PERIODIC ORBITS AROUND GEOSTATIONARY POSITIONS

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Abstract. We generate families of planar periodic orbits emanating from the geostationary points, both stable and unstable. We show that, even for the unstable points, it is possible to have stable periodic orbits.

Key words: periodic orbits, geostationary positions

1. Introduction

As it is well known (Blitzer et al., 1962; Musen, 1962; Blitzer, 1965), the dynamics of a satellite under the action of a truncated potential that takes into account the effect of the second order harmonics (the zonal J_2 and the tesserals C_{22} and S_{22}) has four equilibria, the so-called geostationary points. The linear stability is also well established, two of them stable, while the other two are unstable (Morando, 1963). Even more, recently, Deprit and López (1966) proved that for the Earth and Earth-like planets, the two linear stable points are also stable in the Lyapunov sense. Thus, neighborhoods of the stable points seem to be good candidates to place satellites, a fact that has been realized a long time ago (see e.g. Soop, 1994 and references therein) and some attempts to make analytical theories of orbits around the stable points have been made (Oberti, 1994). The other two equilibria are unstable, and in principle, they are avoided; however, we show here that it is possible to find families of periodic orbits emanating from these unstable points and that some of the orbits are stable.

The motion of synchronous satellites is known to be affected from a drift in the longitude of the satellite produced by the perturbation of the tesseral harmonics, making east-west stationkeeping necessary for practical purposes (Gedeon, 1969). The semimajor axis and longitude are also affected by the luni-solar perturbations, and the east-west stationkeeping strategy must take their effect into account (Kamel et al., 1973). The solar radiation pressure produces long term effects in longitude, inclination and eccentricity for synchronous satellites (Milani et al., 1987). The inclination is also affected by tesserals and luni-solar perturbations, but for all practical purposes the long term effects in inclination are considered independent of the other effects, and inclination maneuvres are prepared independently (Soop, 1994). The drift in longitude produced by tesserals results in the periodic

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solutions computed in this paper. Therefore, they can be of practical interest making the corresponding part of the east-west stationkeeping maneuvres non-necessary.

Briefly, the procedure followed to find families of periodic orbits is the following: starting with a set of initial conditions close to one periodic solution, we correct this initial set to obtain initial conditions for a true periodic orbit. Then, we vary the value of the parameter (the Jacobian constant C in the present case), and by calculating and refining a tangent prediction we obtain new initial conditions corresponding to a periodic orbit for the new value of the chosen parameter. In order to improve the prediction, we must numerically integrate the equations of motion and their tangent and normal variations, that is, the variational equations associated with this solution. The main feature of this method is that it splits the normal displacements along an orbit from the tangent ones: the latter, indeed, are secular in nature. For details, the reader is addressed to (Deprit and Henrard, 1967) and (Lara et al., 1995). As a by product of the method employed, we determine for each orbit of the family a stability index which allow us to check for which values of the parameters the family is stable, and also to determine the value of the stability-instability bifurcation, which is a matter of capital importance.

But to start the method, we need a periodic orbit. As a guideline for our research we consider a similar situation with $1:1$ resonance, namely, the restricted three body problem (RTBP). In this problem, it is well known (see e.g. Szebehely, 1967) that from the linearized system it is possible to find short- and long-period periodic orbits around the Lagrangian points for some values of the mass parameter. Thus, from the linearized equations of motion, we select a set of initial conditions corresponding to an almost periodic orbit; then, taking into account differential corrections, we modify the original initial conditions until we find initial conditions for a true periodic orbit. Once such an orbit is obtained, we apply the method of numerical continuation of families of periodic orbits (Deprit and Henrard, 1967; Lara et al., 1995) to obtain a family that depends on a parameter, namely the Jacobian constant.

As an illustration of the method employed, we present here several families around both stable and unstable points and, also, we determine their stability.

2. Geostationary Points

Let us consider the motion of a satellite referred to a synodic reference frame that is rotating as the planet does. The origin of the reference frame is at the center of mass of the Earth, and the axes coincide with the principal axes of inertia. We consider the satellite as a mass point, and take up to the second order in the potential expansion. We also suppose that the Earth rotates around the z-axis with constant velocity ω. Under these assumptions, the Lagrangian defining the motion is

$$\mathcal{L} = \tfrac{1}{2}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \omega(x\dot{y} - y\dot{x}) + \Omega(x, y, z), \tag{1}$$

where Ω is the effective potential function

$$\Omega = \tfrac{1}{2}\,\omega^2(x^2 + y^2) - \mathcal{V}(x, y, z), \tag{2}$$

and the potential is

$$\mathcal{V} = -\frac{\mu}{r}\left[1 + \left(\frac{\alpha}{r}\right)^2 \left\{3C_{2,2}\frac{x^2 - y^2}{r^2} - \frac{1}{2}C_{2,0}\left(1 - 3\frac{z^2}{r^2}\right)\right\}\right], \tag{3}$$

