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Rota–Baxter Operators on Quadratic Algebras

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Abstract. We prove that all Rota–Baxter operators on a quadratic division algebra are trivial. For nonzero weight, we state that all Rota–Baxter operators on the simple odd-dimensional Jordan algebra of bilinear form are projections on a subalgebra along another one. For weight zero, we find a connection between the Rota–Baxter operators and the solutions to the alternative Yang–Baxter equation on the Cayley–Dickson algebra. We also investigate the Rota–Baxter operators on the matrix algebras of order two, the Grassmann algebra of plane, and the Kaplansky superalgebra.

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1. Introduction

Given an algebra A and a scalar Δ in a field F, a linear operator $R\colon A \to A$ is called a *Rota–Baxter operator* (*RB operator*, shortly) on A of weight Δ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \Delta xy) \tag{1.1}$$

holds for all $x, y \in A$. The algebra A is called the *Rota–Baxter algebra* (*RB algebra*).

The Rota–Baxter algebras were introduced by Baxter [5], and then, they were popularized by Rota and his school [25, 26]. The linear operators with the property (1.1) were independently introduced in the context of Lie algebras by Belavin and Drinfeld [6] and by Semenov-Tyan-Shansky [27]. These operators were connected with the so-called R-matrices, which are solutions to the classical Yang–Baxter equation. Recently, some applications of the Rota–Baxter algebras were found in such areas as the quantum field theory, the Yang–Baxter equations, the cross products, the operads, the Hopf

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algebras, the combinatorics, and the number theory (some references may be found, for example, in [16]).

In 2000, Aguiar established a connection between the Rota–Baxter algebras and the dendriform algebras. He showed that a Rota–Baxter algebra of weight $\Delta = 0$ possesses the structure of a dendriform algebra. Later on, a connection with the dendriform trialgebras was established [8]. Some functors between the categories of the Rota–Baxter algebras and the dendriform dialgebras (trialgebras) were investigated in [16].

In the present article, we are interested in the study (classification) of the structures of Rota–Baxter algebras on some well-known simple (super)algebras. The investigations of this type previously were carried out for the direct sum of the complex numbers field in [7], and the simple three-dimensional Lie algebra $\mathrm{sl}_2(\mathbb{C})$ [22,23]. In [12], Goncharov considered the structures of bialgebra on an arbitrary simple finite-dimensional algebra A over a field of characteristic zero with a semisimple Drinfeld double. He proved that these structures induce on A Rota–Baxter operators of nonzero weight. In addition, for simple Lie algebras and some non skew-symmetric solutions to the classical Yang–Baxter equations, he constructed Rota–Baxter operators of nonzero weight. As a corollary, he constructed Rota–Baxter operators of nonzero weight on the simple non-Lie Malcev algebra.

Some of the results of the present article were proved by the authors independently. Preliminary Sect. 2 consists of the results of Pilar Benito (PB) and Vsevolod Gubarev (VG). The results of Sect. 3 were obtained by PB and Alexander Pozhidaev (AP). The results of Sect. 4.1 were proved by VG and of Sect. 4.3—by PB and VG. The results of Sects. 4.4, 4.5, 5.2, and 5.3 were obtained by VG, and they are actually some applications of the technique developed in Sect. 3. Theorem 5.2, which is a reproof of [28], was proved by VG. The results of Sect. 5.4 were obtained by AP.

In what follows, the characteristic of the main field F is different from two.

2. Preliminaries

By the *trivial* RB operators of weight Δ, we mean the zero operator and $-\Delta\mathrm{id}$, where id denotes the identity map.

Consider some well-known examples of RB operators (see, e.g., [15]).

Example 1. Given an algebra A of continuous functions on $\mathbb{R}$, an integration operator $R(f)(x) = \int_0^x f(t)\,\mathrm{d}t$ is an RB operator on A of weight zero.

Example 2. Given an invertible derivation d of an algebra A, d^{-1} is an RB operator on A of weight zero.

Example 3. Let $A = \{(a_1, a_2, \ldots, a_k, \ldots) \mid a_i \in F\}$ be a countable sum of a field F with the termwise addition, multiplication, and scalar product. An operator R defined as $R(a_1, a_2, \ldots, a_k, \ldots) = \big(a_1, a_1 + a_2, \ldots, \sum_{i=1}^k a_i, \ldots\big)$ is an RB operator on A of weight -1.

Note that the algebra A from Example 3 is not simple as an algebra, but it is simple as an RB algebra. In addition, this example may be generalized for an arbitrary variety of algebras.

Statement 2.1. [15] *Let P be an RB operator of weight Δ. Then*

(a) *the operator $-P-\Delta id$ is an RB operator of weight Δ;*

(b) *the operator $\Delta^{-1}P$ is an RB operator of weight 1, provided that $\Delta\neq 0$.*

Let A be an algebra. In what follows, we fix the notation ϕ for the map defined on the set of all RB operators on A as $\phi(P)=-P-\Delta(P)\mathrm{id}$. It is clear that ϕ^2 coincides with the identity map.

Statement 2.2. *Let P be an RB operator of weight Δ on an algebra A, and let $\psi\in\mathrm{Aut}(A)$. Then, $P^{(\psi)}=\psi^{-1}P\psi$ is an RB operator on A of weight Δ.*

Proof. The proof is straightforward. □

Statement 2.3. [15] *Assume that an algebra A is splitted as a vector space into the direct sum of two subalgebras A_1 and A_2. An operator P defined by the rule*

$$P(x_1+x_2)=-\Delta x_2,\quad x_1\in A_1,x_2\in A_2, \tag{2.1}$$

is an RB operator on A of weight Δ.

The RB operator from Statement 2.3 is a *splitting* RB operator with respect to the subalgebras A_1 and A_2. In [18], such RB operator is called a quasi-idempotent operator.

Remark 2.4. Let P be a splitting RB operator on an algebra A of weight Δ with respect to subalgebras A_1,A_2. Then, $\phi(P)$ is an RB operator of weight Δ:

$$\phi(P)(x_1+x_2)=-\Delta x_1,\quad x_1\in A_1,x_2\in A_2,$$

and $\phi(P)$ is a splitting RB operator with respect to the same subalgebras A_1 and A_2.

Remark 2.5. The set of all splitting RB operators on an algebra A is in bijective correspondence with all decompositions of A into the direct sum of two subalgebras.

Example 4. [18] Let A be an associative algebra, and let $e\in A$ be an element, such that $e^2=-\lambda e$, $\lambda\in F$. A linear map $l_e\colon x\to ex$ is an RB operator of weight λ satisfying $R^2+\lambda R=0$. If $\lambda\neq 0$, then l_e is a splitting RB operator on A with respect to the subalgebras $A_1=(1-e)A$ and $A_2=eA$, and the decomposition $A=A_1\oplus A_2$ is exactly a Pierce one.

In an alternative algebra A with an element e, such that $e^2=-\lambda e$, $\lambda\in F$, the operator l_e is an RB operator if e lies in the associative or commutative center of A. It follows easily using the identities of alternative algebras [30].

Example 5. In [24], there were described all possible linear Rota–Baxter structures on a 0-dialgebra with a bar-unit.

Example 6. In [4], it was proved that every RB algebra of weight Δ in the variety Var with respect to the operations

$$x \succ y = R(x)y, \quad x \prec y = xR(y), \quad x \cdot y = \Delta xy$$

is a post-Var algebra.

In [14], given a post-Var algebra A, its enveloping RB algebra B of weight Δ in the variety Var was constructed. By the construction, $B = A \oplus A'$, where A' is a copy of A as a vector space, and the RB operator R was defined as follows: $R(a') = \Delta a$, $R(a) = -\Delta a$, $a \in A$. From the definition, we have $A_1 = \ker R = \mathrm{span}\{a + a' \mid a \in A\}$, $A_2 = R(B) = A$, and R is a splitting RB operator on B with respect to A_1 and A_2. Therefore, given a post-Var algebra A, there exists an enveloping algebra B with a splitting RB operator R of weight 1.

Lemma 2.6. [13] *Let A be a unital algebra, and let P be an RB operator on A of weight Δ.*

(a) *If $\Delta \neq 0$, then P is splitting if and only if $P(P(x) + \Delta x) = 0$.*

(b) *If $\Delta \neq 0$ and $P(1) \in F$, then P is splitting.*

(c) *If $\Delta = 0$, then $1 \notin \mathrm{Im}\, P$. Moreover, if A is a simple finite-dimensional algebra,* $\dim A > 1$, *then* $\dim \ker P \geq 2$.

(d) *If $\Delta = 0$ and $P(1) \in F$, then $P(1) = 0$, $P^2 = 0$, and* $\mathrm{Im}\, P \subset \ker P$.

Proof. (a) If P is splitting RB operator of weight $\Delta \neq 0$, then $P(P(x)+\Delta x) = 0$ from the definition.

Suppose that $P(P(x) + \Delta x) = 0$. Let us show that $A = \ker P \oplus P(A)$ as the direct sum of vector spaces. On the contrary, assume that there exists a nonzero $x \in \ker P \cap P(A)$. Then, $x = P(y)$ and $P(x) = P^2(y) = 0$. By the hypothesis, $x = P(y) = -(1/\Delta)P^2(y) = 0$, a contradiction.

By (1.1), $\ker P$ and $P(A)$ are some subalgebras of A. From $P(P(x) + \Delta x) = 0$, we have that the restriction of P on $P(A)$ is equal to $-\Delta \mathrm{id}$ and $P(\ker P) = 0$.

(b) By (1.1) for $x = y = 1$, we have $P(1) \in \{0, -\Delta\}$. It suffices to consider only the case $P(1) = 0$. Indeed, if $P(1) = -\Delta$, by Statement 2.1, we can study an RB operator $\phi(P)$ of the same weight, and $\phi(P)(1) = 0$. By Remark 2.4, we are done.

By (1.1), for $x \in A$, we have

$$0 = P(1)P(x) = P(P(1)x + 1 \cdot P(x) + \Delta x) = P(P(x) + \Delta x). \tag{2.2}$$

Therefore, we apply (a).

