EXOTIC MULTIPLICITY FUNCTIONS AND HEAT MAXIMAL OPERATORS IN CERTAIN DUNKL SETTINGS

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Abstract. We consider settings of the Dunkl Laplacian, the Dunkl harmonic oscillator, and the Dunkl Ornstein-Uhlenbeck operator with the underlying group of reflections isomorphic to $\mathbb{Z}_2^d$. In each of these contexts we admit all real-valued multiplicity functions, not necessarily bounded from below, and construct the corresponding 'exotic' transform or orthogonal system. This leads to new Dunkl operator based frameworks, which generalize those yet well known, and in which harmonic analysis can reasonably be developed. To support the last claim, in all the cases we study the associated heat semigroup maximal operators and prove that they are bounded on L^p, $p > 1$, and from L^1 to weak L^1.

1. Introduction and preliminaries

In this paper we consider three settings related to the Dunkl operators in $\mathbb{R}^d$, $d \geq 1$, with the underlying group of reflections isomorphic to $\mathbb{Z}_2^d$. For this reflection group the associated multiplicity functions are represented by multi-parameters $\alpha \in \mathbb{R}^d$. In our notation, which for technical reasons differs from the common one by a shift by $1/2$ in each coordinate of α, the parameter $\alpha_0 = (-1/2, \ldots, -1/2)$ corresponds to the trivial multiplicity function. For basic concepts, facts, and results concerning the Dunkl theory we refer to [1, 2, 3, 4].

Let $d \geq 1$ and $\alpha \in \mathbb{R}^d$. The Dunkl operators

$$T_j^\alpha f(x) = \frac{\partial}{\partial x_j} f(x) + (2\alpha_j + 1) \frac{f(x) - f(\sigma_j x)}{2x_j}, \qquad j = 1, \ldots, d,$$

form a commuting system of difference-differential operators acting on sufficiently regular functions defined on $\mathbb{R}^d$; here σ_j denotes the reflection with respect to the hyperplane in $\mathbb{R}^d$ orthogonal to the jth coordinate vector. The Dunkl Laplacian is naturally defined as

$$[\mathrm{DL}, \mathbb{Z}_2^d] \qquad \Delta_\alpha f(x) = -\sum_{j=1}^d \big(T_j^\alpha\big)^2 f(x)$$
$$= -\sum_{j=1}^d \left(\frac{\partial^2}{\partial x_j^2} f(x) + \frac{2\alpha_j + 1}{x_j} \frac{\partial}{\partial x_j} f(x) - (2\alpha_j + 1) \frac{f(x) - f(\sigma_j x)}{2x_j^2} \right).$$

Closely related to Δ_α there are two other important difference-differential operators occurring in the Dunkl theory, namely the Dunkl harmonic oscillator and the Dunkl Ornstein-Uhlenbeck operator:

$$[\mathrm{DHO}, \mathbb{Z}_2^d] \qquad L_\alpha f(x) = \Delta_\alpha f(x) + |x|^2,$$
$$[\mathrm{DOU}, \mathbb{Z}_2^d] \qquad U_\alpha f(x) = \Delta_\alpha f(x) + 2x \cdot \nabla f(x).$$

A variant of U_α with the standard Euclidean gradient ∇ replaced by the Dunkl one, $\nabla_\alpha = (T_1^\alpha, \ldots, T_d^\alpha)$, also appears in the literature; see e.g. [5].

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When $\alpha \in (-1,\infty)^d$ all the three operators, Δ_α, L_α and U_α, 'govern' harmonic analysis frameworks in which they play roles similar to that of the Euclidean Laplacian in the classical harmonic analysis (actually, $\Delta_{\alpha_0} = -\Delta$, $\nabla_{\alpha_0} = \nabla$, and we recover the classical situations). In case of Δ_α there exists an analogue of the Fourier transform called the Dunkl transform, and for L_α and U_α there are orthogonal bases of eigenfunctions in suitable L^2 spaces consisting of the so-called generalized Hermite functions and polynomials, respectively. This enables one to define in a canonical way self-adjoint extensions of Δ_α, L_α and U_α, and then develop harmonic analysis related to them.

However, if $\alpha \notin (-1,\infty)^d$ this nice picture breaks down. The Dunkl transform is not even defined in the relevant L^2 space, and the same for the above mentioned orthogonal systems. Thus the motivation and the main aim of this paper is to bring in an 'exotic' Dunkl transform and 'exotic' orthogonal systems so that for *any* $\alpha \in \mathbb{R}^d$ (i.e. for *any* real-valued multiplicity function) the difference-differential operators Δ_α, L_α and U_α can be extended to self-adjoint operators in a fashion indicated above. This leads to new harmonic analysis frameworks related to Dunkl operators. Our inspiration comes from similar questions investigated by Hajmirzaahmad [6, 7] in the contexts of Jacobi and Laguerre expansions, as well as from a recent paper in a similar spirit by Sjögren, Szarek and one of the authors [8], where also the (modified) Hankel transform setting was considered.

To show that the exotic situations are in fact by no means exotic in the sense of basic harmonic analysis results, in all the cases we study the associated heat semigroups and their maximal operators. As the main results we prove that the latter are bounded on L^p, $p>1$, and satisfy weak type $(1,1)$ estimates. This implies in a standard way pointwise almost everywhere convergence of the semigroups in L^p, $p \ge 1$, to their initial values. It must be emphasized that in the exotic situations the associated measures are not even locally finite (there exist balls of arbitrarily small radii and infinite measure), hence analysis of the maximal operators is even more delicate than usually. In this aspect we relay heavily on the results, methods, and tools elaborated in [8].

The paper is organized as follows. Below, still in this section, we introduce the basic notation used throughout the paper. Then we also recall some standard facts and conventions concerning Bessel functions needed subsequently. In Sections 2–4 we treat the settings related to Δ_α, L_α and U_α, respectively (in Section 4 also a variant of U_α involving ∇_α is considered). In each case we introduce the exotic situation and prove the maximal theorem. For the sake of clarity and the reader's convenience, the exotic contexts are each time presented first in dimension one.

Notation. We use the following notation and abbreviations:

$$\begin{aligned}
\mathbb{R}_+ &= (0,\infty) \qquad \text{(positive half-line)},\\
\mathbf{1} &= (1,\ldots,1) \in \mathbb{R}^d_+,\\
\langle\alpha\rangle &= \alpha_1 + \ldots + \alpha_d \qquad \text{(length of a multi-parameter } \alpha \in \mathbb{R}^d),\\
\mathbb{1} &\equiv \text{the constant function equal to 1 on } \mathbb{R}^d \text{ or } \mathbb{R}^d_+,\\
|x| &= \sqrt{x_1^2 + \ldots + x_d^2} \qquad \text{(Euclidean norm of } x \in \mathbb{R}^d),\\
x\cdot y &= x_1y_1 + \ldots + x_dy_d \qquad \text{(dot product of } x,y \in \mathbb{R}^d),\\
x^\gamma &= x_1^{\gamma_1}\cdot\ldots\cdot x_d^{\gamma_d}, \qquad x \in \mathbb{R}^d_+, \quad \gamma \in \mathbb{R}^d,\\
xy &= (x_1y_1,\ldots,x_dy_d), \qquad x,y \in \mathbb{R}^d,\\
x \vee y &= \max(x,y), \qquad x,y \in \mathbb{R},\\
x \wedge y &= \min(x,y), \qquad x,y \in \mathbb{R}.
\end{aligned}$$

Given a measure μ, we denote by $\langle\cdot,\cdot\rangle_{d\mu}$ the standard inner product in $L^2(d\mu)$. For definitions of $A(\mathcal{E})$ and $m_{\mathcal{E}}(\alpha)$ see the beginning of Section 2.2. We use standard conventions concerning empty sums and products.