where μ is the gravitational constant, $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance of the satellite, α the equatorial radius and the harmonic coefficients are $C_{2,0} < 0 < C_{2,2}$ because the Earth spins around its axis of greatest inertia. The numerical values we use are $C_{2,0} = -0.1082630 \times 10^{-2}$ and $C_{2,2} = 0.1814964 \times 10^{-5}$, taken from (Deprit and López, 1996). Note that these coefficients are not the usual ones appearing in the literature, as they are computed with respect to the Earth's principal axes.

The equations of motion corresponding to the Lagrangian (1) are

$$\begin{aligned} &\ddot{x} - 2\omega\,\dot{y} = \Omega_x, \\ &\ddot{y} + 2\omega\,\dot{x} = \Omega_y, \\ &\ddot{z} = \Omega_z. \end{aligned} \tag{4}$$

As usually happens with equations referred to rotating frames, the system (4) accepts the Jacobian integral

$$C = 2\Omega - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{5}$$

The stationary solutions of this problem are found by solving the system

$$\Omega_x = \Omega_y = \Omega_z = 0, \tag{6}$$

equations that are fulfilled only on the equatorial plane ($z = 0$) and when either $y = 0$ and

$$\left(\frac{r}{a_k}\right)^5 - \left(\frac{r}{a_k}\right)^2 = 3\epsilon_x, \tag{7}$$

or $x = 0$ and

$$\left(\frac{r}{a_k}\right)^5 - \left(\frac{r}{a_k}\right)^2 = 3\epsilon_y, \tag{8}$$

where the parameters ϵ_x and ϵ_y are defined by

$$\epsilon_x = \left(-\tfrac{1}{2}C_{2,0} + 3C_{2,2}\right)(\alpha/a_k)^2, \tag{9}$$

$$\epsilon_y = \left(-\tfrac{1}{2}C_{2,0} - 3C_{2,2}\right)(\alpha/a_k)^2, \tag{10}$$

and the scaling factor $a_k = (\mu/\omega^2)^{1/3}$ is the semimajor axis of a Keplerian orbit with mean motion $n = \omega$. Note that $\epsilon_x > 0$ for an oblate body.

For the first case, Equation (7), a sufficient condition to have real roots $x = \pm r$ is that $r > a_k$. The corresponding implicit equation can be solved, for instance, by Newton–Raphson starting from $r = a_k$. One simple iteration results accurate to the order of ϵ^4 while two iterations are enough up to $\mathcal{O}(\epsilon^8)$

$$\begin{aligned}\frac{r}{a_k} &= 1 + \epsilon_x - 3\epsilon_x^2 + \frac{44}{3}\epsilon_x^3 - \frac{260}{3}\epsilon_x^4 + \\ &\quad + 567\epsilon_x^5 - \frac{35581}{9}\epsilon_x^6 + \frac{259160}{9}\epsilon_x^7 + \mathcal{O}(\epsilon_x^8),\end{aligned} \tag{11}$$

Note that the series (11) is equivalent to that of (Wytrzyszczak, 1998, p. 16) or to Equation (9) of Deprit and López (1996), that was obtained by solving the corresponding implicit equation by means of Lie transforms.

In the second case, Equation (8), the right-hand member of the quintic equation is positive for the Earth and Equation (11) is applicable again, simply by switching the parameter ϵ_x by ϵ_y. When $\epsilon_y < 0$, two or none real solutions could exist in the y-axis; this case has been studied in detail by Howard (1990).

In sum, for the Earth, Equations (11) have four equilibria, two on the x-axis, that we denote by $E_1(\pm r_1, 0)$ and two on the y-axis that we name $E_2(0, \pm r_2)$.

By denoting $\dot{x} = u$, $\dot{y} = v$, the linearized motion around the equilibria is written as

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{u} \\ \delta\dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{xx} & \Omega_{xy} & 0 & 2\omega \\ \Omega_{yx} & \Omega_{yy} & -2\omega & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta u \\ \delta v \end{pmatrix}, \tag{12}$$

where the Hessian matrix must be evaluated at the equilibria. It is worth noting that the crossed partial derivative

$$\Omega_{xy} = 3\mu\frac{x\,y}{r^5}\left[1 + 5\frac{\alpha^2}{r^2}\left(-\frac{1}{2}C_{2,0} + 7\,C_{2,2}\,\frac{x^2 - y^2}{r^2}\right)\right] \tag{13}$$

vanish along both axes (x and y), whence, the Hessian determinant is $H = \Omega_{xx}\,\Omega_{yy}$, and for the geostationary points its value is

$$H(r_i) = (-1)^i\,12\,C_{2,2}\frac{\alpha^2\mu^2}{r_i^5}\left(\frac{5}{a_k^3} - \frac{2}{r_i^3}\right), \tag{14}$$

and is negative for $i = 1$ (x-axis) and positive for $i = 2$ (y-axis).

