LOWER BOUNDS FOR THE CENTERED HARDY-LITTLEWOOD MAXIMAL OPERATOR ON THE REAL LINE

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Abstract. Let $1 < p < \infty$. We prove that there exists an $\varepsilon_p > 0$ such that for each $f \in L^p(\mathbb{R})$, the centered Hardy-Littlewood maximal operator M on $\mathbb{R}$ satisfies the lower bound $\|Mf\|_{L^p(\mathbb{R})} \geq (1+\varepsilon_p)\|f\|_{L^p(\mathbb{R})}$.

1. Introduction

Given a locally integrable real-valued function f on $\mathbb{R}^d$ define its uncentered maximal function $M_u f(x)$ as

$$M_u f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B \in \mathbb{R}^d$ containing the point x; here $|B|$ denotes the d-dimensional Lebesgue measure of the ball B. The usefulness of this and other maximal functions comes from the fact that they are larger than the original function f, but not much larger, and usually improve regularity. Since $M_u f$ is often used as a close upper bound for f, it is interesting to know precisely how much larger $M_u f$ is, and the same question can be asked about other maximal operators.

It is well known that $M_u f(x) \geq f(x)$ a.e. On the other hand, since an average does not exceed a supremum, $\|M_u f\|_{L^\infty(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)}$. It is shown in [7] that M_u has no nonconstant fixed points. In [6] A. Lerner studied whether given any $1 < p < \infty$, there is a constant $\varepsilon_{p,d} > 0$ such that

$$\|M_u f\|_{L^p(\mathbb{R}^d)} \geq (1+\varepsilon_{p,d})\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d). \tag{1}$$

We note that lack of existence of nonconstant fixed points does not imply (1). Using Riesz's sunrise lemma, Lerner proved for the real line that

$$\|M_u f\|_{L^p(\mathbb{R})} \geq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_{L^p(\mathbb{R})}.$$

A proof of inequality (1) for every dimension $d \geq 1$ and every $1 < p < \infty$ was obtained in [3]. Inequality (1) has been shown to be true for other maximal functions, say, maximal funtions defined taking the supremum over shifts and dilates of a fixed centrally symmetric

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convex body, maximal functions defined over λ-*dense family* of sets, *almost centered* maximal functions (see [3]) and dyadic maximal functions [8].

For the centered maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

Lerner's inequality

$$\|Mf\|_{L^p(\mathbb{R}^d)} \geq (1+\varepsilon_{p,d})\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d), \tag{2}$$

need not hold. First of all, it was shown in [5] that M has a nonconstant fixed point $f \in L^p(\mathbb{R}^d)$ (that is, $Mf = f$) if and only if $d \geq 3$ and $p > d/(d-2)$. But, as was noted before, the lack of nonconstant fixed points does not imply (2). In this context, Ivanisvili and Zbarsky (cf. [4]) noted that (2) is valid for any d when $p \equiv p_d$ is sufficiently close to 1.

The main result in [4] proves for $d = 1$ and every $1 < p < 2$ that (2) is true, in the form

$$\|Mf\|_{L^p(\mathbb{R})} \geq \left(\frac{p}{2(p-1)}\right)^{1/p} \|f\|_{L^p(\mathbb{R})}. \tag{3}$$

They also proved that inequality (2) holds for $d = 1$ and $1 < p < \infty$, if we restrict f to the class of indicator functions or unimodal functions. Besides, they conjectured (see [4, p. 343]) that (2) is valid for $d = 1$ and $1 < p < \infty$ without restrictions on the functions.

In this paper we give an afirmative answer to their conjecture, proving the following

Theorem 1.1. *Let $1 < p < \infty$. Then there exists an $\varepsilon_p > 0$ such that*

$$\|Mf\|_{L^p(\mathbb{R})} \geq (1+\varepsilon_p)\|f\|_{L^p(\mathbb{R})} \quad \text{for any } f \in L^p(\mathbb{R}).$$

Furthermore, if A_p is the best constant for the strong (p,p) inequality satisfied by the centered maximal operator on the real line, and γ_n is as in Definition 2.4, then for every $n \geq 1$ we can select

$$(1+\varepsilon_p)^p = 1 + \left(\frac{A_p - 1}{A_p^n - 1}\right)^p \left[\left(\frac{\gamma_n p}{(p-1)}\right)^{1/p} - 1\right]^p.$$

Let us note that this expression is stricly larger that 1 if we suitably choose n, taking into account that $\gamma_n \uparrow 1$ (see Remark 2.5).