The notation $X \lesssim Y$ is used to indicate that $X \le CY$ with a positive constant C independent of significant quantities. We write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$.

Facts and conventions concerning Bessel functions. In this paper we use the Bessel functions $J_\nu(z)$ and $I_\nu(z)$, see e.g. [9], and the parameter will always satisfy $\nu > -1$ (ν is called sometimes the order).

J_ν is the most standard Bessel function, it has infinitely many oscillations (and zeros) over the half-line $\mathbb{R}_+$. On the other hand, the modified Bessel function $I_\nu(z)$ is strictly positive on $\mathbb{R}_+$. Both $J_\nu(z)$ and $I_\nu(z)$ are smooth for $z > 0$.

In general, the Bessel functions cannot be expressed directly via elementary functions. However, this is possible if (and only if) ν is half-integer, but not integer. In particular,

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z, \qquad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \tag{1.1}$$

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cosh z, \qquad I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh z. \tag{1.2}$$

Basic asymptotics of $J_\nu(z)$ and $I_\nu(z)$ are the following:

$$J_\nu(z) \simeq z^\nu, \quad z \to 0^+, \qquad J_\nu(z) = \mathcal{O}(z^{-1/2}), \quad z \to \infty, \tag{1.3}$$

$$I_\nu(z) \simeq z^\nu, \quad z \to 0^+, \qquad I_\nu(z) \simeq z^{-1/2}e^z, \quad z \to \infty. \tag{1.4}$$

Throughout the paper it is understood that

$$z^{-\nu}J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{2^\nu n!\Gamma(n+\nu+1)}, \qquad z^{-\nu}I_\nu(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{2^\nu n!\Gamma(n+\nu+1)}, \tag{1.5}$$

so that both $z \mapsto z^{-\nu}J_\nu(z)$ and $z \mapsto z^{-\nu}I_\nu(z)$ are even smooth functions of $z \in \mathbb{R}$. Their value at $z = 0$ is $2^{-\nu}/\Gamma(\nu+1)$.

We now list some inequalities satisfied by $I_\nu(z)$. In view of [10, Theorem 1], we have

$$I_{\nu+\varepsilon}(z) < I_\nu(z), \qquad z > 0, \quad \nu \geq -\varepsilon/2, \quad \nu > -1, \quad \varepsilon > 0. \tag{1.6}$$

With $\varepsilon = 1$ this is Soni's inequality (see [11])

$$I_{\nu+1}(z) < I_\nu(z), \qquad z > 0, \quad \nu \geq -1/2. \tag{1.7}$$

In the spirit of (1.7), taking into account (1.7) and (1.4), we can write

$$I_{\nu+1}(z) \leq C_\nu I_\nu(z), \qquad z > 0, \quad \nu > -1, \tag{1.8}$$

where $C_\nu = 1$ for $\nu \geq -1/2$ (this is optimal, as can be easily verified with the aid of more precise than (1.4) large argument asymptotics for I_ν) and necessarily $C_\nu > 1$ in case $-1 < \nu < -1/2$. In fact, see e.g. [12, Section 3.2], one has

$$I_{\nu+1}(z) > I_\nu(z), \qquad z > M_\nu, \quad -1 < \nu < -1/2, \tag{1.9}$$

with sufficiently large constants M_ν.

2. The Dunkl Laplacian context

In this section we focus on the setting related to the Dunkl Laplacian Δ_α. For the sake of clarity of the presentation, we begin with the one-dimensional situation.

2.1. **Classical and exotic $[\mathbf{DL}, \mathbb{Z}_2^d]$ contexts in dimension one.**

We consider the one-dimensional Dunkl Laplacian Δ_α for any $\alpha \in \mathbb{R}$. This operator is formally symmetric in $L^2(\mathbb{R}, dw_\alpha)$, where

$$dw_\alpha(x) = |x|^{2\alpha+1}\,dx.$$

The classical Dunkl Laplacian context occurs when $\alpha > -1$. Then there exists a natural self-adjoint extension of Δ_α (acting initially on, say, $C_c^2(\mathbb{R}\setminus\{0\})$), call it $\Delta_\alpha^{\mathrm{cls}}$, that is given spectrally by the Dunkl transform. The latter is defined for suitable functions f as

$$D_\alpha f(z) = \frac{1}{2}\int_{\mathbb{R}} f(x)\overline{\psi_z^\alpha(x)}\,dw_\alpha(x), \qquad z \in \mathbb{R},$$

where

$$\psi_z^\alpha(x) = \varphi_z^\alpha(x) + \mathrm{i}xz\varphi_z^{\alpha+1}(x), \qquad x, z \in \mathbb{R},$$

with

$$\varphi_z^\alpha(x) = (xz)^{-\alpha}J_\alpha(xz).$$

Observe that (see (1.5)) $\varphi_z^\alpha(x)$ is even with respect to x and also to z. Note that D_α extends to an isometry on $L^2(dw_\alpha)$ and its inverse is, up to a reflection, the identity: $D_\alpha^{-1}f(z) = D_\alpha f(-z)$. Further, one has

$\Delta_\alpha \psi_z^\alpha = z^2 \psi_z^\alpha$, $z \in \mathbb{R}$, and consequently $D_\alpha(\Delta_\alpha f)(z) = z^2 D_\alpha f(z)$ for sufficiently regular functions. Therefore $\Delta_\alpha^{\rm cls}$ is defined by

$$\Delta_\alpha^{\rm cls} f = D_\alpha^{-1}\big(z^2 D_\alpha f(z)\big),$$
$$\operatorname{Dom} \Delta_\alpha^{\rm cls} = \big\{ f \in L^2(dw_\alpha) : z^2 D_\alpha f(z) \in L^2(dw_\alpha)\big\}.$$

The semigroup $\mathbf{W}_t^\alpha = \exp(-t\Delta_\alpha^{\rm cls})$ has in $L^2(dw_\alpha)$ the integral representation

$$\mathbf{W}_t^\alpha f(x) = \frac{1}{2}\int_{\mathbb{R}} \mathbf{W}_t^\alpha(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

where

$$\mathbf{W}_t^\alpha(x,y) = W_t^\alpha(x,y) + xyW_t^{\alpha+1}(x,y).$$

Here

$$W_t^\alpha(x,y) = \frac{1}{(2t)^{\alpha+1}} \exp\Big(-\frac{1}{4t}\big(x^2+y^2\big)\Big)\Big(\frac{xy}{2t}\Big)^{-\alpha} I_\alpha\Big(\frac{xy}{2t}\Big);$$

notice that (see (1.5)) $W_t^\alpha(x,y)$ is an even function of both x and y that is jointly smooth in $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+$. Note also that, see (1.7)–(1.9),

$$|xy| W_t^{\alpha+1}(x,y) \le C_\alpha W_t^\alpha(x,y), \qquad x,y \in \mathbb{R}, \quad t > 0, \tag{2.2}$$

with the optimal $C_\alpha = 1$ in case $\alpha \ge -1/2$ and $C_\alpha > 1$ when $-1 < \alpha < -1/2$.