The eigenvalues of system (12) are computed from the characteristic equation

$$\lambda^4 + 2\Sigma\,\lambda^2 + \Pi = \Lambda^2 + 2\Sigma\,\Lambda + \Pi = 0, \tag{15}$$

where

$$\Sigma = 2\omega^2 - \tfrac{1}{2}[\Omega_{xx}(r_i) + \Omega_{yy}(r_i)], \qquad \Pi = H(r_i). \tag{16}$$

For the points $E_1(\pm r_1, 0)$ on the x-axis, $\Pi = H(r_1) < 0$ and, consequently, the two roots Λ_1, Λ_2 of Equation (15) are

$$\begin{aligned} \Lambda_1 &= -\Sigma + \sqrt{\Sigma^2 - \Pi} > 0, \\ \Lambda_2 &= -\Sigma - \sqrt{\Sigma^2 - \Pi} < 0. \end{aligned} \tag{17}$$

Therefore, as there are two real and two complex (purely imaginary) roots, the points E_1 are unstable.

On the contrary, the eigenvalues corresponding to the stationary points $E_2(0, \pm r_2)$ are purely imaginary,

$$\begin{aligned} \lambda_{1,2} &= \pm\sqrt{-1}\,\sqrt{-\Lambda_1} = \pm\sqrt{-1}\,w_1, \\ \lambda_{3,4} &= \pm\sqrt{-1}\,\sqrt{-\Lambda_2} = \pm\sqrt{-1}\,w_2, \end{aligned} \tag{18}$$

therefore, these equilibria are linearly stable. More than this, by means of the stability Arnold theorem for non-definite quadratic forms (Arnold, 1961) Deprit and López (1996) proved that these points also enjoy Lyapunov stability.

3. Natural Families of Planar Periodic Orbits

To find families of periodic orbits, we use the method of numerical continuation with respect to a parameter. The method is essentially the one given by Deprit and Henrard (1967) with some additions made in (Lara et al., 1995) and (Lara, 1996). The process addresses a boundary value problem for the variational equations relative to conservative dynamical systems with two degrees of freedom.

Let us consider the two degree of freedom system

$$\begin{aligned} \ddot{x} &= 2A(x, y; \sigma)\,\dot{y} + W_x(x, y; \sigma), \\ \ddot{y} &= -2A(x, y; \sigma)\,\dot{x} + W_y(x, y; \sigma), \end{aligned} \tag{19}$$

where A and W are two functions depending on the coordinates and on one parameter σ. This system has the Jacobian integral

$$C = 2W - (\dot{x}^2 + \dot{y}^2). \tag{20}$$

Each manifold $C =$ constant is determined by the initial conditions. By introducing the notation $U \equiv W(x, y; \sigma) - C/2$, system (19) together with the Jacobian integral (20) can be written as

$$\begin{aligned} \ddot{x} &= 2A(x, y; \sigma)\dot{y} + U_x(x, y; \sigma), \\ \ddot{y} &= -2A(x, y; \sigma)\dot{x} + U_y(x, y; \sigma), \\ \mathcal{I} &= \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - U(x, y; \sigma) \equiv 0. \end{aligned} \tag{21}$$

Let us assume that $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \sigma_0)$ – solution of the system (21) for the initial conditions $\boldsymbol{\xi}_0 = (x_0, y_0, \dot{x}_0, \dot{y}_0)$ – is periodic with period T_0. Hence,

$$\boldsymbol{\xi}(t; \sigma_0) = \boldsymbol{\xi}(t + T_0; \sigma_0). \tag{22}$$

Let us choose a variation of the parameter $\sigma = \sigma_0 + \Delta\sigma$. The question is how to find a new set of initial conditions $\boldsymbol{\xi}_0 + \delta\boldsymbol{\xi}_0$ such that the solution of (21) for these initial conditions be periodic with period $T_0 + \Delta T$, that is, such that

$$\boldsymbol{\xi}(t; \sigma_0 + \Delta\sigma) = \boldsymbol{\xi}(t + T; \sigma_0 + \Delta\sigma). \tag{23}$$