Our approach consists of extending the methods in [4] and using the following inequality (see Lemma 2.7 below) for any locally integrable function in $\mathbb{R}$:

$$M^n f \geq \gamma_n M_L f, \tag{4}$$

where M_L denotes the left maximal operator and M^n denotes the iteration of the centered maximal operator n times. This inequality extends the trivial inequality $Mf \geq M_L f/2$.

Using (4), we prove

Theorem 1.2. *Let $n \in \mathbb{N}$ and $f \in L^p(\mathbb{R})$. Then,*

$$\|M^n f\|_p \geq \left(\frac{\gamma_n p}{(p-1)}\right)^{1/p} \|f\|_p.$$

Since $\gamma_1 = 1/2$, this result is an extension of (3).

Let us remark that simultaneously and independently, Zbarsky [9] has proved (2) for $d = 1$ and $d = 2$ and the centered maximal operator associated to centrally symmetric convex bodies. This extends Theorem 1.1, but without an explicit expression for the lower constant ε_p.

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2. Definitions and lemmas

Definition 2.1. For all $n \in \mathbb{N} \cup \{0\}$, define the following functions $g_n : [-1/2, \infty) \longrightarrow [0, 1]$. Let g_0 be the null function and for $n \geq 1$, set

$$g_n(t) := \frac{1 + \int_0^{1+2t} g_{n-1}(u)du}{2(1+t)}, \quad t \geq -\frac{1}{2}. \tag{5}$$

In the next lemma we give an explicit formula for the functions g_n.

Lemma 2.2. *Let $\{g_n\}_{n=0}^{\infty}$ be the functions from Definition 2.1. Then,*

(1) *$0 \leq g_n(t) \leq 1$ for all $n \in \{0\} \cup \mathbb{N}$ and all $t \geq -1/2$.*

(2) *For all $n \geq 0$ and $t \geq -1/2$, we have*

$$g_n(t) = \frac{\log(2+2t)}{1+t} \sum_{j=1}^{n} \frac{\log^{j-2}(2^j(1+t))}{2^j(j-1)!}.$$

(3) *For all $t \geq -1/2$, we have $\lim_{n\to\infty} g_n(t) = 1$.*

Proof. Part 1 of the lemma follows by simple induction in n. To prove part 2, for each $n \in \{0\} \cup \mathbb{N}$, we define $h_n(t) := g_{n+1}(t) - g_n(t)$. Since $g_0(t) = 0$, it holds that

$$g_n(t) = \sum_{j=0}^{n-1} h_j(t).$$

Let us note that $h_0(t) = g_1(t) - g_0(t) = g_1(t) = 1/(2+2t) > 0$. Besides, by (5),

$$h_n(t) = \frac{\int_0^{1+2t} h_{n-1}(u)du}{2(1+t)}, \quad n \geq 1,\ t \geq -1/2. \tag{6}$$

Now we set for each $n \geq 0$,

$$c_n(t) = \frac{\log(2+2t)}{1+t} \frac{1}{2^{n+1}n!} \log^{n-1}(2^{n+1}(1+t)), \quad t \geq -1/2,$$

where $c_0(-1/2)$ is defined by continuity, i.e., $c_0(-1/2) = \lim_{t\to-1/2^+} c_0(t) = \lim_{t\to-1/2^+} 1/(2+2t) = 1$. Then we have that $h_0(t) = c_0(t)$ for all $t \in [-1/2, \infty)$. Moreover, it is a calculus exercise to check that (6) also holds with c_n and c_{n-1} instead of h_n and h_{n-1}, for every $n \geq 1$. As a consequence, we have that $c_n(t) = h_n(t)$ for all $n \geq 0$ and all $t \in [-1/2, \infty)$. Thus, part 2 of the lemma holds.

Finally we will prove part 3 of the lemma. By part 2, we have that

$$\lim_{n\to\infty} g_n(t) = \frac{\log(2+2t)}{1+t}\sum_{j=1}^{\infty}\frac{\log^{j-2}(2^j(1+t))}{2^j(j-1)!}, \quad t \geq -1/2. \tag{7}$$

As a consequence of Lagrange expansion [1, p.206, eq.6.24], we can obtain

$$e^{xy} = \sum_{k=0}^{\infty} x(x+kz)^{k-1}\frac{(ye^{-yz})^k}{k!}, \quad x, y, z \in \mathbb{R}.$$

This equation with $y = 1$, $x = \log(2+2t)$ and $z = \log 2$ implies

$$2 + 2t = \sum_{k=0}^{\infty} \log(2+2t)(\log(2+2t) + k\log 2)^{k-1}\frac{2^{-k}}{k!}.$$

From this equation and (7), part 3 of the lemma follows. □

Remark 2.3. Part 3 of the lemma could also be proved by suitably bounding the functions g_n, using an argument inspired in [4, p.4-5] to obtain:

$$1 \geq g_n(t) \geq 1 - \left(\frac{\sqrt{8}}{3}\right)^n \sqrt{1+t}, \quad n \in \mathbb{N}, t \geq 0. \tag{8}$$

Definition 2.4. For each $n \in \mathbb{N}$, let us denote by $\gamma_n := g_n(0)$, where g_n are the functions from Definition 2.1.