The kernel $\mathbf{W}_t^\alpha(x,y)$ is strictly positive for $\alpha \ge -1/2$, see (1.7), whereas for $-1 < \alpha < -1/2$ it takes both positive and negative values; thus in the latter case there is no positivity in the sense that $\mathbf{W}_t^\alpha f \ge 0$ whenever $f \ge 0$. Further, the asymptotics (1.4) show that the integral representation (2.1) provides a pointwise definition of $\mathbf{W}_t^\alpha f$, $t > 0$, for all $f \in L^p(dw_\alpha)$, $1 \le p \le \infty$. Since the Bessel semigroup is conservative, it follows (see Remark 2.1 below) that $\mathbf{W}_t^\alpha \mathbb{1} = \mathbb{1}$, $t > 0$. Thus $\{\mathbf{W}_t^\alpha\}$ is a Markovian symmetric diffusion semigroup for $\alpha \ge -1/2$. When $-1 < \alpha < -1/2$ each $\mathbf{W}_t^\alpha$, $t > 0$, is not contractive on L^∞ (the counterexample is the odd function $f(y) = \chi_{(0,\infty)}(y) - \chi_{(-\infty,0)}(y)$; see the computations in [13], particularly Lemmas 2.2 and 2.3 there).

Remark 2.1. *The Dunkl Laplacian context is connected with that of the (modified) Hankel transform. The latter expresses as*

$$H_\alpha f(z) = \int_0^\infty f(x)\varphi_z^\alpha(x)\, dw_\alpha(x), \qquad z > 0,$$

and the associated heat kernel is precisely $W_t^\alpha(x,y)$ (considered for $x, y > 0$). It is sometimes also called the Bessel heat kernel, since the corresponding semigroup of operators is the classical Bessel semigroup, see e.g. [13, Section 6].

We now pass to the exotic situation and, to begin with, assume that $\alpha < 0$. Consider the 'reflected' functions

$$\widetilde{\psi}_z^\alpha(x) = \widetilde{\varphi}_z^\alpha(x) + \mathrm{i}xz\widetilde{\varphi}_z^{\alpha+1}(x), \qquad x, z \in \mathbb{R},$$

with

$$\widetilde{\varphi}_z^\alpha(x) = |xz|^{-2\alpha}\varphi_z^{-\alpha}(x)$$

and define the exotic Dunkl transform by

$$\widetilde{D}_\alpha f(z) = \frac{1}{2}\int_{\mathbb{R}} f(x)\overline{\widetilde{\psi}_z^\alpha(x)}\, dw_\alpha(x), \qquad z \in \mathbb{R},$$

for sufficiently regular f. Decomposing f into its even and odd parts and using basic properties of the Hankel/exotic Hankel transforms (see [8]), one finds that $\widetilde{D}_\alpha$ extends to an isometry on $L^2(dw_\alpha)$ and the inverse coincides with its reflection, $\widetilde{D}_\alpha^{-1} f(z) = \widetilde{D}_\alpha f(-z)$. Moreover, $\Delta_\alpha \widetilde{\psi}_z^\alpha = z^2 \widetilde{\psi}_z^\alpha$, as easily verified. Thus $\widetilde{D}_\alpha(\Delta_\alpha f)(z) = z^2 \widetilde{D}_\alpha f(z)$ for suitable f and this leads us to defining an exotic self-adjoint extension of the Dunkl Laplacian (acting initially on $C_c^2(\mathbb{R}\setminus\{0\})$) as

$$\Delta_\alpha^{\rm exo} f = \widetilde{D}_\alpha^{-1}\big(z^2 \widetilde{D}_\alpha f(z)\big),$$
$$\operatorname{Dom} \Delta_\alpha^{\rm exo} f = \big\{ f \in L^2(dw_\alpha) : z^2 \widetilde{D}_\alpha f(z) \in L^2(dw_\alpha)\big\}.$$

The exotic semigroup $\widetilde{\mathbf{W}}_t^\alpha = \exp(-t\Delta_\alpha^{\rm exo})$ has in $L^2(dw_\alpha)$ the integral representation

$$\widetilde{\mathbf{W}}_t^\alpha f(x) = \frac{1}{2}\int_{\mathbb{R}} \widetilde{\mathbf{W}}_t^\alpha(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}, \quad t>0, \tag{2.3}$$

with the kernel

$$\widetilde{\mathbf{W}}_t^\alpha(x,y) = \widetilde{W}_t^\alpha(x,y) + xy\widetilde{W}_t^{\alpha+1}(x,y),$$

being

$$\widetilde{W}_t^\alpha(x,y) = |xy|^{-2\alpha} W_t^{-\alpha}(x,y).$$

The kernel $\widetilde{\mathbf{W}}_t^\alpha(x,y)$ is not non-negative, it takes both positive and negative values for any given $\alpha<0$. By (1.4) we see that the integral representation (2.3) provides a pointwise definition of $\widetilde{\mathbf{W}}_t^\alpha f$, $t>0$, for $f \in L^p(dw_\alpha)$, all $1\le p\le\infty$, provided that $\alpha \le -1/2$. Furthermore, for such α the operators $\{\widetilde{\mathbf{W}}_t^\alpha\}$ satisfy the semigroup property on each $L^p(dw_\alpha)$, $1\le p\le\infty$. When $\alpha\in(-1/2,0)$ the so-called pencil phenomenon occurs, that is, for any given $t>0$, the requirement that $\widetilde{\mathbf{W}}_t^\alpha f$ is well defined by (2.3) for all $f\in L^p(dw_\alpha)$ forces the restriction $p>2\alpha+2$. Furthermore, if we in addition require that $\widetilde{\mathbf{W}}_t^\alpha f$ is in $L^p(dw_\alpha)$ then the dual restriction $p<(2\alpha+2)/(2\alpha+1)$ comes into play.

For further reference we note that, see (1.7)–(1.9),

$$\widetilde{W}_t^\alpha(x,y) \le C_\alpha |xy| \widetilde{W}_t^{\alpha+1}(x,y), \qquad x,y\in\mathbb{R}, \quad t>0, \tag{2.4}$$

and here the optimal $C_\alpha = 1$ for $\alpha\le -1/2$ whereas $C_\alpha>1$ whenever $\alpha\in(-1/2,0)$. Moreover, in the range where the classical and exotic settings overlap with no pencil phenomenon interfering, one has the control

$$|xy|\widetilde{W}_t^{\alpha+1}(x,y) \le C_\alpha W_t^\alpha(x,y), \qquad x,y\in\mathbb{R}, \quad t>0, \quad \alpha\in(-1,-1/2], \tag{2.5}$$

with $C_\alpha=1$ and equality for $\alpha=-1/2$. This follows from (1.6).