The Poincaré method of continuation ensures that, under certain conditions, there exist initial conditions

$$\boldsymbol{\xi}_0 + \sum_{k \geqslant 1} \boldsymbol{\xi}_{0,k}\,(\Delta\sigma)^k \tag{24}$$

such that the solution of the ordinary differential Equations (21) is periodic with period

$$T = T_0 + \sum_{k \geqslant 1} T_{0,k}\,(\Delta\sigma)^k. \tag{25}$$

In general it is not possible to compute all the coefficients in the series Equations (24), (25). Contrary, assuming $\Delta\sigma$ small enough, we only consider the first order approximation for the initial conditions

$$\boldsymbol{\xi}_0 + \Delta\sigma\,\delta\boldsymbol{\xi}_0, \tag{26}$$

the corrections $\delta\boldsymbol{\xi}_0$ are solutions of the variational system

$$\begin{aligned}
\delta\ddot{x} &= 2A\delta\dot{y} + (U_{xx} + 2A_x\dot{y})\delta x + (U_{xy} + 2A_y\dot{y})\delta y + U_{x\sigma} + 2A_\sigma\dot{y},\\
\delta\ddot{y} &= -2A\delta\dot{x} + (U_{xy} - 2A_x\dot{x})\delta x + (U_{yy} - 2A_y\dot{x})\delta y + U_{y\sigma} - 2A_\sigma\dot{x},\\
\delta\mathcal{I} &= \dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - U_x\delta x - U_y\delta y - U_\sigma \equiv 0.
\end{aligned} \tag{27}$$

As system (27) is linear, the general solution is formed by a linear combination of particular solutions that depend on arbitrary integration constants. The desired variations will be determined by computing those values of the integration constants that fulfill the periodicity condition (23). Notice from (23) that the unknown ΔT is an implicit variable; we overcome this inconvenience by expanding (23) up to the first order.

The Cartesian formulation of the variational Equations (27) mix secular terms, produced by displacements along the orbit, with periodic terms. An alternative formulation can be given by using intrinsic coordinates, namely the tangent p and the normal n displacements obtained from the rotation

$$\begin{aligned}
\delta x &= p\cos\phi - n\sin\phi,\\
\delta y &= p\sin\phi + n\cos\phi,\\
\delta\dot{x} &= \dot{p}\cos\phi - \dot{n}\sin\phi - \dot{\phi}\,\delta y,\\
\delta\dot{y} &= \dot{p}\sin\phi + \dot{n}\cos\phi + \dot{\phi}\,\delta y,
\end{aligned} \tag{28}$$

where

$$\tan\phi = \frac{\dot{y}}{\dot{x}}. \tag{29}$$

When using intrinsic coordinates, the variational differential equations are separable (Deprit and Henrard, 1967); indeed, the normal displacements are obtained from the generalized Hill equation

$$\ddot{n} + \Theta n = -2U_\sigma \frac{A+\dot{\phi}}{V} - 2A_\sigma V + U_{y\sigma}\cos\phi - U_{x\sigma}\sin\phi, \tag{30}$$

where

$$\begin{aligned}\Theta \;=\; & \frac{\ddot{V}}{V} + 2(A+\dot{\phi})^2 + \\ & + 2A^2 - U_{xx} - U_{yy} - 2V(A_x\sin\phi - A_y\cos\phi),\end{aligned} \tag{31}$$

and

$$V^2 = \dot{x}^2 + \dot{y}^2. \tag{32}$$

The next step is the computation of the tangent displacement p by integrating the quadrature

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{V}\right) = 2\,\frac{A+\dot{\phi}}{V} n + \frac{U_\sigma}{V^2}. \tag{33}$$

The corrections computed from (26) are tangent approximations, hence the algorithm must consist of two steps: the tangent predictor followed by an isoenergetic corrector (see Deprit and Henrard, 1967; Lara et al., 1995 for practical details on the implementation).

The knowledge of orbit stability is of capital importance. The stability of each orbit is determined by means of the index $k = |\mathrm{Tr}(T)|$, with $\mathrm{Tr}(T)$ the trace of the resolvent of Hill's Equation (30). If $k > 2$ the orbit is unstable, if $k < 2$ the orbit is stable, and when $k = 2$ we have indifferent stability.

4. Families of Periodic Orbits

To initialize the algorithm for numerically propagating periodic orbits, we need one periodic orbit to begin with. We determine such an orbit from the linearized system (12). Generally, a periodic orbit of the linear equation will not be periodic in the general system due to non-linearities. However, differential corrections can modify the initial conditions to satisfy the periodicity conditions.