Remark 2.5. It follows from the previous definition and Lemma 2.2 that

$$\gamma_n = \frac{1}{2}\sum_{j=1}^{n}\frac{j^{j-2}}{(j-1)!}\left(\frac{\log 2}{2}\right)^{j-1}, \quad n \in \mathbb{N}.$$

Besides, $\gamma_1 = 1/2$ and γ_n increases to 1 when $n \to \infty$.

Definition 2.6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally integrable function. We define the left maximal function $M_L f$ as

$$M_L f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(u)|du, \quad x \in \mathbb{R}.$$

It is easy to see that $Mf \geq M_L f/2$. In the next lemma we extend this inequality to the iterated centered maximal operator, defined via $M^1 f := Mf$, and for $n \geq 2$, $M^n f := M(M^{n-1}f)$.

Lemma 2.7. *Let $\{\gamma_n\}_{n=1}^{\infty}$ be the sequence from Definition 2.4. Then, for all $n \in \mathbb{N}$ and all f in $L^1_{loc}(\mathbb{R})$, $M^n f \geq \gamma_n M_L f$.*

Proof. Let us assume that $f \geq 0$. Fix $x \in \mathbb{R}$ and $h > 0$. Define $F(x,h) := \frac{1}{h}\int_{x-h}^{x} f(t)dt$. Now, using an inductive process in $n \in \mathbb{N}$, we will prove that for all $y \geq x$,

$$M^n f(y) \geq F(x,h)\; g_n\left(\frac{y-x}{h}\right), \tag{9}$$

where g_n comes from Definition 2.1. Indeed, for $n = 1$ and every $y \geq x$, we have

$$Mf(y) \geq \frac{1}{2(y-x+h)} \int_{x-h}^{2y-x+h} f(t)dt \geq \frac{hF(x,h)}{2(y-x+h)} = \frac{F(x,h)}{2\left(1+\frac{y-x}{h}\right)} = F(x,h)g_1\left(\frac{y-x}{h}\right).$$

Hence, by induction hypothesis, for all $n \geq 2$ and all $y \geq x$,

$$M^n f(y) \geq \frac{1}{2(y-x+h)} \int_{x-h}^{2y-x+h} M^{n-1}f(t)dt \geq \frac{\int_{x-h}^{x} f(t)dt + \int_{x}^{2y-x+h} M^{n-1}f(t)dt}{2(y-x+h)} \geq$$

$$\frac{hF(x+h) + F(x,h)\int_{x}^{2y-x+h} g_{n-1}\left(\frac{t-x}{h}\right)dt}{2(y-x+h)} = hF(x,h)\frac{1+\int_0^{1+2\frac{y-x}{h}} g_{n-1}(z)dz}{2(y-x+h)} =$$

$$= F(x,h)\frac{1+\int_0^{1+2\frac{y-x}{h}} g_{n-1}(z)dz}{2(1+\frac{y-x}{h})} = F(x,h)g_n\left(\frac{y-x}{h}\right),$$

so (9) is proved. As a consequence, $M^n f(x) \geq F(x,h)g_n(0) = F(x,h)\gamma_n$, and taking the supremum over $h > 0$ we obtain

$$M^n f(x) \geq M_L f(x) \cdot \gamma_n, \quad n \in \mathbb{N}.$$

□

Remark 2.8. It is known that for every $f \in L^p(\mathbb{R})$, we have $\|M_L f\|_p \geq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p$ (see [2, p.93, 2.1.11(a)] and integrate). This inequality, together with the previous lemma, leads inmediately to

$$\|M^n f\|_p \geq \gamma_n \|M_L f\|_p \geq \gamma_n \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p. \tag{10}$$

This inequality is enough to prove Theorem 1.1, but with a smaller ε_p. Indeed, inequality (10) will be improved in Theorem 1.2 by the use of the following lemma, which is an extension of [4, Lemma 3] and uses the same arguments. We include it here for the reader's convenience.