Remark 2.2. *The connection of the exotic Dunkl Laplacian situation with the exotic Hankel one is the following, cf.* [8]. *The exotic (modified) Hankel transform is*

$$\widetilde{H}_\alpha f(z) = \int_0^\infty f(x)\widetilde{\varphi}_z^\alpha(x)\, dw_\alpha(x), \qquad z>0,$$

and the corresponding exotic Bessel semigroup has the integral kernel $\widetilde{W}_t^\alpha(x,y)$ *restricted to* $x,y>0$.

Remark 2.3. *The case* $\alpha=-1/2$ *corresponds to the trivial multiplicity function. Then, see* (1.1),

$$\psi_z^{-1/2}(x) = \sqrt{\frac{2}{\pi}}\big(\cos(xz)+i\sin(xz)\big) = \sqrt{\frac{2}{\pi}}e^{ixz},$$

$D_{-1/2}$ *is the Fourier transform, and* $\frac{1}{2}\mathbf{W}_t^{-1/2}(x,y)$, *see* (1.2), *is the Gauss-Weierstrass kernel. However, in the exotic situation*

$$\widetilde{\psi}_z^{-1/2}(x) = \sqrt{\frac{2}{\pi}}\,\mathrm{sign}(xz)\big(\sin(xz)+i\cos(xz)\big) = \sqrt{\frac{2}{\pi}}\,\mathrm{sign}(xz) i e^{-ixz}$$

so, in a sense, $\widetilde{D}_{-1/2}$ *arises from the Fourier transform by exchanging the roles of contributions of the cosine and sine transforms. Furthermore,* $\widetilde{\mathbf{W}}_t^{-1/2}(x,y) = \mathrm{sign}(xy)\mathbf{W}_t^{-1/2}(x,y)$, *so* $\frac{1}{2}\widetilde{\mathbf{W}}_t^{-1/2}(x,y)$ *is the Gauss-Weierstrass kernel multiplied by* $\mathrm{sign}(xy)$.

2.2. **Multi-dimensional $[\mathbf{DL}, \mathbb{Z}_2^d]$ situation and the maximal theorem.** The general Dunkl Laplacian context arises by taking a product of one-dimensional classical and exotic Dunkl Laplacian situations in the following way. Let $d\ge1$ and $\alpha\in\mathbb{R}^d$ be a multi-parameter. Consider the product measure in $\mathbb{R}^d$

$$w_\alpha = w_{\alpha_1}\otimes\ldots\otimes w_{\alpha_d}.$$

We assume that $\alpha\in A(\mathcal{E})$ for some fixed $\mathcal{E}\subset\{1,\ldots,d\}$, where the set $A(\mathcal{E})$ is defined by

$$A(\mathcal{E}) = \big\{\alpha\in\mathbb{R}^d : \alpha_i<0 \ \text{ for } \ i\in\mathcal{E} \ \text{ and } \ \alpha_i>-1 \ \text{ for } \ i\in\mathcal{E}^c\big\};$$

here and elsewhere $\mathcal{E}^c$ is the complement of $\mathcal{E}$ in $\{1,\ldots,d\}$. The set $\mathcal{E}$ indicates which axes are exotic, thus the particular $\mathcal{E}=\emptyset$ corresponds to the classical multi-dimensional $[\mathrm{DL},\mathbb{Z}_2^d]$ situation. For $\alpha\in A(\mathcal{E})$ denote its maximal exotic coordinate by

$$m_{\mathcal{E}}(\alpha)=\begin{cases}\max\{\alpha_i : i\in\mathcal{E}\}, & \mathcal{E}\neq\emptyset,\\ -\infty, & \mathcal{E}=\emptyset.\end{cases}$$

For $z\in\mathbb{R}^d$ define

$$\Psi_z^{\alpha,\mathcal{E}}=\bigotimes_{i=1}^{d}\begin{cases}\widetilde{\psi}_{z_i}^{\alpha_i}, & i\in\mathcal{E},\\ \psi_{z_i}^{\alpha_i}, & i\in\mathcal{E}^c.\end{cases}$$

These are eigenfunctions of the d-dimensional Dunkl Laplacian Δ_α with the corresponding eigenvalues $|z|^2$. For suitable f, the generalized Dunkl transform is defined by

$$\mathbb{D}_{\alpha,\mathcal{E}}f(z)=\frac{1}{2^d}\int_{\mathbb{R}^d}f(x)\overline{\Psi_z^{\alpha,\mathcal{E}}(x)}\,dw_\alpha(x),\qquad z\in\mathbb{R}^d.$$

By the corresponding properties of the one-dimensional classical and exotic Dunkl transforms it follows that $\mathbb{D}_{\alpha,\mathcal{E}}$ is an isometry on $L^2(\mathbb{R}^d,dw_\alpha)$ and its inverse is $\mathbb{D}_{\alpha,\mathcal{E}}^{-1}f(z)=\mathbb{D}_{\alpha,\mathcal{E}}f(-z)$.

We consider the self-adjoint extension of Δ_α (acting initially on $C_c^2((\mathbb{R}\setminus\{0\})^d)$) given by

$$\begin{aligned}\Delta_{\alpha,\mathcal{E}}f&=\mathbb{D}_{\alpha,\mathcal{E}}^{-1}\big(|z|^2\mathbb{D}_{\alpha,\mathcal{E}}f(z)\big),\\ \operatorname{Dom}\Delta_{\alpha,\mathcal{E}}&=\big\{f\in L^2(dw_\alpha): |z|^2\mathbb{D}_{\alpha,\mathcal{E}}f(z)\in L^2(dw_\alpha)\big\}.\end{aligned}$$

The semigroup $\mathbb{W}_t^{\alpha,\mathcal{E}}=\exp(-t\Delta_{\alpha,\mathcal{E}})$ has in $L^2(dw_\alpha)$ the integral representation

$$\mathbb{W}_t^{\alpha,\mathcal{E}}f(x)=\frac{1}{2^d}\int_{\mathbb{R}^d}\mathbb{W}_t^{\alpha,\mathcal{E}}(x,y)f(y)\,dw_\alpha(y),\qquad x\in\mathbb{R}^d,\quad t>0,\tag{2.6}$$

where the kernel has the product structure

$$\mathbb{W}_t^{\alpha,\mathcal{E}}(x,y)=\prod_{i\in\mathcal{E}}\widetilde{\mathbf{W}}_t^{\alpha_i}(x_i,y_i)\prod_{i\in\mathcal{E}^c}\mathbf{W}_t^{\alpha_i}(x_i,y_i),\qquad x,y\in\mathbb{R}^d,\quad t>0.$$

This kernel is strictly positive if $\mathcal{E}=\emptyset$ and $\alpha\in[-1/2,\infty)^d$; otherwise it takes both positive and negative values.