The general solution of the variational Equation (12) is

$$\delta x = \sum_{1\leqslant j\leqslant 4} A_j e^{\lambda_j t}, \qquad \delta y = \sum_{1\leqslant j\leqslant 4} \alpha_j A_j e^{\lambda_j t}, \tag{34}$$

where A_j $(1 \leqslant j \leqslant 4)$ are arbitrary integration constants and the coefficients α_j (cf. Szebehely, 1967, p. 244, Eq. (12)) are

$$\alpha_j = \frac{\lambda_j^2 - \Omega_{xx}}{2\omega\lambda_j}. \tag{35}$$

4.1. MOTION NEAR THE y-AXIS EQUILIBRIA

As proved in Section 2, the eigenvalues corresponding to the points $E_2(0, \pm r_2)$ are pure imaginary,

$$\lambda_{1,2} = \pm\sqrt{-1}\, w_1, \qquad \lambda_{3,4} = \pm\sqrt{-1}\, w_2, \tag{36}$$

thus, Equations (34) may be rewritten as

$$\begin{aligned} \delta x &= C_1 \cos w_1 t + S_1 \sin w_1 t + C_2 \cos w_2 t + S_2 \sin w_2 t, \\ \delta y &= (S_1 \cos w_1 t - C_1 \sin w_1 t)\, q(w_1) + \\ &\quad + (S_2 \cos w_2 t - C_2 \sin w_2 t)\, q(w_2), \end{aligned} \tag{37}$$

where

$$q(w_i) = \frac{w_i + \Omega_{xx}/w_i}{2\omega}, \qquad i = 1, 2. \tag{38}$$

For the stable equilibria at the Earth $w_1 \ll w_2$, hence Equations (37) are comprised of long- and short-period terms. As the integration constants (C_1, S_1, C_2, S_2) can be expressed in terms of the initial conditions a proper selection of the initial conditions will remove either the long- or short-period terms from the solution.

4.1.1. *Short-period periodic motion*

Setting $C_1 = S_1 = 0$ and replacing the integration constants by the initial conditions, the linear periodic solution is

$$\begin{aligned} \delta x &= \delta x_0 \cos w_2 t + \frac{1}{q(w_2)} \delta y_0 \sin w_2 t, \\ \delta y &= \delta y_0 \cos w_2 t - q(w_2)\, \delta x_0 \sin w_2 t, \end{aligned} \tag{39}$$

and consequently, only short-period terms are retained.

However, taking into account the nonlinear part of the problem, this solution (39) no longer represents a periodic orbit, but for δx_0, δy_0 small enough and for the corresponding initial velocities

$$\delta \dot{x}_0 = \frac{1}{q(w_2)} \delta y_0\, w_2, \qquad \delta \dot{y}_0 = -q(w_2)\, \delta x_0\, w_2, \tag{40}$$

the solution of the equations of motion (4) for the initial conditions $(\delta x_0, r_2 + \delta y_0, \delta\dot{x}_0, \delta\dot{y}_0)$ is approximately periodic with period

$$T = \frac{2\pi}{w_2}. \tag{41}$$

The use of the differential corrector algorithm described in Section 3 improves those initial conditions and the period in an iterative way until finding an exact periodic orbit.

As an illustration we set $\delta x_0 = 0$, $\delta y_0 = 0.003$, and compute $\delta\dot{x}_0$ and $\delta\dot{y}_0$ from Equation (40). The units of length and time are such that $\mu = 1$ and $\omega = 1$. For the period $T \approx 6.2834227$ computed from Equation (41) these initial conditions produce $\max|\xi(0) - \xi(T)| \approx 10^{-4}$, where $\xi = x, y, \dot{x}, \dot{y}$. After differential corrections the periodicity condition is within 10^{-13} for every coordinate ξ. Figure 1 shows the initial approximation and the improved periodic orbit.

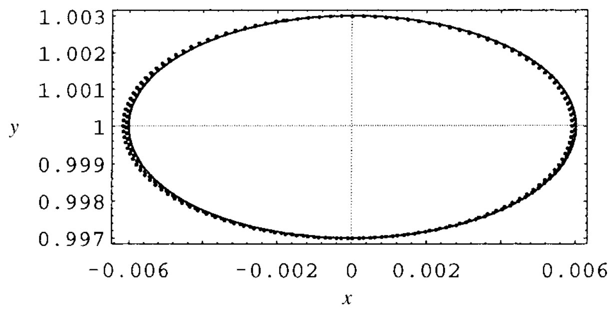

Figure 1. Short-period periodic motion around the $E_2(0, +r_2)$ geostationary equilibrium. The dashed line corresponds to a quasi periodic solution; the solid line is the improved periodic solution.