(c) Suppose $R(x) = 1$ for some $x \in A$. By (1.1), $1 = R(x)R(x) = 2R(x) = 2$, a contradiction.

Let A be a simple finite-dimensional algebra, $\dim A = n$. By a), we have $\dim \mathrm{Im}\, P \leq n - 1$. Assume that $\dim \mathrm{Im}\, P = n - 1$. By (1.1), $\ker P$ is an $\mathrm{Im}\, P$-bimodule. Since $A = \mathrm{span}\{1, \mathrm{Im}\, P\}$, $\ker P$ is a proper ideal of A, a contradiction with the simplicity of A.

(d) By (c), $P(1) = 0$. Other assertions follow from

$$0 = P(1)P(x) = P(P(1)x + 1 \cdot P(x)) = P(P(x)).$$

□

Lemma 2.7. *Let A be an algebra, and let R be an RB operator on A of weight zero.*

(a) *A nonzero element $e \in A$, such that $e^2 = \alpha e$, $\alpha \in F^*$, could not be an eigenvector of R with nonzero eigenvalue.*

(b) *If A is a unital finite-dimensional algebra,* $\mathrm{Im}\,(R)$ *is a subalgebra with trivial multiplication, and F is algebraically closed, then R is nilpotent.*

Proof. (a) If $R(e) = ke$ with $k \in F^*$, then

$$\alpha k^2 e = k^2 e^2 = R(e)R(e) = R(R(e)e + eR(e)) = 2kR(e^2) = 2\alpha kR(e) = 2\alpha k^2 e,$$

a contradiction.

(b) Suppose that v is an eigenvector of R with nonzero eigenvalue k. We have

$$0 = R(1)R(v) = R(R(1)v + R(v)) = R((1/k)R(1)R(v) + R(v)) = R^2(v) = k^2 v,$$

a contradiction. □

3. Quadratic Algebras

Let A be a quadratic algebra, i.e., every element $x \in A$ satisfies the equation:

$$x^2 - t(x)x + n(x)1 = 0, \tag{3.1}$$

where 1 is a unit of A, the *trace* $t(x)$ is linear on A, and the *norm* $n(x)$ is quadratic on A [30].

Putting $f(x, y) = n(x + y) - n(x) - n(y)$, we get

$$x \circ y = t(x)y + t(y)x - f(x, y)1. \tag{3.2}$$

We have $A = F1 \oplus A_0$, where $A_0 = \{x \in A \mid t(x) = 0\}$.

Let R be an RB operator on A of weight Δ. Setting $x = y$ in (1.1) and applying (3.1), we infer that

$$-n(R(x))1 = R(t(x)R(x) - f(x, R(x)) + \Delta t(x)x - \Delta n(x)). \tag{3.3}$$

Taking $x \in A_0$ in (3.3), we obtain

$$n(R(x))1 = (f\big(x, R(x)\big) + \Delta n(x))R(1). \tag{3.4}$$

Applying Lemma 2.6, we arrive at the following statement.

Lemma 3.1. *Let A be a quadratic algebra with an RB operator R of weight Δ.*

(a) *If $R(1) = 0$ or $R(1) \notin F$, then $n(R(x)) = 0$ and $R(x)(R(x) - t(R(x))1) = 0$ for all $x \in A_0$.*

(b) *For $\Delta = 0$, either $R(1) = 0$ or $n(x + R(x)) = n(x)$ for all $x \in A_0$.*

(c) *For $\Delta \neq 0$, if $n(R(x)) \neq 0$ for some $x \in A_0$, then R is splitting.*

Theorem 3.2. *All RB operators on a quadratic division algebra are trivial.*

Proof. If a quadratic division algebra A coincides with F, then the statement is obvious. Let $\dim_F(A) \geq 2$.

If $R(1) \in F$, then we have $R^2 = -\Delta R$ by Lemma 2.6. For $\Delta = 0$, by Lemma 2.6 (d), $R(1) = 0$ and by Lemma 3.1 (a), $R(x)(R(x) - t(R(x))1) = 0$

for all $x \in A$. Since A has no zero divisors, $R(x) \in F$ for all $x \in A$. By (1.1), $R = 0$.

For $\Delta \neq 0$, by Lemma 2.6 (b), R is splitting with respect to some subalgebras A_1 and A_2, i.e., $R(A_1) = 0$, and R is equal to $-\Delta\mathrm{id}$ on A_2. Up to ϕ, we have $1 \in A_1$. For each $x \in A_2$, we have $x \in R(A_2)$; by (3.1) and Lemma 3.1 (a), we obtain $x(x - t(x)1) = 0$. As $x \notin F$, $x = 0$. Therefore, $R = 0$.

Let $R(1) \notin F$. By Lemma 3.1 (a), $R(x)(R(x) - t(R(x))1) = 0$ for all $x \in A_0$. When $\Delta = 0$, $R(x) \in F$ for all $x \in A_0$, and $R(x) = 0$ for every $x \in A_0$ by (1.1). Therefore, $\operatorname{Im} R$ is the linear span of $R(1)$. By (1.1), $R(1)^2 = \alpha R(1)$ for some $\alpha \in F$. We have either $R(1) = 0$ and $R = 0$ or $R(1) \in F$, a contradiction. For $\Delta \neq 0$, we have $R(x) \in F$ for all $x \in A_0$, and by (1.1), we infer that A_0 is a proper ideal of A, a contradiction with the divisibility. □

Corollary 3.3. *Given a quadratic division algebra A, there are no representations of A as a sum (as vector spaces) of its proper subalgebras.*

Proof. Assume that A is equal to $A_1 \oplus A_2$, where A_1, A_2 are some subalgebras of A. Hence, by Statement 2.3, there exist nontrivial RB operators on A of nonzero weight. By Theorem 1, we arrive at a contradiction. □

Lemma 3.4. *Let A be a quadratic commutative algebra. Then, the RB operators R of weight 0 on A, such that $R(1) = 0$ are in one-to-one correspondence with the linear maps R on A, such that $R(1) = 0$, $\operatorname{Im} R \subseteq \ker R \cap \ker n$.*

Proof. Let R be an RB operator of weight 0 on A, such that $R(1) = 0$. By Lemma 2.6 d), $R^2 = 0$. By (3.4), $n(R(A)) = 0$. Thus, $\operatorname{Im} R \subseteq \ker n$, and $\operatorname{Im} R \subseteq \ker R$.

Conversely, let R be a map on A as above. Then, $R^2 = 0$, and $n(R(A)) = 0$. By (3.1), $R(x)R(x) = t(R(x))R(x)$. By (3.2)

$$R(x \circ R(x)) = R(t(x)R(x) + t(R(x))x) = t(R(x))R(x) = R(x)R(x).$$

Thus, A is a Rota–Baxter algebra of weight zero. □

Let A be an algebra over a field F, let S be a subalgebra of A, let I be a subspace of A, such that $SI + IS \subseteq I$, and let D be a nondegenerate derivation from S to A modulo I, (i.e., $D(xy) - D(x)y - xD(y) \in I$ for all $x, y \in S$) with the property $A = D(S) \oplus I$. In this case, we say that (S, I, D) is an *RB-triple* on A. Denote the space of all derivations from S to A modulo I by $\mathrm{Der}_F(S, I, A)$.

Lemma 3.5. *Let A be an algebra over a field F. Then, the RB operators of weight 0 on A are in one-to-one correspondence with the RB triples on A.*

Proof. Let R be an RB operator of weight 0 on A. Put $I = \ker R$ and $S = \operatorname{Im} R$. Choose a basis for I and complete it to a basis of A by some $a_j \in A$, $j \in J$ for some set of indexes J, such that $S = \operatorname{span}\{R(a_j) \mid j \in J\}$. Put $A_0 = \operatorname{span}\{a_j \mid j \in J\}$. Then, $A = I \oplus A_0$. Define a linear map $D\colon S \to A$ by the rule $D(R(a_j)) = a_j$ for all $j \in J$. Then, $A = D(S) \oplus I$. Note that if $a \in A$, then $a = i + a_0$ for some uniquely defined $i \in I$, $a_0 \in A_0$; therefore,

$D(R(x)) \equiv x\,(\mathrm{mod}\,I)$. Take arbitrary $a_i, a_j \in A_0$ and put $s_1 = R(a_i)$, $s_2 = R(a_j)$. Then, by (1.1) with $\Delta = 0$, we have $s_1 s_2 = R(s_1 a_j + a_i s_2)$, and

$$D(s_1 s_2) = D(R(s_1 D(s_2) + D(s_1)s_2) \equiv s_1 D(s_2) + D(s_1)s_2\,(\mathrm{mod}\,I).$$

It is easy to see that (S, I, D) is an RB triple on A.

Conversely, let (S, I, D) be an RB triple on A. Define an operator R on A by the rule:

$$\ker R = I, \quad R(D(s)) = s,\ s \in S.$$

If either $x \in I$ or $y \in I$, then (1.1) holds. Take $x = D(s_1)$, $y = D(s_2)$ for arbitrary $s_1, s_2 \in S$. Then

$$R(x)R(y) = s_1 s_2,$$
$$R(R(x)y + xR(y)) = R(s_1 D(s_2) + D(s_1)s_2) = R(D(s_1 s_2)) = s_1 s_2,$$

and (1.1) holds again. □

Corollary 3.6. *Let $\mathcal{V}$ be a variety of algebras over a field F. Let A be a $\mathcal{V}$ algebra, and let V be an A module in the sense of Eilenberg. Assume that there exists a nondegenerate derivation D from A into $B = A \oplus V$ modulo V, such that $B = D(A) \oplus V$. Then, (A, V, D) is an RB triple on B.*

Proof. By the definition of module in the sense of Eilenberg, we have $A \leq B$, $AV + VA \subseteq V$. Now, apply Lemma 3.5. □

Remark 3.7. The hypotheses of Corollary 3.6 hold if D is a nondegenerate derivation D of A, such that $D(A) = A$.