Lemma 2.9. *Let $0 < \lambda < \infty$ and $n \in \mathbb{N}$. For every locally integrable function $f \geq 0$ defined on the real line, it holds that*

$$|\{M^n f > \lambda\}| \geq \frac{\gamma_n}{\lambda} \int_{\{f>\lambda\}} f.$$

Proof. Since $M^n f \geq f$ almost everywhere and, by Lemma 2.7, $M^n f \geq \gamma_n M_L f$, we have (with the exception of a null set) that

$$\{M^n f > \lambda\} \supseteq \{f > \lambda\} \cup \{M_L f > \frac{\lambda}{\gamma_n}\}.$$

Then, we separate this into two disjoint sets, take Lebesgue measure, apply $M_L f > f$ a.e. and using Riesz's rising sun lemma [2, p.93] we obtain,

$$|\{M^n f > \lambda\}| \geq |\{f > \lambda\} \setminus \{M_L f > \frac{\lambda}{\gamma_n}\}| + |\{M_L f > \frac{\lambda}{\gamma_n}\}| \geq$$

$$\geq \frac{\gamma_n}{\lambda}\int_{\{f>\lambda\}\setminus\{M_Lf>\frac{\lambda}{\gamma_n}\}} f + \frac{\gamma_n}{\lambda}\int_{\{M_Lf>\frac{\lambda}{\gamma_n}\}} f \geq \frac{\gamma_n}{\lambda}\int_{\{f>\lambda\}\cup\{M_Lf>\frac{\lambda}{\gamma_n}\}} f \geq \frac{\gamma_n}{\lambda}\int_{\{f>\lambda\}} f.$$

□

3. Proofs of the theorems

To prove the theorems one just has to use the previous lemmas and some arguments from [4]. We include here the proofs for the reader's convenience.

Proof of Theorem 1.2. Without loss of generality we assume that $f \geq 0$. By Lemma 2.9 we have:

$$|\{M^n f > \lambda\}| \geq \frac{\gamma_n}{\lambda}\int_{\mathbb{R}} f(x)\chi_{(\lambda,\infty)}(f(x))dx.$$

We multiply both sides of the previous inequality by $p\lambda^{p-1}$ and integrate:

$$\int_{\mathbb{R}}(M^n f(x))^p dx \geq \int_0^\infty \gamma_n p\lambda^{p-2}\int_{\mathbb{R}} f(x)\chi_{(\lambda,\infty)}(f(x))dxd\lambda =$$

$$\gamma_n p\int_{\mathbb{R}} f(x)\int_0^{f(x)}\lambda^{p-2}d\lambda dx = \frac{\gamma_n p}{(p-1)}\int_{\mathbb{R}} f(x)^p dx.$$

□

Proof of Theorem 1.1. First, we have that

$$\|Mf\|_p^p \geq \|f\|_p^p + \|Mf-f\|_p^p. \tag{11}$$

Besides, if we denote by $A_p > 1$ the best constant for the strong (p,p) inequality satisfied by M, it holds that

$$\|M^n f - f\|_p \leq \sum_{i=1}^n \|M^i f - M^{i-1}f\|_p \leq \sum_{i=1}^n A_p^{i-1}\|Mf-f\|_p = \frac{A_p^n-1}{A_p-1}\|Mf-f\|_p. \tag{12}$$

Furthermore, by Theorem 1.2

$$\left(\frac{\gamma_n p}{(p-1)}\right)^{1/p}\|f\|_p \leq \|M^n f\|_p \leq \|M^n f - f\|_p + \|f\|_p.$$

Then

$$\left[\left(\frac{\gamma_n p}{(p-1)}\right)^{1/p} - 1\right]\|f\|_p \leq \|M^n f - f\|_p. \tag{13}$$

Now, putting (11), (12) and (13) together we get

$$\|Mf\|_p^p \geq \|f\|_p^p + \left(\frac{A_p-1}{A_p^n-1}\right)^p\|M^n f - f\|_p^p \geq$$

$$\|f\|_p^p + \left(\frac{A_p-1}{A_p^n-1}\right)^p\left[\left(\frac{\gamma_n p}{(p-1)}\right)^{1/p} - 1\right]^p\|f\|_p^p =$$

$$= \|f\|_p^p \left\{ 1 + \left(\frac{A_p - 1}{A_p^n - 1} \right)^p \left[\left(\frac{\gamma_n p}{(p-1)} \right)^{1/p} - 1 \right]^p \right\}.$$

Let us note that, by Remark 2.5, for n big enough, $\gamma_n p/(p-1) > 1$. □

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