4.1.2. *Long-period periodic motion*

Now the short-period terms affected by the coefficients C_2 and S_2, are removed from Equation (37), which results again in Equations (39) but now with w_1 instead of w_2. The initial velocities and period are also obtained from (40) and (41) by switching w_2 by w_1. We make $\delta x_0 = 0$ and $\delta y_0 = 0.0001$; the corresponding initial velocities are $\dot{x}_0 \approx 0.00015$ and $\dot{y}_0 = 0$; the approximate period is $T \approx 5138.49$ close to 2.5 years. The initial approximation has a periodicity $\max|\xi(0) - \xi(T)| \approx 3 \times 10^{-3}$. Figure 2 shows the long-period improved solution around the $(0, +r_2)$ equilibrium. In this case the periodicity condition is of the order 10^{-10}.

4.2. MOTION NEAR THE x-AXIS EQUILIBRIA

The points $E_1(\pm r_1, 0)$ are unstable, but despite this fact, there exists short-period periodic motion around these equilibria, as it happens with the collinear points of the RTBP (see Szebehely, 1967, p. 242 ff.).

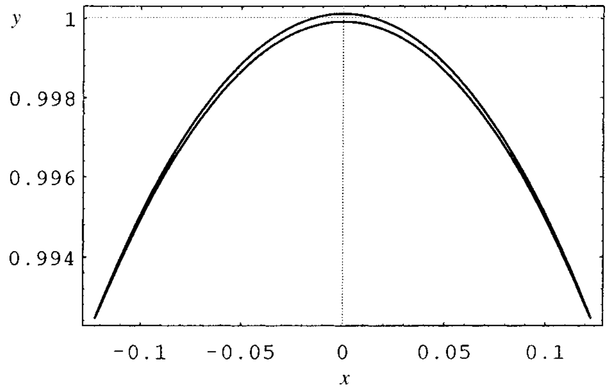

Figure 2. Long-period periodic motion around the $(0, +r_2)$ geostationary point. The period of the orbit is close to two and a half years. Note the different scales for abscissas and ordinates.

In linear approximation, the solution is again given by (39). The initial velocities corresponding to a small displacement δx_0, δy_0 are again those of (40). Then, the state

$$(r_1 + \delta x_0, \delta y_0, \delta \dot{x}_0, \delta \dot{y}_0)$$

corresponds to an approximately periodic solution of Equations (4) with period given by Equation (41).

The same initial conditions and period used in Section 4.1.1. for the short-period periodic motion around the E_2 equilibrium produce now max $|\xi(0)-\xi(T)| \approx 10^{-3}$; after the use of the corrector algorithm the periodicity condition is of the order 10^{-14}. Figure 3 presents both the initial and the improved solution.

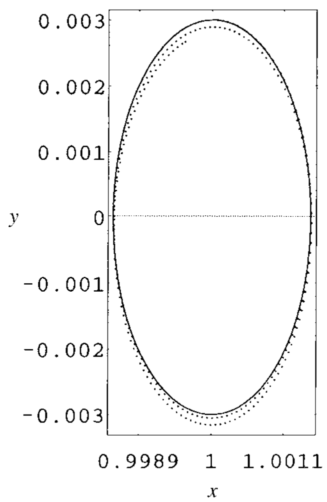

Figure 3. Periodic motion around the $E_1(+r_1, 0)$ geostationary equilibrium. The dashed line corresponds to a quasi-periodic solution; the solid line is the improved periodic solution.

4.3. FAMILIES OF PERIODIC ORBITS

Once we find initial conditions for periodic orbits around the equilibria, we are in situation to apply the method of continuation of periodic orbits families discussed in Section 3. We propagate the long-period and the two short-period orbits found above.

The long-period family is made of almost circular orbits that in the rotating frame are seen as periodic oscillations around the stable equilibrium. Both the amplitude of the oscillation and the period of the orbit grow with the Jacobian constant, and apparently this family ends with an orbit that oscillates between both unstable equilibrium with an infinite period, thus corresponding to a heteroclinic orbit. As can be appreciated in Figure 4 the behavior of the stability index is highly oscillatory between the critical values $k = \pm 2$. The rightmost part of the figure presents one orbit close to the termination of this family.

The family that is made of short-period periodic orbits around the E_2 geostationary equilibrium is named here as the E_2-family. This family starts with orbits that are small ellipses around E_2 with the semimajor axis in the x-direction. For increasing values of C, the eccentricity of the orbits grows up, and for the Earth shape, part of this family is made of collision orbits. We focus our study on non-collision orbits of the Earth. Figure 5 presents the evolution of the stability index k versus the Jacobian constant C. Note that the orbits of this family change from stable ($k < 2$) to unstable ($k > 2$) at the value $C \approx 2.46315258$. Therefore, in the notation of Hénon (1965) there is a critical point of the first kind, where $k = 2$, which points towards possible bifurcations in the plane. The orbits of this family have a period close to one sidereal day that slightly increases with eccentricity. Figure 6 presents several non-collision orbits of the family and the evolution of the period.