Example 7. Consider the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ with the standard basis h, e, f. Put $S = \mathrm{span}\{h\}$, $D = \mathrm{ad}\,(e + f)$, $I = \mathrm{span}\{h, e\}$. Then, the operator R, such that $R(f) = h/2$ and $R(I) = 0$ gives the RB operator on $\mathrm{sl}_2(\mathbb{C})$ of weight zero. It is exactly the case (R5) [19] from six possible variants of RB operators on $\mathrm{sl}_2(\mathbb{C})$ of weight zero.

Example 8. Let A be an algebra. Assume that S is a subalgebra of A with trivial multiplication, $A = S \oplus I$, and S acts on I. (For example, one may consider a Lie algebra and its Cartan subalgebra as S.) Then, every nondegenerate mapping on S with the kernel I determines an RB operator on A of weight zero.

Example 9. Consider a semisimple finite-dimensional Lie algebra L over a field F of characteristic 0. Assume that there are some nonzero roots α, β, such that $\beta + \alpha$ belongs to the set Γ of nonzero roots of L but $\beta - \alpha \notin \Gamma$. Take h in the Cartan subalgebra H of L, such that $\beta(h) \neq 0$. Put $S = \mathrm{span}\{h, e_\alpha\}$, $I = H \oplus \sum_{\gamma \in \Gamma \setminus \{\beta, \alpha+\beta\}} \mathrm{span}\{e_\gamma\}$, and $D = \mathrm{ad}(e_\beta)$. Consider the operator R on L, such that $R(e_\beta) = -\beta(h)^{-1}h$, $R(e_{\alpha+\beta}) = c_{\alpha,\beta}^{-1} e_\alpha$, where $[e_{\alpha,\beta}] = c_{\alpha,\beta} e_{\alpha+\beta}$ for $c_{\alpha,\beta} \in F$, and $R(I) = 0$. By Lemma 3.5, R gives an RB operator of weight 0 on L.

Statement 3.8. (a) *Let $D \in \mathrm{Der}_F(S, I, A)$, $f(x, y) = D(xy) - D(x)y - xD(y)$. Assume that there exists $\theta\colon S \to I$, such that $-f(x, y) = \theta(xy) - \theta(x)y - x\theta(y)$ for all $x, y \in S$. Then, $D + \theta \in \mathrm{Der}_F(S, I, A)$.*

(b) *Let A be an algebra over a field F, and let $A = S \oplus I$ for some subalgebra S of A and an ideal I of A. Then,* $\mathrm{Der}_F(S, I, A) = \mathrm{Der}_F(S) + \mathrm{End}_F(S, I)$*; i.e., every derivation D from S in A modulo I is a sum of a derivation $D_1 \in \mathrm{Der}(S)$ and a linear map $\theta\colon S \to I$, and conversely.*

Proof. Proof of (a) is straightforward.

(b) Take $D \in \mathrm{Der}(S, I, A)$. Put $D_1 = \pi \circ D$, $D_1\colon S \to S$, where π is the projection on S; i.e., $D_1(s) = \pi(D(s)) \in S$ for all $s \in S$. Then, $\pi \in \mathrm{Hom}_F(A, S)$. Now, it suffices to put $\theta = D - D_1$.

The converse assertion is immediate. □

4. RB Operators of Nonzero Weight

4.1. The Simple Jordan Algebra of Bilinear Form

Let $J_{n+1}(f) = F \cdot 1 \oplus V$ be the direct sum of F and a linear n-dimensional space V, $n \geq 2$, and let f be a nondegenerate symmetric bilinear form on V. With respect to the product

$$(\alpha\cdot 1+a)(\beta\cdot 1+b) = (\alpha\beta+f(a,b))\cdot 1+(\alpha b+\beta a), \quad \alpha,\beta \in F,\ a,b \in V, \tag{4.1}$$

the space $J_{n+1}(f)$ is a simple Jordan algebra [30].

The algebra $J_{n+1}(f)$ is quadratic, since for every $x = \alpha\cdot 1+a \in J_{n+1}(f)$, $\alpha \in F$, $a \in V$, we have $x^2 - 2\alpha x + (\alpha^2 - f(a,a)) \cdot 1 = 0$. Hence, $t(x) = 2\alpha$, $n(x) = \alpha^2 - f(a,a)$.

Choose a basis e_1, e_2, …, e_n for V, such that the matrix of the form f in this basis is diagonal with some elements $d_1, d_2, \ldots, d_n \in F$ on the main diagonal. Since f is nondegenerate, $d_i \neq 0$ for each i.

Given an RB operator R of weight Δ on $J_{n+1}(f)$, assume that R is defined by a matrix $(r_{ij})_{i,j=0}^{n}$ in the basis $1, e_1, e_2, \ldots, e_n$.

The identity (1.1) is equivalent to the system of equations, which is quadratic with respect to r_{ij}. Due to the symmetricity of f, it suffices to consider the equations arising from the equalities by the products x_0y_0, x_sy_s (let us denote them as $\underline{00}$ and $\underline{ss}$ for $s > 0$, respectively) and $x_0y_k + x_ky_0$, $x_ky_l + x_ly_k$, $k \neq l$ (notation: $\underline{0k}$ for $k > 0$ and $\underline{kl}$ for $k \neq l$, $k,l > 0$). There are eight series of equations: (the bold number denotes the projection of (1.1) either on 1 or on e_i):

$$\begin{aligned}
\mathbf{0},\ \underline{00} &: d_1r_{10}^2 + \cdots + d_nr_{n0}^2 = r_{00}^2 + \Delta r_{00} + 2(r_{01}r_{10} + \cdots + r_{0n}r_{n0}),\\
\underline{ss} &: d_1r_{1s}^2 + \cdots + d_nr_{ns}^2 = r_{0s}^2 + d_sr_{00}(2r_{ss} + \Delta),\\
\underline{0k} &: d_1r_{10}r_{1k} + \cdots + d_nr_{n0}r_{nk} = \Delta r_{0k} + d_kr_{00}r_{k0} + r_{00}r_{0k} + \cdots + r_{0n}r_{nk},\\
\underline{kl} &: d_1r_{1k}r_{1l} + \cdots + d_nr_{nk}r_{nl} = r_{0k}r_{0l} + r_{00}(d_kr_{kl} + d_lr_{lk}),\\
\mathbf{i{>}0},\ \underline{00} &: 2(r_{i1}r_{10} + \cdots + r_{in}r_{n0}) + \Delta r_{i0} = 0,\\
\underline{ss} &: r_{i0}(2r_{ss} + \Delta) = 0,\\
\underline{1k} &: d_kr_{i0}r_{k0} + r_{i1}r_{1k} + \cdots + r_{in}r_{nk} + \Delta r_{ik} = 0,\\
\underline{kl} &: r_{i0}(d_kr_{kl} + d_lr_{lk}) = 0.
\end{aligned}$$

Assume that R is an RB operator on $J_{n+1}(f)$, such that $R(1) \notin F$ and F is algebraically closed. Therefore, we have

$$r_{ss} = -\Delta/2,\ s > 0, \tag{4.2}$$

$$d_k r_{kl} + d_l r_{lk} = 0,\ k, l > 0, k \neq l. \tag{4.3}$$

Then, the system of quadratic equations written above is equivalent to the following:

$$d_1 r_{10}^2 + \cdots + d_n r_{n0}^2 = r_{00}^2 + \Delta r_{00} + 2(r_{01}r_{10} + \cdots + r_{0n}r_{n0}), \tag{4.4}$$

$$d_1 r_{1s}^2 + \cdots + d_n r_{ns}^2 = r_{0s}^2,\ s > 0, \tag{4.5}$$

$$d_1 r_{10} r_{1k} + \cdots + d_n r_{n0} r_{nk} = \Delta r_{0k} + d_k r_{00} r_{k0} + r_{00} r_{0k} + \cdots + r_{0n} r_{nk},\ k > 0, \tag{4.6}$$

$$d_1 r_{1k} r_{1l} + \cdots + d_n r_{nk} r_{nl} = r_{0k} r_{0l},\ k, l > 0,\ k \neq l, \tag{4.7}$$

$$2(r_{i1} r_{10} + \cdots + r_{in} r_{n0}) + \Delta r_{i0} = 0,\ i > 0, \tag{4.8}$$

$$d_k r_{i0} r_{k0} + r_{i1} r_{1k} + \cdots + r_{in} r_{nk} + \Delta r_{ik} = 0,\ i, k > 0. \tag{4.9}$$

By (4.2) and (4.9) for $i = k = s > 0$ by (4.5), we have

$$r_{0s}^2 = \sum_{i=1}^{n} d_i r_{is}^2 = -\sum_{i=1}^{n} d_s r_{is} r_{si} + d_s \frac{\Delta^2}{2} = d_s(d_s r_{s0}^2 + \Delta r_{ss}) + d_s \frac{\Delta^2}{2} = d_s^2 r_{s0}^2.$$

Therefore

$$r_{0s} = z_s d_s r_{s0} \tag{4.10}$$

with $z_s \in \{-1, +1\}$. Therefore, (4.9) could be derived from (4.5) with the help of (4.2) and (4.10).

By (4.2), (4.3), and (4.7), we have

$$\begin{aligned} r_{0k} r_{0l} &= \sum_{i=1}^{n} d_i r_{ik} r_{il} = -d_k \sum_{i=1}^{n} r_{ki} r_{il} + 2 d_k r_{kk} r_{kl} \\ &= d_k(\Delta r_{kl} + d_l r_{0k} r_{l0}) + 2 d_k r_{kk} r_{kl} = d_k d_l r_{k0} r_{l0}, \end{aligned}$$

whence $z_s = z \in \{-1, +1\}$ for all $s > 0$ by (4.10).

