\|_r) = \|f\|_r^{1-r/s} |f|^{r/s} \operatorname{sign} f.$$

By definition, if $\lambda > 0$ then $\psi_{r,s}(\lambda f) = \lambda \psi_{r,s}(f)$. This entails that if $C \subset L^r$ is a cone, then $\psi_{r,s}(C) \subset L^s$ is a cone. Given a subset $A \subset L^r$, we denote by $|A|$ the set $|A| := \{|f| : f \in A\}$. Observe that if A is a cone then so is $|A|$.

Theorem 4.1. *Let $1 < p < \infty$ and denote by $q := p/(p-1)$ its conjugate exponent. Let $C_1 \subset L^p$ and $C_2 \subset L^q$ be cones, let $S(L^p)$ stand for the unit sphere of L^p and let $\overline{|C_1|}$ and $\overline{|C_2|}$ denote the topological closures of $|C_1|$ and $|C_2|$. If $\overline{|C_1|} \cap S(L^p)$ is norm compact, and $\psi_{p,2}(\overline{|C_1|}) \cap \psi_{q,2}(\overline{|C_2|}) = \{0\}$, then there exists a constant $\gamma = \gamma(C_1, C_2) \in [0, 1)$ such that for every $f \in C_1$ and every $g \in C_2$,*

$$\|fg\|_1 \leq \gamma \|f\|_p \|g\|_q. \tag{4.1}$$

In the proof we use the following result, which is part of [1, Theorem 2.2].

Theorem 4.2. *Let $1 < p < \infty$, let $q = p/(p-1)$ be its conjugate exponent, and let $M = \max\{p, q\}$. If $f \in L^p$, $g \in L^q$, and $\|f\|_p, \|g\|_q > 0$, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \left(1 - \frac{1}{M} \left\| \frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{q/2}}{\|g\|_q^{q/2}} \right\|_2^2 \right). \tag{4.2}$$

A different proof of inequality (4.2) (with the slightly weaker constant $M = p + q$, but sufficient for the purposes of this note) can be found in [2]. Next we prove Theorem 4.1.

Proof. If either $C_1 = \{0\}$ or $C_2 = \{0\}$ there is nothing to show, so assume otherwise. Note that since $|C_1|$ and $|C_2|$ are cones, the same happens with their topological closures. The cones $\psi_{p,2}(\overline{|C_1|})$ and $\psi_{q,2}(\overline{|C_2|})$ are also closed, as the following argument shows: The Mazur maps $\psi_{r,s}$ are uniform homeomorphisms between closed balls, and also between spheres, of any fixed (bounded) radius (cf. [3, Proposition 9.2, p. 198], and the paragraph before the said proposition). In particular, if $\{f_n\}$ is a Cauchy sequence in $\psi_{q,2}(\overline{|C_2|})$ (for instance) then it is a bounded sequence in L^2, so $\psi_{q,2}^{-1} = \psi_{2,q}$ maps it to a Cauchy sequence in $\overline{|C_2|}$, with limit, say, h. Then $\lim_n f_n = \psi_{q,2}(h) \in \psi_{q,2}(\overline{|C_2|})$. Likewise, $\psi_{p,2}(\overline{|C_1|})$ is closed.

The rest of the proof proceeds as before. Let $v \in \psi_{p,2}(\overline{|C_1|}) \cap S(L^2)$ and denote by $F(v)$ the distance from v to $\psi_{q,2}(\overline{|C_2|}) \cap S(L^2)$. Then $F(v) > 0$, so F achieves a minimum value $\kappa > 0$ over the compact set $\psi_{p,2}(\overline{|C_1|}) \cap S(L^2)$, and now (4.1) follows from (4.2). □

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