When $m_{\mathcal{E}}(\alpha)\le -1/2$ the integral formula (2.6) provides a definition of the semigroup $\mathbb{W}_t^{\alpha,\mathcal{E}}$, $t>0$, on all $L^p(dw_\alpha)$ spaces, $1\le p\le\infty$. On the other hand, in case $m_{\mathcal{E}}(\alpha)>-1/2$ a pencil type phenomenon occurs. For each $t>0$ fixed, $\mathbb{W}_t^{\alpha,\mathcal{E}}$ is defined by (2.6) in $L^p(dw_\alpha)$ and maps this space into itself if and only if p satisfies

$$2m_{\mathcal{E}}(\alpha)+2<p<1+\frac{1}{2m_{\mathcal{E}}(\alpha)+1}.$$

Now bring in the maximal operator

$$\mathbb{W}_*^{\alpha,\mathcal{E}}f=\sup_{t>0}\big|\mathbb{W}_t^{\alpha,\mathcal{E}}f\big|.$$

Our objective is to study L^p mapping properties of $\mathbb{W}_*^{\alpha,\mathcal{E}}$, especially the weak type $(1,1)$ estimate (the latter makes sense only when $m_{\mathcal{E}}(\alpha)\le -1/2$). The more subtle and much less standard case $m_{\mathcal{E}}(\alpha)> -1/2$ requires qualitatively different analysis that is beyond the scope of this paper; cf. [14, 15].

Our main result concerning the $[\mathrm{DL},\mathbb{Z}_2^d]$ context reads as follows.

Theorem 2.4. *Let $d\ge 1$ and $\alpha\in A(\mathcal{E})$ for some $\mathcal{E}\subset\{1,\ldots,d\}$. Assume that $m_{\mathcal{E}}(\alpha)\le -1/2$. Then $\mathbb{W}_*^{\alpha,\mathcal{E}}$ is bounded on $L^p(\mathbb{R}^d,dw_\alpha)$ for $1<p\le\infty$, and satisfies the weak type $(1,1)$ estimate*

$$w_\alpha\big\{x\in\mathbb{R}^d : \mathbb{W}_*^{\alpha,\mathcal{E}}f(x)>\lambda\big\}\le\frac{C}{\lambda}\int_{\mathbb{R}^d}|f(x)|\,dw_\alpha(x)$$

with a constant C independent of $\lambda>0$ and $f\in L^1(\mathbb{R}^d,dw_\alpha)$.

In the classical multi-dimensional Dunkl Laplacian setting (the case of $\mathcal{E} = \emptyset$) this result is already known, as well as its weighted generalization, see e.g. [16, Theorem 2.1] and also [17].

Let us introduce an auxiliary maximal operator, acting on functions on $\mathbb{R}^d_+$,

$$\mathbb{V}^{\alpha,\mathcal{E}}_* \colon f(x) \mapsto \sup_{t>0} \int_{\mathbb{R}^d_+} \mathbb{V}^{\alpha,\mathcal{E}}_t(x,y)|f(y)|\, dw_\alpha(y),$$

with the kernel

$$\mathbb{V}^{\alpha,\mathcal{E}}_t(x,y) = \prod_{i\in\mathcal{E}} x_i y_i \widetilde{W}^{\alpha_i+1}_t(x_i,y_i) \prod_{i\in\mathcal{E}^c} W^{\alpha_i}_t(x_i,y_i), \qquad x,y\in\mathbb{R}^d_+, \quad t>0.$$

In view of (2.2), (2.4) and the symmetries involved, it is clear that the following result implies Theorem 2.4.

Lemma 2.5. *Let d, α and $\mathcal{E}$ be as in Theorem 2.4. Then $\mathbb{V}^{\alpha,\mathcal{E}}_*$ is bounded on $L^p(\mathbb{R}^d_+, dw_\alpha)$ for $1 < p \le \infty$ and from $L^1(\mathbb{R}^d_+, dw_\alpha)$ to weak $L^1(\mathbb{R}^d_+, dw_\alpha)$.*

Proving Lemma 2.5 we first make several reductions. Observe that we can assume that $f \ge 0$, since the kernel $\mathbb{V}^{\alpha,\mathcal{E}}_t(x,y)$ is positive. Further, we can consider only α satisfying $m_{\mathcal{E}}(\alpha) \le -1$, because of the majorization (2.5); then $\mathcal{E}$ can be dropped from the notation. Finally, in view of the known results in the non-exotic situation we let $\mathcal{E} \neq \emptyset$ and furthermore assume, for symmetry reasons, that $\mathcal{E} = \{1,\ldots,d'\}$ for some $1 \le d' \le d$. Then $\alpha = (\alpha',\alpha'') \in (-\infty,-1]^{d'} \times (-1,\infty)^{d-d'}$ and our kernel can be written as

$$\mathbb{V}^{\alpha}_t(x,y) = (x'y')^{-2\alpha'-\mathbf{1}'} W^{-\alpha'-\mathbf{1}'}_t(x',y') W^{\alpha''}_t(x'',y''), \qquad x,y\in\mathbb{R}^d_+, \quad t>0,$$

where for $z\in\mathbb{R}^d$ we denote $z' = (z_1,\ldots,z_{d'}) \in \mathbb{R}^{d'}$ and $z'' = (z_{d'+1},\ldots,z_d) \in \mathbb{R}^{d-d'}$, and $W^{-\alpha'-\mathbf{1}'}_t(x',y')$ and $W^{\alpha''}_t(x'',y'')$ are d'- and d''-dimensional classical Bessel heat kernels that are simply products of one-dimensional Bessel heat kernels. Notice that the double prime part may be void.

We will prove that the maximal operator $\mathbb{V}^\alpha_*$ is bounded on $L^\infty(\mathbb{R}^d_+)$ and from $L^1(\mathbb{R}^d_+, dw_\alpha)$ to weak $L^1(\mathbb{R}^d_+, dw_\alpha)$. Then the rest of Lemma 2.5 will follow by interpolation.

Considering the L^∞-boundedness, since the Bessel semigroup is conservative, it follows immediately from the one-dimensional lemma below.

Lemma 2.6. *Let $\beta \le -1/2$. Then*

$$\int_0^\infty (xy)^{-2\beta-1} W^{-\beta-1}_t(x,y)\, dw_\beta(y) \le C < \infty$$

with C independent of $x>0$ and $t>0$.

Proof. The integral in question, denote it by $\mathcal{I}$, can be expressed in terms of a function $H_{\eta,\gamma}$ introduced in [13, p. 440]. Namely, by [13, Lemma 2.2]

$$\mathcal{I} = H_{1/2,-\beta}\Big(\frac{x^2}{4t}\Big).$$

Since [13, Lemma 2.3] implies $\|H_{1/2,-\beta}\|_{L^\infty(\mathbb{R}_+)} < \infty$, the conclusion follows. □

The proof of the weak type $(1,1)$ of $\mathbb{V}^\alpha_*$, in view of the product structure of $\mathbb{V}^\alpha_t(x,y)$ and weak type $(1,1)$ of the Bessel heat maximal operator, reduces to showing the following two lemmas.