Applying (4.2), (4.3), (4.8), we get from (4.6)

$$\begin{aligned} \Delta r_{0k} + d_k r_{00} r_{k0} + r_{00} r_{0k} &= \sum_{i=1}^{n} d_i r_{i0} r_{ik} - \sum_{i=1}^{n} r_{0i} r_{ik} \\ &= (1 - z) \sum_{i=1}^{n} d_i r_{i0} r_{ik} = -d_k (1 - z) \left(\sum_{i=1}^{n} r_{i0} r_{ki} \right) \\ &\quad + (1 - z) d_k (r_{k0} r_{kk} + r_{k0} r_{kk}) \\ &= (1/2) d_k (1 - z) \Delta r_{k0} - d_k \Delta (1 - z) r_{k0} \\ &= (1/2) d_k (z - 1) \Delta r_{k0}. \end{aligned}$$

Thus, $(1 + z) r_{k0} (2 r_{00} + \Delta) = 0$. Since $R(1) \notin F$, $(1 + z)(2 r_{00} + \Delta) = 0$.

Summarizing, we have the following system on $\bar{r}_{ij} = \frac{\sqrt{d_i}}{\sqrt{d_j}} r_{ij}$ satisfying $\bar{r}_{kk} = -\Delta/2$, $\bar{r}_{kl} = -\bar{r}_{lk}$, $\bar{r}_{0k} = z \bar{r}_{k0}$ for $k, l > 0$, $k \neq l$, $z \in \{-1, +1\}$:

$$(1 + z)(2 \bar{r}_{00} + \Delta) = 0, \tag{4.11}$$

$$(1-2z)\sum_{p=1}^{n}\bar{r}_{p0}^{2}=\bar{r}_{00}(\bar{r}_{00}+\Delta),\quad \sum_{p=1}^{n}\bar{r}_{pi}\bar{r}_{p0}=-\frac{\Delta z}{2}\bar{r}_{0i},\ i>0, \tag{4.12}$$

$$\sum_{p=1}^{n}\bar{r}_{pk}\bar{r}_{pl}=\bar{r}_{0k}\bar{r}_{0l},\ k,l>0,k\neq l,\quad \sum_{p=1}^{n}\bar{r}_{pk}^{2}=\bar{r}_{0k}^{2},\ k>0. \tag{4.13}$$

Consider the first case: (I) $z=1$, $\bar{r}_{00}=-\Delta/2$. Then, the system (4.12)–(4.13) is of the form:

$$\sum_{p=1}^{n}\bar{r}_{p0}^{2}=\frac{\Delta^2}{4},\quad \sum_{p=0}^{n}\bar{r}_{pi}\bar{r}_{p0}=-\Delta\bar{r}_{0i},\ i>0, \tag{4.14}$$

$$\sum_{p=1}^{n}\bar{r}_{pk}\bar{r}_{pl}=\bar{r}_{0k}\bar{r}_{0l},\ k,l>0,k\neq l,\quad \sum_{p=1}^{n}\bar{r}_{pk}^{2}=\bar{r}_{0k}^{2},\ k>0. \tag{4.15}$$

The second case is the following: (II) $z=-1$ (in what follows, we assume that $\operatorname{char} F\neq 3$):

$$\sum_{p=1}^{n}\bar{r}_{p0}^{2}=\frac{\bar{r}_{00}(\bar{r}_{00}+\Delta)}{3},\quad \sum_{p=1}^{n}\bar{r}_{pi}\bar{r}_{p0}=\frac{\Delta}{2}\bar{r}_{0i},\ i>0, \tag{4.16}$$

$$\sum_{p=1}^{n}\bar{r}_{pk}\bar{r}_{pl}=\bar{r}_{0k}\bar{r}_{0l},\ k,l>0,k\neq l,\quad \sum_{p=1}^{n}\bar{r}_{pk}^{2}=\bar{r}_{0k}^{2},\ k>0. \tag{4.17}$$

Applying (4.16)–(4.17), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\bar{r}_{i0}^{2}&=\frac{4}{\Delta^2}\sum_{i=1}^{n}\left(\sum_{p=1}^{n}\bar{r}_{pi}\bar{r}_{p0}\right)^2\\&=\frac{4}{\Delta^2}\sum_{p=1}^{n}\bar{r}_{p0}^{2}\left(\sum_{i=1}^{n}\bar{r}_{pi}^{2}\right)+\frac{8}{\Delta^2}\sum_{p,q=1,\,p\neq q}^{n}\bar{r}_{p0}\bar{r}_{q0}\left(\sum_{i=1}^{n}\bar{r}_{pi}\bar{r}_{qi}\right)\\&=\frac{4}{\Delta^2}\sum_{p=1}^{n}\bar{r}_{p0}^{4}+\frac{8}{\Delta^2}\sum_{p,q=1,\,p\neq q}^{n}\bar{r}_{p0}^{2}\bar{r}_{q0}^{2}=\frac{4}{\Delta^2}\left(\sum_{p=1}^{n}\bar{r}_{p0}^{2}\right)^2,\end{aligned}$$

whence $A=\sum_{i=1}^{n}\bar{r}_{i0}^{2}$ is equal to 0 or $\Delta^2/4$.

Suppose that $A=0$. By (4.16), up to action of ϕ, we may assume that $\bar{r}_{00}=0$. Therefore, $R(1)R(1)=0$. By (1.1), we have

$$\begin{gathered}0=R(1)R(1)=2R^2(1)+\Delta R(1),\\0=R(1)R(1)R(1)=2R^2(1)R(1)=2R^3(1)+2\Delta R^2(1)=\Delta R^2(1),\end{gathered} \tag{4.18}$$

whence $R^2(1)=0=R(1)$, a contradiction to the assumption $R(1)\notin F$.

Thus, $A=\Delta^2/4$ and by (4.16), we arrive at the following subcases.

(II a) $z=-1$, $\bar{r}_{00}=\Delta/2$. In this case, the system (4.16)–(4.17) is of the form:

$$\sum_{p=1}^{n}\bar{r}_{p0}^{2}=\frac{\Delta^2}{4},\quad \sum_{p=0}^{n}\bar{r}_{pi}\bar{r}_{p0}=\Delta\bar{r}_{0i},\ i>0, \tag{4.19}$$

$$\sum_{p=1}^{n}\bar r_{pk}\bar r_{pl}=\bar r_{0k}\bar r_{0l},\ k,l>0,k\neq l,\qquad \sum_{p=1}^{n}\bar r_{pk}^2=\bar r_{0k}^2,\ k>0. \tag{4.20}$$

(II b) $z=-1$, $\bar r_{00}=-3\Delta/2$,

$$\sum_{p=1}^{n}\bar r_{p0}^2=\frac{\Delta^2}{4},\quad \sum_{p=0}^{n}\bar r_{pi}\bar r_{p0}=-\Delta\bar r_{0i},\ i>0, \tag{4.21}$$

$$\sum_{p=1}^{n}\bar r_{pk}\bar r_{pl}=\bar r_{0k}\bar r_{0l},\ k,l>0,k\neq l,\qquad \sum_{p=1}^{n}\bar r_{pk}^2=\bar r_{0k}^2,\ k>0. \tag{4.22}$$

Note that the numbers $\bar r_{ij}$ satisfying (I) and (II a) could be obtained from each other by multiplying the first row by -1. Furthermore, for both cases (II a) and (II b), we define

$$s_{kl}=\begin{cases}\Delta/2, & k=l,\\ \bar r_{kl}, & k,l>0,\ k\neq l,\\ i\bar r_{kl}, & k\ or\ l=0,\ k\neq l,\end{cases}$$

where i is a root of $x^2+1=0$.

It is easy to prove that the systems (4.19)–(4.20) and (4.21)–(4.22) in the terms of s_{ij} have the same form:

$$\sum_{p=0}^{n}s_{pk}s_{pl}=0,\quad 0\le k,l\le n. \tag{4.23}$$

We can represent the matrix $S=\{s_{kl}\}$ as $S=\frac{\Delta}{2}E+M$ for the skew-symmetric matrix M and the identity matrix E.

The system (4.23) is equivalent to the equality $(\frac{\Delta}{2}E+M)^T(\frac{\Delta}{2}E+M)=0$, or applying the skew-symmetricity of M, we have $M^2=\frac{\Delta^2}{4}E$.

Theorem 4.1. *Let $J_{n+1}(f)$ be the simple Jordan algebra of bilinear form f. If n is even, then all RB operators on $J_{n+1}(f)$ of nonzero weight are splitting.*

Proof. Let R be a non-splitting RB operator of weight $\Delta\neq 0$, which is defined by a matrix $(r_{ij})_{i,j=0}^n$ in a basis $1,e_1,\dots,e_n$. By Lemma 2.6 (b), $R(1)\notin F$. Let $\bar F$ be an algebraical closure of F.

Assume that $\operatorname{char}F\neq 3$. Then, as it was stated above, we can construct a skew-symmetric matrix $M\in M_{n+1}(\bar F)$, such that $M^2=\frac{\Delta^2}{4}E$. Hence, the rank of M is equal to $n+1$. It is well known that the rank of a skew-symmetric matrix over the field of characteristic different from 2 is even [10]. We arrive at a contradiction.