The behavior of the E_1-family – the family made of short-period periodic orbits around E_1 – is quite similar to the previous one. The family starts with orbits that

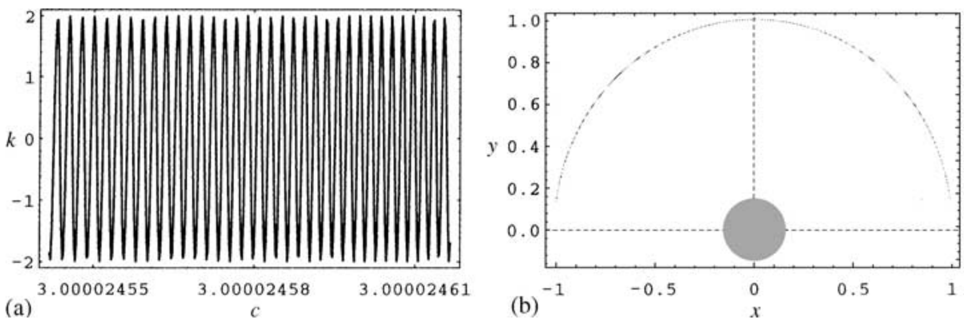

Figure 4. (a) Evolution of the stability index k of the family made of long-period periodic orbits around the $E_2(0, +r_2)$ geostationary equilibrium. For clarity only a small part of the family is presented. (b) One orbit close to the termination. The period is about 5 years.

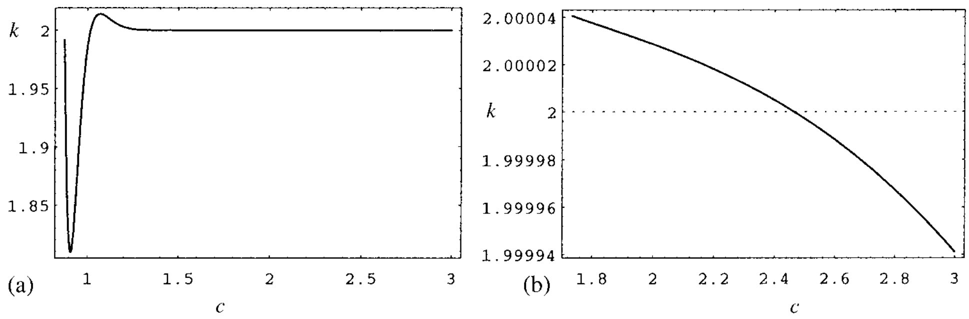

Figure 5. Stability index k of the E_2-family versus the Jacobian constant C. (a) The whole family, (b) part made of non-collision orbits of the Earth.

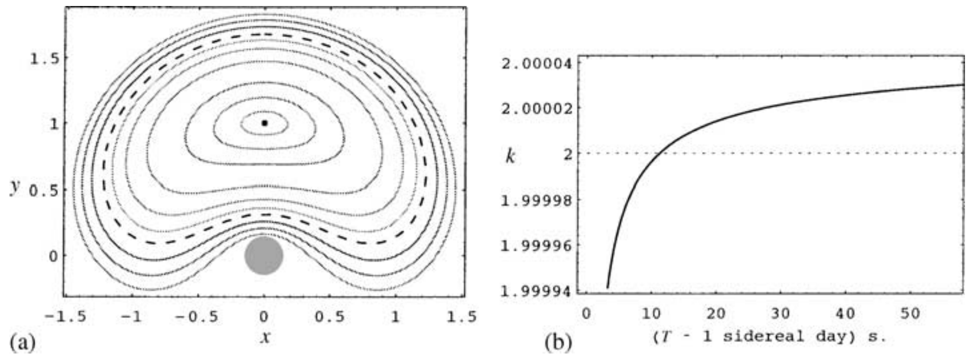

Figure 6. (a) Some orbits of the E_2-family; the dashed line marks the change in stability. (b) Evolution of the period.

are small ellipses around E_1 with the semimajor axis in the y-direction, and the eccentricity grows as C does. Figure 7 presents the evolution of k versus C for non-collision orbits of the Earth. Figure 8 provides some orbits of this family and the evolution of the period.