Lemma 2.7. *For any $d\ge 1$, $1 \le d' \le d$ and each $\alpha \in (-\infty,-1]^{d'} \times (-1,\infty)^{d-d'}$ the maximal operator*

$$f(x) \mapsto \sup_{t>0} \int_{\mathbb{R}^d_+} \chi_{\{x_i/2 \le y_i \le 2x_i \text{ for } i=1,\ldots,d'\}} \mathbb{V}^\alpha_t(x,y) f(y)\, dw_\alpha(y), \qquad f \ge 0,$$

is weak type $(1,1)$ with respect to dw_α.

Proof. We follow the proof of [8, Lemma 3.3], see also the proof of [8, Lemma 4.2]. According to the notation used there, the operator we need to estimate is

$$\begin{aligned} f(x) &\mapsto \sup_{t>0} \int_{y'\sim x'} (x'y')^{-2\alpha'-\mathbf{1}'} W^{-\alpha'-\mathbf{1}'}_t(x',y') W^{\alpha''}_t(x'',y'') f(y)\, dw_\alpha(y) \\ &\simeq \sup_{t>0} \int_{y'\sim x'} W^{\breve{\alpha}}_t(x,y) f(y)\, dw_{\breve{\alpha}}(y), \end{aligned}$$

where $\breve{\alpha} = (-\alpha' - \mathbf{1}', \alpha'') \in (-1,\infty)^d$. From here the argument is exactly as in [8], using weak type $(1,1)$ of the Bessel heat maximal operator. We leave the details to the reader. $\square$

Lemma 2.8. *For each $\beta \le -1$ the one-dimensional operator*

$$N^\beta f(x) = \int_0^\infty \chi_{\{y<x/2 \text{ or } y>2x\}} \Big[(xy)^{-2\beta-1} \sup_{t>0} W_t^{-\beta-1}(x,y) \Big] f(y)\, dw_\alpha(y), \qquad f \ge 0,$$

is bounded on $L^1(\mathbb{R}_+, dw_\beta)$.

Proof. We proceed as in the proof of [8, Lemma 4.3]. The kernel of N^β is comparable with

$$\begin{aligned} N(x,y) &= \chi_{\{y<x/2 \text{ or } y>2x\}} (xy)^{-2\beta-1} \sup_{t>0} (xy+t)^{\beta+1/2} \frac{1}{\sqrt{t}} \exp\Big(- \frac{(x-y)^2}{4t} \Big) \\ &\lesssim (xy)^{-2\beta-1} (x \vee y)^{2\beta}. \end{aligned}$$

Then it suffices to verify that $\int_0^\infty (xy)^{-2\beta-1}(x\vee y)^{2\beta}\, dw_\beta(x) \lesssim 1$, which is straightforward. $\square$

Now Lemma 2.5, hence also Theorem 2.4, are proved.

3. The Dunkl harmonic oscillator context

This section is devoted to the Dunkl harmonic oscillator framework. In contrast with the previous setting, here we deal with discrete orthogonal expansions rather than continuous ones (in terms of the classical and exotic Dunkl transforms). As we shall see, the maximal theorem in this context can be concluded without much effort from the analysis in Section 2.

3.1. **Classical and exotic $[\mathbf{DHO}, \mathbb{Z}_2^d]$ contexts in dimension one.** We consider the one-dimensional Dunkl harmonic oscillator $L_\alpha = \Delta_\alpha + x^2$ with all $\alpha \in \mathbb{R}$ admitted. For any α, this operator is formally symmetric in $L^2(\mathbb{R}, dw_\alpha)$.

The classical Dunkl harmonic oscillator context occurs when $\alpha > -1$. Then there is a natural self-adjoint extension of L_α (acting initially on $C_c^2(\mathbb{R} \setminus \{0\})$), denote it by L_α^{cls}, given spectrally in terms of eigenfunction expansions. To make it more precise, let $\{h_n^\alpha : n \in \mathbb{N}\}$ be the system of generalized Hermite functions, which is an orthonormal basis in $L^2(dw_\alpha)$ of eigenfunctions of the Dunkl harmonic oscillator, $L_\alpha h_n^\alpha = (2n+2\alpha+2)h_n^\alpha$. Then the self-adjoint operator L_α^{cls} is defined by

$$L_\alpha^{\text{cls}} f = \sum_{n=0}^\infty (2n+2\alpha+2) \langle f, h_n^\alpha \rangle_{dw_\alpha} h_n^\alpha$$

on the domain $\operatorname{Dom} L_\alpha^{\text{cls}}$ consisting of all $f \in L^2(dw_\alpha)$ for which the above series converges in the L^2 sense.

The generalized Hermite functions h_n^α can be represented in terms of Laguerre functions of convolution type

$$\ell_k^\alpha(x) = c_k^\alpha e^{-x^2/2} L_k^\alpha(x^2), \qquad k \ge 0,$$

where L_k^α are the classical Laguerre polynomials and $c_k^\alpha > 0$ are normalizing constants so that $\{\ell_k^\alpha : k \in \mathbb{N}\}$ is an orthonormal basis in $L^2(\mathbb{R}_+, dw_\alpha)$. We have

$$h_{2k}^\alpha(x) = (-1)^k 2^{-1/2} \ell_k^\alpha(x), \qquad h_{2k+1}^\alpha(x) = (-1)^k 2^{-1/2} x \ell_k^{\alpha+1}(x),$$

where $k \ge 0$ and $x \in \mathbb{R}$.

The semigroup $\mathbf{G}_t^\alpha = \exp(-tL_\alpha^{\text{cls}})$ has in $L^2(dw_\alpha)$ the integral representation

$$\mathbf{G}_t^\alpha f(x) = \frac{1}{2} \int_{\mathbb{R}} \mathbf{G}_t^\alpha(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}, \quad t > 0, \tag{3.1}$$

with the kernel given by

$$\mathbf{G}_t^\alpha(x,y) = G_t^\alpha(x,y) + xy G_t^{\alpha+1}(x,y),$$

and here

$$G_t^\alpha(x,y) = \frac{1}{(\sinh 2t)^{\alpha+1}} \exp\Big(- \frac{\cosh 2t}{2 \sinh 2t}(x^2+y^2) \Big) \Big(\frac{xy}{\sinh 2t} \Big)^{-\alpha} I_\alpha\Big(\frac{xy}{\sinh 2t} \Big).$$

The latter is an even function of both x and y that is jointly smooth in $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+$.

Remark 3.1. *The Dunkl harmonic oscillator context is deeply connected with the setting of Laguerre function expansions of convolution type. In particular, expansions of even functions with respect to the system* $\{h_n^\alpha\}$ *reduce to expansions with respect to* $\{\ell_n^\alpha\}$ *on* $\mathbb{R}_+$*. Furthermore,* $G_t^\alpha(x,y)$ *restricted to* $x,y>0$ *is exactly the heat kernel in the Laguerre situation; see e.g.* [18].