If $\operatorname{char}F=3$, then in the case (II), we have the following system of equations:

$$\bar r_{00}(\bar r_{00}+\Delta)=0,\quad \sum_{p=1}^{n}\bar r_{pi}\bar r_{p0}=\frac{\Delta}{2}\bar r_{0i},\ i>0, \tag{4.24}$$

$$\sum_{p=1}^{n}\bar r_{pk}\bar r_{pl}=\bar r_{0k}\bar r_{0l},\ k,l>0,k\neq l,\qquad \sum_{p=1}^{n}\bar r_{pk}^2=\bar r_{0k}^2,\ k>0. \tag{4.25}$$

Up to action of ϕ, we may assume that $\bar{r}_{00} = 0$. By the same reasons as above, from (4.24)–(4.25), we see that $A = \sum_{i=1}^{n} \bar{r}_{i0}^2$ is equal to 0 or $\Delta^2/4$. As it was proved above, the case $A = 0$ is contradictory. For $A = \Delta^2/4$, we define the matrix $Q = (q_{kl}) \in M_{n+1}(\bar{F})$ with the entries:

$$q_{kl} = \begin{cases} -\Delta/2, & k = l, \\ \bar{r}_{kl}, & k, l > 0,\ k \neq l, \\ i\bar{r}_{kl}, & k \text{ or } l = 0,\ k \neq l. \end{cases}$$

Analogously, we obtain $Q^T Q = 0$ and $Q = -\frac{\Delta}{2} E + M$ for a skew-symmetric matrix M. The final arguments are the same as in the case $\operatorname{char} F \neq 3$. □

Actually, we have proved even more than Theorem 4.1 states:

Corollary 4.2. *Let $J_{n+1}(f)$ be the simple Jordan algebra of bilinear form f, and let R be an RB operator on $J_{n+1}(f)$ of nonzero weight. If n is even, then we have $R(1) = 0$ up to ϕ.*

Remark 4.3. Note that for the simple Jordan algebra $J_{n+1}(f)$ of bilinear form f and odd n, there is the correspondence between the set X_Δ of all RB operators of nonzero weight Δ on $J_{n+1}(f)$ with the property $R(1) \notin F$ for all $R \in X_\Delta$ and the set Y_Δ of all skew-symmetric matrices from $M_{n+1}(F)$ satisfying $S^2 = \frac{\Delta^2}{4} E$ for $S \in Y_\Delta$. It is interesting to compare with the weight zero case. In [13], it was proved that over an algebraically closed field F, we have the correspondence between the set X_0 of RB operators of weight zero on $J_{n+1}(f)$ satisfying $R(1) \notin F$ and $R^2 = 0$ for $R \in X_0$ and the set Y_0 of all skew-symmetric matrices from $M_{n+1}(F)$, whose squares are zero.

The following example says about the situation in even dimension over an algebraically closed field.

Example 10. Let $J_{2n}(f)$ be the simple Jordan algebra of bilinear from f over an algebraically closed field F. The following operator $\frac{\Delta}{2} R$ defined by nonzero matrix entries of R as

$$r_{00} = -3, \quad r_{01} = \sqrt{d_1}, \quad r_{10} = -\frac{1}{\sqrt{d_1}}, \quad r_{jj} = -1,\ j = 1, \dots, 2n-1,$$

$$r_{i\,i+1} = \frac{d_{i+1}}{d_i} \sqrt{-\frac{d_i}{d_{i+1}}}, \quad r_{i+1\,i} = -\sqrt{-\frac{d_i}{d_{i+1}}},\ i = 2, \dots, 2n-2,$$

is a non-splitting RB operator on $J_{2n}(f)$ of weight Δ. This RB operator arises from the case (II b).

Example 10 may be generalized for the simple countable-dimensional Jordan algebra of diagonalized bilinear form.

The next example shows that non-splitting RB operators of nonzero weight on the simple even-dimensional Jordan algebra of bilinear form can be not block diagonal (as in Example 10).

Example 11. Consider $J_4(f)$ over $\mathbb{Z}_5$ with the form f having the identity matrix in the basis $1, e_1, e_2, e_3$. Then, the following operator on $J_4(f)$ (arisen from the case (II b))

$$R(1) = 4 + 4e_1 + 3e_2 + 3e_3, \quad R(e_1) = 1 + 3e_1 + 4e_2 + e_3,$$
$$R(e_2) = 2R(e_1), \quad R(e_3) = 2 + 4e_1 + 3e_2 + 3e_3$$

is a non-splitting RB operator of weight -1.

We can see that there are also splitting RB operators using all RB operators from the cases (I) or (II).

Example 12. Consider $J_4(f)$ over $\mathbb{Z}_{13}$ with the form f having the identity matrix in the basis $1, e_1, e_2, e_3$. Then, the following operator on $J_4(f)$ [arisen from the case (I)]

$$R(1) = R(e_1) = 7 + 7e_1 + 7e_2 + 9e_3,\ R(e_2) = 7 + 6e_1 + 7e_2 + 4e_3,\ R(e_3) = 5R(e_2)$$

is a splitting RB operator of weight -1, although $R(1) \notin F$. Here, we have $\ker R = \operatorname{span}\{1 - e_1, e_2 - 5e_3\}$ and $\operatorname{Im} R = \operatorname{span}\{1 + e_2, e_1 + 5e_3\}$.

Statement 4.4. *Let A be the simple Jordan algebra of bilinear form, and let R be an RB operator on A of nonzero weight Δ. If $R(1) = 0$, then* $\dim\ker R \geq 2$.

Proof. By Lemma 2.6 (b), R is splitting. Therefore, $1 \in \ker R$ and $1 \notin \operatorname{Im} R$.

Suppose that $\dim\ker R = 1$. From $0 = R(R(e_i) + \Delta e_i)$, $i = 1, \dots, n-1$, we deduce that $R(e_i) = r_i \cdot 1 - \Delta e_i$ for all $i = 1, \dots, n-1$ and for some $r_i \in F$. Since $R(e_1)R(e_2) = r_1 r_2 \cdot 1 \in \operatorname{Im} R$, we obtain either $r_1 = 0$ or $r_2 = 0$. Taking $r_1 = 0$, one has $R(e_1)R(e_1) = d_1\Delta^2 \cdot 1 \in \operatorname{Im} R$ with nonzero $d_1 \in F$, a contradiction. □

In [13], all RB operators on $J_3(f)$ of weight zero were described. We have very close result for nonzero weight.

Example 13. Let $J_3(f)$ be the simple three-dimensional Jordan algebra of bilinear form $f = (d_1, d_2)$, and let R be a nontrivial RB operator on $J_3(f)$ of nonzero weight Δ. By Corollary 4.2, up to ϕ we have $R(1) = 0$. By Statement 4.4, $\dim\operatorname{Im} R = 1$. Thus, $R(e_1) = k(\alpha_0 \cdot 1 + \alpha_1 e_1 + \alpha_2 e_2)$, $R(e_2) = l(\alpha_0 \cdot 1 + \alpha_1 x + \alpha_2 y)$ for some $k, l, \alpha_i \in F$, k, and l are nonzero simultaneously as well as α_i. We have $le_1 - ke_2 \in \ker R$, so R is splitting with respect to the subalgebras $A_1 = \langle 1, le_1 - ke_2\rangle$ and $A_2 = \langle \alpha_0 \cdot 1 + \alpha_1 e_1 + \alpha_2 e_2\rangle$. The image of R is a subalgebra of $J_3(f)$, so $\alpha_0^2 - d_1\alpha_1^2 - d_2\alpha^2 = 0$. By (1.1), $k\alpha_1 + l\alpha_2 + \Delta = 0$ (it corresponds to the fact that $J_3(f) = A_1 \oplus A_2$). Thus, we described all RB operators on A of nonzero weight up to ϕ.

4.2. (Anti)Commutator Algebras

Given an algebra A with a product $\cdot$, define the operations $\circ$ and $[\,,]$ on the vector space of A by the rule:

$$a \circ b = a \cdot b + b \cdot a, \quad [a, b] = a \cdot b - b \cdot a.$$

We denote the space A with $\circ$ as $A^{(+)}$ and the space A with $[\,,]$ as $A^{(-)}$.

Statement 4.5. *Given an RB operator R of weight Δ on an algebra A, R is an RB operator on $A^{(+)}$ and $A^{(-)}$ of weight Δ.*

Proof. Proof is immediate by (1.1). □

Corollary 4.6. *Given an algebra A, if all RB operators on $A^{(+)}$ (or $A^{(-)}$) of nonzero weight are splitting, then all RB operators on A of nonzero weight are splitting.*

Proof. Let R be an RB operator of nonzero weight Δ on A. By Statement 4.5, R is an RB operator of weight $\Delta \neq 0$ on $A^{(+)}$ and $A^{(-)}$. By hypothesis, $R(R+\Delta\mathrm{id}) = 0$ on A. Thus, R is splitting on A by Lemma 2.6 (a). □

4.3. The Matrix Algebra of Order 2

Example 14. Define a linear map R on $M_n(F)$ as follows: R is zero on all strictly upper (lower) triangular matrices; R is equal to $-\mathrm{id}$ on all strictly lower (upper) triangular matrices; and R is an RB operator on the algebra of diagonal matrices of weight 1 [3]. Then, R is an RB operator on $M_n(F)$ of weight 1.

For example, a linear map R on $M_2(F)$, such that $R(e_{11}) = R(e_{12}) = 0$, $R(e_{22}) = e_{11}$, and $R(e_{21}) = -e_{21}$ is an RB operator on $M_2(F)$ of weight 1.

Due to [3], the set of all RB operators of Example 14 is invariant under ϕ.

Lemma 4.7. *Let A be a quadratic algebra with a unit 1, and let R be an RB operator on A of weight 1, which is non-splitting. If $R(1) = \alpha \cdot 1 + p$ with $t(p) = 0$, then one of three following cases occurs:*

(I) $R(1) = -\frac{1}{2} + p$, $R(p) = \frac{1}{4} - \frac{p}{2}$;

(II) $R(1) = \frac{1}{2} + p$, $R(p) = -\frac{1}{4} - \frac{p}{2}$;

(III) $R(1) = -\frac{3}{2} + p$, $R(p) = -\frac{1}{4} - \frac{p}{2}$.

