The Lindemann theorem for matrices

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Abstract

In this note, we prove a Lindemann theorem about the transcendence of the exponential of square matrices.

In 1882, Ferdinand von Lindemann [1] proved that, if $\alpha \in \mathbb{C}$ is an algebraic number, then e^α is algebraic if and only if $\alpha = 0$. In particular, this result implies the transcendence of π because $e^{i\pi} = -1$. This note states a Lindemann theorem for the exponential of square matrices, that are defined as

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(The convergence of this series is well known.) In what follows, we will use I to denote the identity matrix, and O the zero matrix.

Theorem 1. *Let m be a positive integer, let $\mathbb{A}$ be the field of algebraic complex numbers, and let $M_m(\mathbb{A})$ be the set of $m \times m$ square matrices whose elements are in $\mathbb{A}$, and take $A \in M_m(\mathbb{A})$. Then $\exp(A) \in M_m(\mathbb{A})$ if and only if $A^m = O$.*

Proof. Let us define the function $F : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ by means of the series

$$F(X) = \exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

Thus, we want to prove that $F(A) \in M_m(\mathbb{A})$ if and only if $A^m = O$. The "if" part is clear, because $F(A)$ becomes a finite sum, so let us analyze the "only if" part.

The eigenvalues of A are the roots of the polynomial $\det(A - xI) = 0$ and, being that $\mathbb{A}$ is an algebraically closed field, these eigenvalues are in $\mathbb{A}$.

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On the other hand, $A^m = O$ is equivalent to saying that 0 is the unique eigenvalue of A. (It is enough to use the Cayley–Hamilton theorem.) Let us prove that 0 is the unique eigenvalue by *reductio ad absurdum*.

Let us suppose that $\lambda \neq 0$ in $\mathbb{A}$ is an eigenvalue of A. An eigenvector associated to λ will be a nonzero vector solution $\mathbf{v} \in \mathbb{C}^m$ of the linear homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. The existence of nonzero solutions is guaranteed because $\det(A - \lambda I) = 0$. The general solution of the system depends on $m - r$ free parameters, with $r = \operatorname{rank}(A - \lambda I)$, and it can be obtained by means of gaussian elimination, a process that only uses algebraic operations. As $A - \lambda I$ is in $M_m(\mathbb{A})$, by choosing algebraic values, not all zero, for the free parameters, we will have an eigenvector $\mathbf{v} \in \mathbb{A}^m$. But, by hypothesis, $F(A) \in M_m(\mathbb{A})$, so also $F(A)\mathbf{v} \in \mathbb{A}^m$.

From $A\mathbf{v} = \lambda\mathbf{v}$, it follows that $A^k\mathbf{v} = \lambda^k\mathbf{v}$ for any $k \geq 1$. Consequently,

$$F(A)\mathbf{v} = \exp(A)\mathbf{v} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \mathbf{v} = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \mathbf{v} = (\exp(\lambda))\mathbf{v}.$$

From this equality, and because $\mathbf{v} \neq \mathbf{0}$, it follows that $\exp(\lambda) \in \mathbb{A}$. But, by the Lindemann theorem, the unique $\lambda \in \mathbb{A}$ such that $\exp(\lambda)$ is in $\mathbb{A}$ is $\lambda = 0$, so we have found a contradiction. □

Well-known consequences of the Lindemann theorem, or the more general Lindemann–Weierstrass theorem, are that, for $\alpha \neq 0$ an algebraic number, $\log(1+\alpha)$, $\sin(\alpha)$, $\cos(\alpha)$, $\sinh(\alpha)$, $\cosh(\alpha)$ are trascendental numbers.

Moreover, it is standard to extend an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with radius of convergence $r > 0$ to a matrix function $F(X) = \sum_{k=0}^{\infty} a_k X^k$ that converges for matrices X whose spectral radius $\rho(X)$ (i.e., the maximum of the absolute value of the eigenvalues of X) satisfies $\rho(X) < r$. In particular, this can be done with log, sin, cos, sinh and cosh. Then, with the same proof as in Theorem 1, we have the following.

Theorem 2. *For $A \in M_m(\mathbb{A})$, any of $\log(I + A)$ (with $\rho(A) < 1$), $\sin(A)$, $\cos(A)$, $\sinh(A)$ or $\cosh(A)$ is in $M_m(\mathbb{A})$ if and only if $A^m = O$.*

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References

[1] Lindemann, F. (1882). Über die Zahl π, *Math. Ann.* 20: 213–225.