Observe that the kernels $W_t^\alpha(x,y)$ and $G_t^\alpha(x,y)$ are very similar. Excluding the exponential factors (which do not depend on α), the latter one is just the first one with $2t$ replaced by $\sinh 2t$. Thus an analogue of (2.2) holds,

$$|xy|G_t^{\alpha+1}(x,y) \le C_\alpha G_t^\alpha(x,y), \qquad x,y\in\mathbb{R}, \quad t>0, \tag{3.2}$$

with the optimal $C_\alpha = 1$ in case $\alpha \ge -1/2$ and $C_\alpha > 1$ when $-1<\alpha<-1/2$. Moreover, it is well known that the Bessel heat kernel controls pointwise the Laguerre one,

$$G_t^\alpha(x,y) < W_t^\alpha(x,y), \qquad x,y,t>0, \quad \alpha>-1. \tag{3.3}$$

This is clear at least at a formal level, since the Laguerre and Bessel Laplacians differ only by the (positive) harmonic confinement x^2. A direct verification of (3.3) is straightforward by means of (1.5) and the elementary bounds $\sinh 2t > 2t$ and $\coth 2t > 1/(2t)$ for $t>0$. The estimate (3.3) is crucial for our purposes since it makes it possible to reduce analysis of the heat maximal operator in the Dunkl harmonic oscillator context to what we have already proved in the Dunkl Laplacian situation.

The kernel $\mathbf{G}_t^\alpha(x,y)$ has similar properties to $\mathbf{W}_t^\alpha(x,y)$, which are justified essentially in the same way. In particular, it is strictly positive for $\alpha \ge -1/2$, but it takes both positive and negative values when $-1<\alpha<-1/2$; in the latter case $f\ge 0$ does not imply $\mathbf{G}_t^\alpha f \ge 0$. Further, the integral representation (3.1) provides a pointwise definition of $\mathbf{G}_t^\alpha f$, $t>0$, for all $f\in L^p(dw_\alpha)$, $1\le p\le\infty$. In view of (3.3), $\mathbf{G}_t^\alpha \mathbb{1}(x) < 1$, $x\in\mathbb{R}$, $t>0$. Accordingly, $\{\mathbf{G}_t^\alpha\}$ is a submarkovian (but not Markovian) symmetric diffusion semigroup when $\alpha\ge -1/2$. On the other hand, for $-1<\alpha<-1/2$ the operators $\mathbf{G}_t^\alpha$ are not contractive on L^∞ when $t>0$ is sufficiently small (the relevant counterexample is the odd function $f(y) = \chi_{(0,\infty)}(y) - \chi_{(-\infty,0)}(y)$).

Next, we consider the exotic situation. To this end, we assume that $\alpha<0$ and bring in the 'reflected' system $\{\widetilde{h}_n^\alpha : n\in\mathbb{N}\}$,

$$\widetilde{h}_{2k}^\alpha(x) = (-1)^k 2^{-1/2}\widetilde{\ell}_k^\alpha(x), \qquad \widetilde{h}_{2k+1}^\alpha(x) = (-1)^k 2^{-1/2} x\widetilde{\ell}_k^{\alpha+1}(x),$$

where $k\ge 0$, $x\in\mathbb{R}$, and

$$\widetilde{\ell}_k^\alpha(x) = |x|^{-2\alpha}\ell_k^{-\alpha}(x).$$

One verifies that $\{\widetilde{h}_n^\alpha\}$ is an orthonormal basis in $L^2(dw_\alpha)$ and, moreover, these are eigenfunctions of the Dunkl harmonic oscillator,

$$L_\alpha \widetilde{h}_n^\alpha = \big(2n - 2\alpha + (-1)^n 2\big)\widetilde{h}_n^\alpha, \qquad n\ge 0.$$

This leads to the following self-adjoint extension of L_α (acting initially on $C_c^2(\mathbb{R}\setminus\{0\})$):

$$L_\alpha^{\text{exo}} f = \sum_{n=0}^\infty \big(2n-2\alpha+(-1)^n 2\big)\langle f, \widetilde{h}_n^\alpha\rangle_{dw_\alpha}\widetilde{h}_n^\alpha,$$

$\operatorname{Dom} L_\alpha^{\text{exo}}$ consisting of all $f\in L^2(dw_\alpha)$ for which this series converges in $L^2(dw_\alpha)$.

The exotic semigroup $\widetilde{\mathbf{G}}_t^\alpha = \exp(-tL_\alpha^{\text{exo}})$ has in $L^2(dw_\alpha)$ the integral representation

$$\widetilde{\mathbf{G}}_t^\alpha f(x) = \frac{1}{2}\int_{\mathbb{R}} \widetilde{\mathbf{G}}_t^\alpha(x,y) f(y)\, dw_\alpha(y), \qquad x\in\mathbb{R}, \quad t>0, \tag{3.4}$$

the kernel being

$$\widetilde{\mathbf{G}}_t^\alpha(x,y) = \widetilde{G}_t^\alpha(x,y) + xy\widetilde{G}_t^{\alpha+1}(x,y),$$

with

$$\widetilde{G}_t^\alpha(x,y) = |xy|^{-2\alpha} G_t^{-\alpha}(x,y).$$

One can check that the kernel $\widetilde{G}_t^\alpha(x,y)$ here is actually (after restricting to $x,y>0$) the exotic Laguerre heat kernel in the context of expansions in Laguerre functions of convolution type, cf. [8, 18].

All the comments preceding Remark 2.2 carry over into the present situation. Thus the kernel $\widetilde{\mathbf{G}}_t^\alpha(x,y)$ takes both positive and negative values for any given $\alpha<0$. The integral representation (3.4) provides a pointwise definition of $\widetilde{\mathbf{G}}_t^\alpha f$, $t>0$, for $f\in L^p(dw_\alpha)$, all $1\le p\le\infty$, provided that $\alpha\le -1/2$. Further, for such α the operators $\{\widetilde{\mathbf{G}}_t^\alpha\}$ satisfy the semigroup property on each $L^p(dw_\alpha)$, $1 \le p \le \infty$. When $\alpha \in (-1/2, 0)$ the pencil phenomenon occurs leading to the restrictions $2\alpha + 2 < p < (2\alpha+2)/(2\alpha+1)$. Moreover,

$$\widetilde{G}_t^\alpha(x,y) \le C_\alpha |xy| \widetilde{G}_t^{\alpha+1}(x,y), \qquad x,y \in \mathbb{R}, \quad t>0, \tag{3.5}$$

and here the optimal $C_\alpha = 1$ for $\alpha \le -1/2$ whereas $C_\alpha > 1$ whenever $\alpha \in (-1/2, 0)$. One also has

$$|xy| \widetilde{G}_t^{\alpha+1}(x,y) \le C_\alpha G_t^\alpha(x,y), \qquad x,y \in \mathbb{R}, \quad t>0, \quad \alpha \in (-1,-1/2],$$

with $C_\alpha = 1$ and equality for $\alpha = -1/2$.