Proof. By Lemma 2.6 (b), $p \notin F$. Let $R(p) = \Delta \cdot 1 + s$, where $t(s) = 0$. Then

$$\begin{aligned}(\alpha^2 - n(p)) \cdot 1 + 2\alpha p &= R(1)R(1) = 2R(R(1)) + R(1)\\ &= (2\alpha^2 + \alpha) \cdot 1 + (2\alpha + 1)p + 2R(p).\end{aligned} \tag{4.26}$$

By (4.26), we conclude

$$\Delta = -\frac{1}{2}(n(p) + \alpha(\alpha + 1)), \quad s = -\frac{p}{2}. \tag{4.27}$$

Considering

$$\begin{aligned}\left(\alpha\Delta + \frac{1}{2}n(p)\right) \cdot 1 + \left(\Delta - \frac{\alpha}{2}\right) p &= (\alpha \cdot 1 + p)\left(\Delta \cdot 1 - \frac{p}{2}\right)\\ &= R(1)R(p) = R(R(1)p + R(p) + p)\\ &= R((\alpha \cdot 1 + p)p + \Delta \cdot 1 - \frac{p}{2} + p)\\ &= (\Delta - \det p)(\alpha \cdot 1 + p) + \left(\alpha + \frac{1}{2}\right)\left(\Delta \cdot 1 - \frac{p}{2}\right)\\ &= \left(2\alpha\Delta + \frac{\Delta}{2} - \alpha n(p)\right) \cdot 1 + \left(\Delta - \frac{\alpha}{2} - \frac{1}{4} - n(p)\right) p,\end{aligned} \tag{4.28}$$

we have $n(p) = -1/4$, and $\left(\alpha + \frac{1}{2}\right)\left(\Delta + \frac{1}{4}\right) = 0$. Solutions to the last equation give exactly the required cases I, II, and III. □

Theorem 4.8. *All RB operators on $M_2(F)$ of nonzero weight either are splitting or are defined by Example 14 up to conjugation by an automorphism of $M_2(F)$.*

Proof. Suppose that R is an RB operator on $M_2(F)$, which is non-splitting, and $R(1) = \alpha \cdot 1 + p$, where $\alpha \in F$, $(0 \neq)p \in \mathrm{sl}_2(F)$. Apply Lemma 4.7. The case III is equivalent to the case II by $\phi(R) = -R - \mathrm{id}$.

Since $\det\left(\frac{1}{2} + p\right) = 0$ for $p \in \mathrm{sl}_2(F)$ and the square of $\left(\frac{1}{2} + p\right)$ is proportional to itself in both cases I and II, we can consider an RB operator $P = R^{(\varphi)}$ with $\varphi \in \mathrm{Aut}(M_2(F))$, such that $\varphi\left(\frac{1}{2}+p\right) = e_{11}$. Hence, $P(e_{11}) = 0$. Let $P(e_{12}) = s$ and $P(e_{21}) = t$.

Case I. $P(e_{22}) = -e_{22}$. We have

$$0 = P(e_{11})P(e_{12}) = P(e_{11}s + e_{12}) = (1 + s_{12})s. \tag{4.29}$$

If $s = 0$, then

$$0 = P(e_{21})P(e_{12}) = P(te_{12} + e_{22}) = t_{21}t - e_{22},$$

a contradiction. Hence, $s_{12} = -1$. Since $\mathrm{Im}\,(P)$ is a subalgebra, $se_{22} \in \mathrm{Im}\,(P)$ and $-e_{12} + s_{22}e_{22} \in \mathrm{Im}\,(P)$. Therefore, $e_{12} \in \mathrm{Im}\,(P)$.

Consider

$$0 = P(e_{21})P(e_{11}) = (1 + t_{21})t. \tag{4.30}$$

If $t_{21} = -1$, then $e_{12}t = -e_{11} + t_{22}e_{12} \in \mathrm{Im}\,(P)$ and $e_{11} \in \mathrm{Im}\,(P)$. Furthermore, $e_{21} \in \mathrm{Im}\,(P)$ and $\mathrm{Im}\,(P) = M_2(F)$, a contradiction. Hence, $t = 0$. As $\dim\ker P = \dim \mathrm{Im}\,(P) = 2$, we have $s = -e_{12} + s_{22}e_{22}$. Comparing the expressions

$$\begin{aligned} &P(e_{12})P(e_{22}) = (-e_{12} + s_{22}e_{22})(-e_{22}) = e_{12} - s_{22}e_{22}, \\ &P(P(e_{12})e_{22} + e_{12}(P(e_{22}) + e_{22})) \\ &\quad = P((-e_{12} + s_{22}e_{22})e_{22}) = P(-e_{12} + s_{22}e_{22}) = e_{12} - 2s_{22}e_{22}, \end{aligned}$$

we have $s_{22} = 0$, and P is splitting.

Case II. $P(e_{22}) = e_{11}$. Since $\mathrm{tr}(e_{12}) = \mathrm{tr}(e_{21}) = 0$, $\det s = \det t = 0$ by Lemma 3.1 (a). From

$$\begin{aligned} P(e_{12})P(e_{22}) = se_{11} = s_{11}e_{11} + s_{21}e_{21} &= P(P(e_{12})e_{22} + e_{12}P(e_{22}) + e_{12}e_{22}) \\ &= P(se_{22} + e_{12}) = s_{22}e_{11} + (1 + s_{12})s \end{aligned}$$

we see that $s = -e_{12}$ or $s = s_{11}e_{11} + s_{21}e_{21}$. Analogously, considering $P(e_{22})P(e_{21})$, we have either $t = -e_{21}$ or $t = t_{11}e_{11} + t_{12}e_{12}$. Together with (4.29) and (4.30), we have either $s = -e_{12}$ or $s = 0$, and either $t = -e_{21}$ or $t = 0$. The case $s = -e_{12}$ and $t = -e_{21}$ leads to $\mathrm{Im}\,(P) = M_2(F)$, a contradiction. The case $s = t = 0$ leads to $e_{22} = e_{21}e_{12} \in \ker P$, a contradiction. The cases $s = -e_{12}$, $t = 0$ and $s = 0$, $t = -e_{21}$ give the RB operators from Example 14. $\square$

4.4. The Grassmann Algebra of Plane

Denote by Gr_2 the Grassmann algebra of plane $\mathrm{span}\{e_1, e_2\}$, i.e., the elements $1, e_1, e_2, e_1 \wedge e_2$ form a linear basis for Gr_2.

The algebra Gr_2 is quadratic, since for $x = \alpha\cdot 1+\beta e_1+\gamma e_2+\delta e_1\wedge e_2 \in \mathrm{Gr}_2$ we have $x^2 = \alpha^2\cdot 1 + 2\alpha\beta e_1 + 2\alpha\gamma e_2 + 2\alpha\delta e_1\wedge e_2 = 2\alpha x - \alpha^2\cdot 1$. Hence, $t(x) = 2\alpha$, $n(x) = \alpha^2$. Let $A_0 = \mathrm{span}\{e_1, e_2, e_1\wedge e_2\}$.

Theorem 4.9. *All RB operators of nonzero weight on* Gr_2 *are splitting.*

Proof. Suppose that R is a non-splitting RB operator of weight 1. On the contrary, by Lemma 3.1, we have $n(R(x)) = 0$ for every $x \in A_0$. Therefore, $t(R(x)) = 0$, $x \in A_0$.

Let $R(1) = \alpha\cdot 1 + p$, where $\alpha \in F$ and p is nonzero element in A_0. By Lemma 4.7, we have $n(R(p)) \neq 0$ in all three cases, a contradiction. □

4.5. The Simple Jordan Superalgebra K_3

The simple Jordan superalgebra K_3 is defined as follows: $\mathrm{K}_3 = A_0 \oplus A_1$, $A_0 = \mathrm{span}\{e\}$ (the even part), $A_1 = \mathrm{span}\{x, y\}$ (the odd part):

$$e^2 = e, \quad ex = xe = \frac{x}{2}, \quad ey = ye = \frac{y}{2}, \quad xy = -yx = \frac{e}{2}, \quad x^2 = y^2 = 0.$$

The superalgebra K_3 is quadratic because of $z^2 - t(z)z = 0$ for each $z \in \mathrm{K}_3$, and $t(\alpha e + \beta x + \gamma y) = \alpha$.

Theorem 4.10. *All RB operators of nonzero weight on* K_3 *are splitting.*

Proof. Let R be a non-splitting RB operator on K_3 of weight 1. Applying (3.2), we have

$$\begin{aligned} t(R(z))R(z) &= R(z)R(z) = R(z\circ R(z) + z^2) \\ &= t(z)R(R(z)) + t(R(z))R(z) + t(z)R(z), \end{aligned} \tag{4.31}$$

whence $t(z)R(R(z) + z) = 0$ for all $z \in \mathrm{K}_3$.

Hence, $R(R(e) + e) = 0$ and $R(R(e+s) + e + s) = R(R(s) + s) = 0$ for each $s \in A_1$. Combining the last two equalities we obtain $R(R(z) + z) = 0$ for all $z \in \mathrm{K}_3$. The statement follows by Lemma 2.6 (a). □

4.6. Derivations of Nonzero Weight

Given an algebra A and $\Delta \in F$, a linear operator $d\colon A \to A$ is called a *derivation of weight* Δ [17] provided that the following equality holds for all $x, y \in A$:

$$d(xy) = d(x)y + xd(y) + \Delta d(x)d(y). \tag{4.32}$$

Let us call the zero operator and $(-1/\Delta)\mathrm{id}$ (for $\Delta \neq 0$) as *trivial derivations of weight* Δ.

Statement 4.11. [20] *Given an algebra A and an invertible derivation d on A of weight Δ, the operator d^{-1} is an RB operator on A of weight Δ.*

Proof. Let $x = d^{-1}(a)$ and $y = d^{-1}(b)$. Then, d^{-1} acts on both sides of (4.32) by the rule:

$$d^{-1}(a)d^{-1}(b) = d^{-1}(ad^{-1}(b) + d^{-1}(a)b + \Delta ab).$$

□

Corollary 4.12. *There are no nontrivial invertible derivations of nonzero weight on quadratic division algebras, the simple odd-dimensional Jordan algebras of bilinear form, the matrix algebra $M_2(F)$, the Grassmann algebra* Gr_2*, and the Kaplansky superalgebra* K_3*.*

Proof. Proof follows from Theorems 3.2, 4.1, 4.8–4.10. □

5. The RB Operators of Weight Zero

5.1. The Matrix Algebra of Order 2

Lemma 5.1. *Let R be an RB operator on $M_n(F)$ of weight zero, and let $\mathrm{char}\, F = 0$. Then,* $\mathrm{Im}\, R$ *consists only of degenerate matrices, and* $\dim(\mathrm{Im}\, R) \leq n^2 - n$*.*

Proof. If $\mathrm{Im}\, R$ contains an invertible matrix, then $1 \in \mathrm{Im}\, R$ by the Cayley–Hamilton theorem, a contradiction with Lemma 2.6 (c). Thus, by [21], we have $\dim(\mathrm{Im}\, R) \leq n^2 - n$. □

Theorem 5.2. [28] *All nonzero RB operators of weight zero on $M_2(F)$ over an algebraically closed field F up to conjugation by automorphisms of $M_2(F)$, transposition and multiplication by a nonzero scalar are the following:*

(M1) $R(e_{21}) = e_{12}$, $R(e_{11}) = R(e_{12}) = R(e_{22}) = 0$;
(M2) $R(e_{21}) = e_{11}$, $R(e_{11}) = R(e_{12}) = R(e_{22}) = 0$;
(M3) $R(e_{21}) = e_{11}$, $R(e_{22}) = e_{12}$, $R(e_{11}) = R(e_{12}) = 0$;
(M4) $R(e_{21}) = -e_{11}$, $R(e_{11}) = e_{12}$, $R(e_{12}) = R(e_{22}) = 0$.