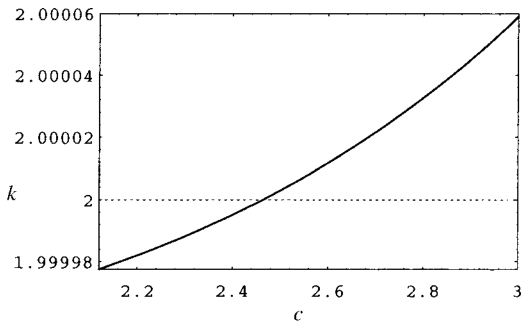

Figure 7. Evolution of the stability index k for non-collision orbits of the E_1-family.

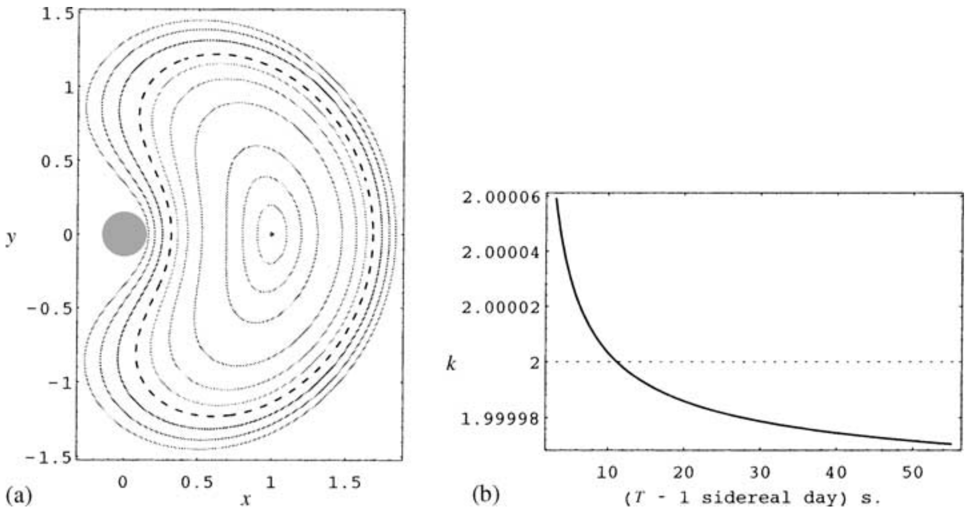

Figure 8. (a) Some orbits of the E_1-family; the dashed line marks the change in stability. (b) Evolution of the period.

The orbits of the E_1-family change from instability ($k > 2$) to stability ($k < 2$) at the value $C \approx 2.46315062$. After a search in the vicinity of this critical value we found a bifurcated family that in the narrow range $\Delta C \approx 2 \times 10^{-6}$ migrates from the critical orbit of the E_2-family to the critical one of the E_1-family. Figure 9 shows several orbits of this family. Within the numerical precision, the orbits of this bifurcated family remain very close to the state of indifferent stability ($k = 2$). Of course, due to the symmetries in our model, four of these families exist.

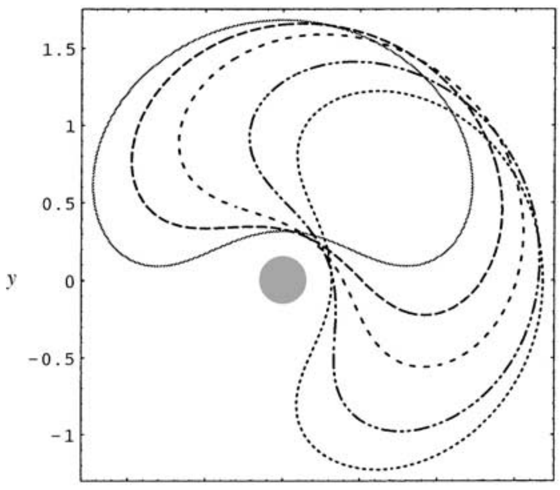

Figure 9. Some orbits of the bifurcated family linking the critical orbits of the E_2- and E_1-families.

5. Conclusions

The study of the linearized motion in the vicinity of the geostationary positions shows that there exist periodic motions around them. The contribution of the non-linear terms break the periodicity of those orbits. However, the use of a differential corrector algorithm enables the computation of families of true periodic orbits emanating from the linearized solutions. The families of short-period periodic orbits emanating from the unstable equilibria are made of unstable orbits, but at a certain point a bifurcation occurs and the orbits become stable. At that critical point new families of periodic orbits appear. These new families are made of orbits that migrate from periodic motion around the unstable equilibria to periodic motion around the stable ones.

The need of east-west stationkeeping efforts on the satellite orbits is mainly due to tesseral harmonics in the potential field. Therefore, the solutions presented here could be useful as nominal solutions when recovering perturbations of the same order of the Earth's $C_{2,2}$ gravity perturbation, namely the solar radiation pressure and tidal attractions from the Sun and Moon, that are neglected in our model.

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