Proof. Let R be a nonzero RB operator on $M_2(F)$ of weight zero. We have $\dim(\mathrm{Im}\, R) \leq 2$ by Lemma 2.6 (d) or by Lemma 5.1.

Let $\dim(\mathrm{Im}\, R) = 1$. If $\mathrm{Im}\, R = \mathrm{span}\{v\}$, then $\det v = 0$ by Lemma 5.1. We may assume that either $\mathrm{tr}(v) = 1$ or $\mathrm{tr}(v) = 0$; thus, v equals e_{11} or e_{12}, respectively. We get $R^2 = 0$ in both cases. Indeed, for $v = e_{11}$, we apply Lemma 2.7a, and for $v = e_{12}$, we apply Lemma 2.7b. Up to conjugation by an automorphism of $M_2(F)$, we may assume that either $\mathrm{Im}\, R = F \cdot e_{11}$ or $\mathrm{Im}\, R = F \cdot e_{12}$.

Consider the case $\mathrm{Im}\, R = F \cdot e_{12}$. If $R(1) = \alpha e_{12}$ for $\alpha \in F^*$, then from $0 = R(1)R(x) = R(R(1)x + R(x)) = R(R(1)x)$ and $0 = R(xR(1))$ for $x = e_{21}$ we have $R(e_{11}) = R(e_{22}) = 0$, a contradiction. Therefore, $R(1) = 0$. If $R(e_{11}) = ke_{12} \neq 0$, then $0 = R(e_{11})R(e_{21}) = R(ke_{12}e_{21} + e_{11}R(e_{21}) = k^2 e_{12}$, a contradiction. Thus, $R(e_{11}) = R(e_{22}) = R(e_{12}) = 0$ and $R(e_{21}) = \alpha e_{12}$ for some $\alpha \in F^*$, and we arrive at (M1).

Let $\mathrm{Im}\, R = F \cdot e_{11}$. If $R(1) = \alpha e_{11}$ for $\alpha \in F^*$, then considering $(1/\alpha)R(1)R(x) = e_{11}R(x) = R(e_{11}x)$ for $x = e_{22}$, we get $R(1) = R(e_{22}) = 0$. If $R(e_{12}) = \alpha e_{11}$ and $R(e_{21}) = \beta e_{11}$ for $\alpha\beta \neq 0$, then the equality $\alpha\beta e_{11} = R(e_{12})R(e_{21}) = R(\alpha e_{11}e_{21} + \beta e_{12}e_{11}) = 0$ gives a contradiction. Hence, $R(e_{12}) = 0$, $R(e_{21}) = \alpha e_{11}$ or $R(e_{21}) = 0$, $R(e_{12}) = \alpha e_{11}$ for some $\alpha \in F^*$, this is (M2).

Let $\dim(\mathrm{Im}\, R) = 2$. If $\mathrm{Im}\, R$ is nilpotent, then up to conjugation by $\mathrm{Aut}(M_2(F))$, we can consider $e_{12} \in \mathrm{Im}\, R$ and nonzero $x = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \mathrm{Im}\, R$.

Since $\mathrm{tr}(x) = \det(x) = 0$, we have $a = c = 0$ and $e_{21} \in \mathrm{Im}\, R$. Thus, $e_{12}e_{21} \in \mathrm{Im}\, R$, a contradiction with $\dim(\mathrm{Im}\, R) = 2$.

Therefore, $\mathrm{Im}\, R$ contains an idempotent. We may assume $e_{11} \in \mathrm{Im}\, R$ up to conjugation by $\mathrm{Aut}(M_2(F))$. Since $\mathrm{Im}\, R$ is a subspace of $M_2(F)$ consisting only of degenerate matrices of maximal possible dimension; therefore, $\mathrm{Im}\, R = \mathrm{span}\{e_{11}, e_{12}\}$ up to transposition by [9]. Assume that R is not nilpotent, so $R(1) = \alpha e_{11} + \beta e_{12} \neq 0$. If $\alpha = 0$, then $R(1)R(1) = 2R^2(1) = 0$, and we get $R(1) = 0$ by (4.18). For $\alpha \neq 0$, applying Lemma 2.7 (a) we arrive at a contradiction. So, R is nilpotent, and $\mathrm{Im}\, R \cap \ker R \neq (0)$.

(a) $\mathrm{Im}\, R = \ker R$. Let $R(x_0 = \alpha e_{21} + \beta e_{22}) = e_{11}$, and $R(y_0 = \gamma e_{21} + \delta e_{22}) = e_{12}$. From $R(x_0)R(x_0) = e_{11} = R(\alpha e_{21})$ we have $\beta = 0$. Thus, $R(e_{21}) = (1/\alpha)e_{11}$ and $\delta \neq 0$. Considering $R(x_0)R(y_0) = e_{12} = \alpha R(e_{22})$, we conclude that $R(e_{22}) = (1/\alpha)e_{12}$ and $\gamma = 0$. This is (M3).

(b) $\dim(\mathrm{Im}\, R \cap \ker R) = 1$. Assume that there exists $a \in \mathrm{Im}\, R \cap \ker R$, such that $a^2 = \alpha a$ for $\alpha \in F^*$. Up to conjugation by $\mathrm{Aut}(M_2(F))$, $a = e_{11}$. As above, $\mathrm{Im}\, R = \mathrm{span}\{e_{11}, e_{12}\}$ up to transposition. Let a nonzero $x = \beta e_{21} + \gamma e_{22} + \delta e_{12}$ belongs to $\ker R$. Since $\ker R$ is an $\mathrm{Im}\, R$-module, we get $e_{11}x = \delta e_{12} \in \ker R$, and $xe_{12} = \beta e_{22} \in \ker R$. So, $\delta = 0$ and $e_{22} \in \ker R$. Hence, $R(1) = 0$, and by Lemma 2.6 (c) $R^2 = 0$ and $\mathrm{Im}\,(R) \subset \ker(R)$, a contradiction.

Therefore, $\mathrm{Im}\, R \cap \ker R$ is nilpotent, and it is equal to $F \cdot e_{12}$. Let a nonzero $x_0 = \alpha e_{21} + \beta e_{22} + \gamma e_{11}$ belongs to $\ker R$. From $e_{11}x_0 = \gamma e_{11} \in \ker R$ we have $\gamma = 0$. If $\alpha \neq 0$, then $x_0 e_{12} = \alpha e_{22} \in \ker R$. Hence, $e_{12}, e_{22} \in \ker R$, a contradiction. Thus, $\alpha = 0$ and $e_{22} \in \ker R$. Let $R(z_0 := \alpha e_{11} + \beta e_{21}) = e_{11}$, and $R(t_0 := \gamma e_{11} + \delta e_{21}) = e_{12}$. From $e_{11} = R(z_0)R(z_0) = R(2\alpha e_{11} + \beta e_{21})$ we obtain $\alpha = 0$. From $e_{12} = R(z_0)R(t_0) = R(\gamma e_{11} + \beta e_{22}) = R(\gamma e_{11})$ we have $\delta = 0$. Finally, $0 = R(e_{11})R(e_{21}) = R((1/\gamma)e_{11} + (1/\beta)e_{11})$, whence $\gamma = -\beta$, and we arrive at (M4). $\square$

Corollary 5.3. *The set of all RB operators of weight zero on an n-dimensional algebra A up to conjugation by automorphisms of A and multiplications on nonzero scalars may be considered as a projective variety $RB(A)$ in $\mathbb{P}^{n^2-1}$ defined by n^3 relations obtained from (1.1), which is written on a linear basis of A. Thus, by Theorem 5.2, $RB(M_2(F))$ has four fixed points under the action by conjugation by an (anti)automorphism. Indeed, $(M4)$ is the only one that does not satisfy $R^2 = 0$. Furthermore, $(M1)$ and $(M2)$ but not $(M3)$ satisfy the condition that all minors of order 2 are zero in the image. Finally, the image of $(M1)$ in $M_2(\mathbb{C})$ is a subalgebra with trivial multiplication, but not the image of $(M2)$. Thus, the corresponding linear and quadratic relations distinguish $(M1)$ and $(M2)$.*

5.2. The Grassmann Algebra of Plane

Statement 5.4. *Up to conjugation by an automorphism of Gr_2 an arbitrary RB operator R of weight zero on Gr_2 with a linear basis $1, e_1, e_2, e_1 \wedge e_2$ is the following one: $R(1), R(e_1) \in \mathrm{span}\{e_2, e_1 \wedge e_2\}$, $R(e_2) = R(e_1 \wedge e_2) = 0$.*

Proof. (a) Take $x = \alpha \cdot 1 + x' \in R(\mathrm{Gr}_2)$, where $x' \in \mathrm{span}\{e_1, e_2, e_1 \wedge e_2\}$. Then, $(x - \alpha \cdot 1)^2 = 0$. Since $R(\mathrm{Gr}_2)$ is a subalgebra of Gr_2, $\alpha^2 \cdot 1 \in R(\mathrm{Gr}_2)$. By Lemma 2.6 (c), $\alpha = 0$.

Given $x = \alpha \cdot 1 + x'$ with $x' \in \mathrm{span}\{e_2, e_1 \wedge e_2\}$, we have $0 = R(x)R(x)$

$$= R(R(x)x + xR(x)) = R(R(x)x' + x'R(x)) + 2\alpha R(R(x)) = 2\alpha R(R(x)).$$

At first $R(R(1)) = 0$; at second $R(R(x)) = 0$ for all $x \in \mathrm{Gr}_2$. Hence, $\mathrm{Im}\, R \subset \ker R$, and $\dim(\mathrm{Im}\, R) \leq 2$.

Assume that there exist x and y, such that $x_1e_1 + x_2e_2$ and $y_1e_1 + y_2e_2$ are linearly independent, $R(x) = x_1e_1 + x_2e_2 + x_{12}e_1 \wedge e_2$, and $R(y) = y_1e_1 + y_2e_2 + y_{12}e_1 \wedge e_2$. By (1.1), $R(x)R(y) = (x_1y_2 - x_2y_1)e_1 \wedge e_2 \in R(\mathrm{Gr}_2)$. From here $R(\mathrm{Gr}_2) = \mathrm{span}\{e_1, e_2, e_1 \wedge e_2\}$, which contradicts to the fact that $\dim(\mathrm{Im}\, R) \leq 2$. Therefore, $R(\mathrm{Gr}_2)$ is an algebra with trivial multiplication.

Show that $e_1 \wedge e_2 \in \ker(R)$. Otherwise, there is $x \in \mathrm{Gr}_2$ that $R(e_1 \wedge e_2)x = e_1 \wedge e_2$. Hence

$$0 = R(e_1 \wedge e_2)R(x) = R(R(e_1 \wedge e_2)x + e_1 \wedge e_2 R(x)) = R(e_1 \wedge e_2),$$

a contradiction.

Let $R(e_1) = k\alpha e_1 + k\beta e_2 + \gamma e_1 \wedge e_2$, and $R(e_2) = l\alpha e_1 + l\beta e_2 + \delta e_1 \wedge e_2$. Then

$$0 = R(e_1)R(1) = R(R(e_1) + e_1R(1)) = R(R(e_1)) = \alpha R(e_1) + \beta R(e_2),$$

whence $R(e_1)$ and $R(e_2)$ are linearly dependent. Up to conjugation by an automorphism of Gr_2, we may assume that $R(e_2) = 0$ and $R(e_1) \in \mathrm{span}\{e_2, e_1 \wedge e_2\}$. It is immediate that a linear map R, such that $R(1), R(e_1) \in \mathrm{span}\{e_2, e_1 \wedge e_2\} \subseteq \ker R$ is an RB operator on Gr_2. □

5.3. The Simple Jordan Superalgebra K_3

Statement 5.5. *An arbitrary RB operator R of weight zero on K_3 up to conjugation by $\mathrm{Aut}(\mathrm{K}_3)$ is the following one:*

$$R(e) = R(x) = 0, \quad R(y) = ae + bx, \ a, b \in F.$$

Proof. Let R be a nonzero RB operator on K_3 of weight zero. By analogy with the proof of Theorem 4.10 and (4.31), we have $R(R(z)) = 0$ for all $z \in \mathrm{K}_3$. So, $\mathrm{Im}\, R \subset \ker R$, and $\dim \mathrm{Im}\, R = 1$, $\dim \ker R = 2$.

Let $R(e) = \alpha e + \beta x + \gamma y$. Suppose that $\alpha \neq 0$. For all $z \in \ker R$,

$$0 = R(e)R(z) = R(R(e)z) \tag{5.1}$$

hold, whence $R(e)z \in \ker R$. Since $R(e) \in \ker R$, there exists $z = \Delta x + \mu y \in \ker R$, such that z and $\beta x + \gamma y$ are linearly independent. From $R(e)z = \alpha z + (\beta\mu - \gamma\Delta)e \in \ker R$, we have $e \in \ker R$.

By analogy with (5.1), for $\alpha = 0$, considering $z = \chi e + \Delta x + \mu y \in \ker R$, we obtain $R(e) = 0$. From here and $\mathrm{Im}\, R \subset \ker R$, the assertion follows. □

5.4. Connection with the Yang–Baxter Equation

Let A be an associative algebra, $r = \sum a_i \otimes b_i \in A \otimes A$. The tensor r is called a solution of the associative Yang–Baxter equation (AYBE, [2,29]) if

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0, \tag{5.2}$$

where

$$r_{12} = \sum a_i \otimes b_i \otimes 1, \quad r_{13} = \sum a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum 1 \otimes a_i \otimes b_i$$

are elements from $A^{\otimes 3}$.

Example 15. [1] Let $r = \sum a_i \otimes b_i$ be a solution of AYBE on an associative algebra A. A linear map $R\colon A \to A$ defined as $R(x) = \sum a_i x b_i$ is an RB operator of weight zero on A.

The image of an RB operator of weight zero on an algebra A is a subalgebra of A. The following example shows that the kernel of an RB operator of weight zero on A is not a subalgebra of A in general (even in the associative case).

Example 16. Consider the following solution to (5.2) on $A = M_4(F)$ with an arbitrary field F:

$$r = e_{11} \otimes e_{12} - e_{12} \otimes e_{11} + e_{33} \otimes e_{34} - e_{34} \otimes e_{33}.$$

By Example 15

$$R(x) = e_{11} x e_{12} - e_{12} x e_{11} + e_{33} x e_{34} - e_{34} x e_{33}$$

is an RB operator on A, and its kernel consists of the matrices (a_{ij}) in A, such that $a_{11} = a_{21} = a_{33} = a_{43} = 0$. It is easy to see that $\ker R$ is not a subalgebra of A.

Let A be an algebra. Assume that $r = \sum a_i \otimes b_i \in A \otimes A$ satisfies the *nonassociative Yang–Baxter equation* over A ($r_{12} = \sum a_i \otimes b_i \otimes 1, r_{13} = \sum a_i \otimes 1 \otimes b_i$ and so on):

$$r_{12}r_{13} - r_{23}r_{12} - r_{13}r_{32} = 0. \tag{5.3}$$

(Note that the classical Yang–Baxter equation [6] on Lie algebras is written as $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.) We have

$$\sum a_i a_j \otimes b_i \otimes b_j - a_i \otimes a_j b_i \otimes b_j - a_i \otimes b_j \otimes b_i a_j = 0,$$

and if $\varphi\colon A \to A^*$ is one-to-one, then

$$\sum a_i a_j b_i^*(x) b_j^*(y) - a_i (a_j b_i)^*(x) b_j^*(y) - a_i b_j^*(x) (b_i a_j)^*(y) = 0. \tag{5.4}$$

On the other hand, let $R(x) = \sum a_i b_i^*(x)$ be an RB operator on A of weight zero. Then, (1.1) gives

$$\sum a_i a_j b_i^*(x) b_j^*(y) - a_i b_j^*(y) b_i^*(x a_j) - a_i b_j^*(x) b_i^*(a_j y) = 0. \tag{5.5}$$

Let A be a finite-dimensional algebra, such that the map φ: $A \mapsto A^*$, $\varphi(a) = a^*$, is an A-bimodule isomorphism, where the action on A^* is defined by the rule

$$a \cdot b^*(x) = b^*(xa), \quad b^* \cdot a(x) = b^*(ax).$$

Let R be an RB operator on A of weight zero. If $a_1, \dots, a_n$ is a basis for A, then $R(a_i) = \sum_{j=1}^n \alpha_{ij} a_j$. Define a linear functional b_j^* in A^* by the rule $b_j^*(a_i) = \alpha_{ij}$. Then, $R(a_i) = \sum_{j=1}^n b_j^*(a_i) a_j$, whence $R(x) = \sum_{j=1}^n b_j^*(x) a_j$ for every $x \in A$.

If $\varphi\colon A \to A^*$, $\varphi(a) = a^*$, is an A-bimodule isomorphism, where the action on A^* is defined as above, then it is easy to see that (5.5) is equivalent to (5.4). Thus, we have

Theorem 5.6. *Let A be a finite-dimensional algebra, such that the map $\varphi\colon A \to A^*$, $\varphi(a) = a^*$, is an A-bimodule isomorphism, where the action on A^* is defined by the rule*

$$a \cdot b^*(x) = b^*(xa), \quad b^* \cdot a(x) = b^*(ax).$$

Then, a solution $r = \sum a_i \otimes b_i$ of the nonassociative Yang–Baxter equation (5.3) over A defines an RB operator on A of weight zero by the rule $R(x) = \sum a_i b_i^*(x)$. Conversely, if R is an RB operator on A, then $R(x) = \sum_{j=1}^n b_j^*(x) a_j$, and $r = \sum a_i \otimes b_i$ is a solution of the nonassociative Yang–Baxter equation (5.3) over A.

Let A be a simple finite-dimensional algebra in some variety $\mathcal{M}$. Assume that A^* is an A-$\mathcal{M}$-bimodule (with the action as above). Then, A^* is an irreducible A-bimodule. Indeed, if V is a submodule of A^* and $V \neq A^*$, then there is $x \in A$, such that $f(x) = 0$ for all $f \in V$. Then, $a \cdot f(x) = 0$ and $f \cdot a(x) = 0$ for every $a \in A$, whence $f(I) = 0$, where $I = \mathrm{ideal}\langle x \rangle = A$, i.e., $f = 0$.

Corollary 5.7. *Let A be a simple finite-dimensional self-adjoined algebra, i.e., $A \cong A^*$ as A-bimodules. Then, the solutions of the nonassociative Yang–Baxter equation (5.3) over A are in one-to-one correspondence with the RB operators on A of weight zero.*

If C is Cayley–Dickson algebra, then C is an alternative D-bialgebra [11], and C is a self-adjoined algebra. Therefore, we obtain the following

Corollary 5.8. *The solutions of the nonassociative Yang–Baxter equation (5.3) over C are in one-to-one correspondence with the RB operators on C of weight zero.*

Note that all skew-symmetric ($r = \sum a_i \otimes b_i = -\sum b_i \otimes a_i$) solutions of the Yang–Baxter equation over the Cayley–Dickson matrix algebra were described in [11].

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