Remark 3.2. *The case $\alpha = -1/2$ corresponds to the trivial multiplicity function and so $L_{-1/2} = -\Delta + x^2$ is the classic harmonic oscillator. Then in the non-exotic situation $\{h_n^{-1/2}\}$ is the system of classical Hermite functions satisfying $L_{-1/2} h_{2k}^{-1/2} = (4k+1) h_{2k}^{-1/2}$, $L_{-1/2} h_{2k+1}^{-1/2} = (4k+3) h_{2k+1}^{-1/2}$, $k \ge 0$. On the other hand, in the exotic situation*

$$\widetilde{h}_{2k}^{-1/2}(x) = \operatorname{sign}(x) h_{2k+1}^{-1/2}(x), \qquad \widetilde{h}_{2k+1}^{-1/2}(x) = \operatorname{sign}(x) h_{2k}^{-1/2}(x),$$

and $L_{-1/2} \widetilde{h}_{2k}^{-1/2} = (4k+3) \widetilde{h}_{2k}^{-1/2}$, $L_{-1/2} \widetilde{h}_{2k+1}^{-1/2} = (4k+1) \widetilde{h}_{2k+1}^{-1/2}$, $k \ge 0$. Furthermore, the relation between the heat kernels is $\widetilde{\mathbf{G}}_t^{-1/2}(x,y) = \operatorname{sign}(xy) \mathbf{G}_t^{-1/2}(x,y)$.

3.2. **Multi-dimensional $[\mathbf{DHO}, \mathbb{Z}_2^d]$ situation and the maximal theorem.** As in Section 2.2, we now let $d \ge 1$, $\alpha \in \mathbb{R}^d$ and consider the product measure w_α in $\mathbb{R}^d$. We assume that $\alpha \in A(\mathcal{E})$ for some fixed $\mathcal{E} \subset \{1, \ldots, d\}$.

For $n \in \mathbb{N}^d$ define

$$\mathfrak{h}_n^{\alpha,\mathcal{E}} = \bigotimes_{i=1}^d \begin{cases} \widetilde{h}_{n_i}^{\alpha_i}, & i \in \mathcal{E}, \\ h_{n_i}^{\alpha_i}, & i \in \mathcal{E}^c. \end{cases}$$

The system $\{\mathfrak{h}_n^\alpha : n \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d, dw_\alpha)$ consisting of eigenfunctions of the d-dimensional Dunkl harmonic oscillator $L_\alpha = \Delta_\alpha + |x|^2$. One has $L_\alpha \mathfrak{h}_n^{\alpha,\mathcal{E}} = \lambda_n^{\alpha,\mathcal{E}} \mathfrak{h}_n^{\alpha,\mathcal{E}}$, where

$$\lambda_n^{\alpha,\mathcal{E}} = 2|n| + \sum_{i \in \mathcal{E}} \big(-2\alpha_i + (-1)^{n_i} 2 \big) + \sum_{i \in \mathcal{E}^c} (2\alpha_i + 2).$$

We consider the self-adjoint extension of L_α (acting initially on $C_c^2((\mathbb{R} \setminus \{0\})^d)$) defined by

$$L_{\alpha,\mathcal{E}} f = \sum_{n \in \mathbb{N}^d} \lambda_n^{\alpha,\mathcal{E}} \langle f, \mathfrak{h}_n^{\alpha,\mathcal{E}} \rangle_{dw_\alpha} \mathfrak{h}_n^{\alpha,\mathcal{E}}$$

on the domain $\operatorname{Dom} L_{\alpha,\mathcal{E}}$ consisting of all $f \in L^2(dw_\alpha)$ for which the above series converges in the L^2 sense.

The semigroup $\mathbb{G}_t^{\alpha,\mathcal{E}} = \exp(-t L_{\alpha,\mathcal{E}})$ has in $L^2(dw_\alpha)$ the integral representation

$$\mathbb{G}_t^{\alpha,\mathcal{E}} f(x) = \frac{1}{2^d} \int_{\mathbb{R}^d} \mathbb{G}_t^{\alpha,\mathcal{E}}(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}^d, \quad t > 0, \tag{3.6}$$

where the kernel has the product structure

$$\mathbb{G}_t^{\alpha,\mathcal{E}}(x,y) = \prod_{i \in \mathcal{E}} \widetilde{\mathbf{G}}_t^{\alpha_i}(x_i, y_i) \prod_{i \in \mathcal{E}^c} \mathbf{G}_t^{\alpha_i}(x_i, y_i), \qquad x,y \in \mathbb{R}^d, \quad t>0.$$

This kernel is strictly positive if $\mathcal{E} = \emptyset$ and $\alpha \in [-1/2, \infty)^d$; otherwise it takes both positive and negative values. When $m_{\mathcal{E}}(\alpha) \le -1/2$ the integral formula (3.6) provides a pointwise definition of the semigroup $\mathbb{G}_t^{\alpha,\mathcal{E}}$, $t>0$, on all $L^p(dw_\alpha)$ spaces, $1 \le p \le \infty$. On the other hand, in case $m_{\mathcal{E}}(\alpha) > -1/2$ a pencil type phenomenon occurs leading to the restriction

$$2 m_{\mathcal{E}}(\alpha) + 2 < p < 1 + \frac{1}{2 m_{\mathcal{E}}(\alpha) + 1}.$$

We have the following result for the maximal operator

$$\mathbb{G}_*^{\alpha,\mathcal{E}} f = \sup_{t>0} \big| \mathbb{G}_t^{\alpha,\mathcal{E}} f \big|.$$

Theorem 3.3. *Let $d \geq 1$ and $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1, \dots, d\}$. Assume that $m_{\mathcal{E}}(\alpha) \leq -1/2$. Then $\mathbb{G}_*^{\alpha,\mathcal{E}}$ is bounded on $L^p(\mathbb{R}^d, dw_\alpha)$ for $1 < p \leq \infty$, and from $L^1(\mathbb{R}^d, dw_\alpha)$ to weak $L^1(\mathbb{R}^d, dw_\alpha)$.*

Proof. In view of (3.2), (3.5), the symmetries involved and the crucial majorization (3.3), the result follows from Lemma 2.5. □

In the classical multi-dimensional Dunkl harmonic oscillator setting (i.e. when $\mathcal{E} = \emptyset$) the result of Theorem 3.3 and its weighted extension are known, see [18, Theorem 3.1] and also [19, Theorem 1.9(a)].

4. The Dunkl Ornstein-Uhlenbeck operator context

The last framework we consider is that of the Dunkl Ornstein-Uhlenbeck operator U_α. Here we are more brief comparing to the previous two sections, since the reader is already familiar with the whole procedure. In the last subsection we treat also an alternative Dunkl Ornstein-Uhlenbeck setting related to the variant of U_α defined via the Dunkl gradient ∇_α. The corresponding maximal theorem concerns a restricted maximal operator only, but this also suffices to infer almost everywhere convergence of the semigroup to L^p initial data for $1 \leq p \leq \infty$.

4.1. **Classical and exotic $[\mathbf{DOU}, \mathbb{Z}_2^d]$ contexts in dimension one.** Consider $U_\alpha = \Delta_\alpha + 2x\frac{d}{dx}$ with $\alpha \in \mathbb{R}$. This operator is formally symmetric in $L^2(\mathbb{R}, d\gamma_\alpha)$, where

$$d\gamma_\alpha(x) = |x|^{2\alpha+1} e^{-x^2}\, dx.$$

The classical Dunkl Ornstein-Uhlenbeck context occurs when $\alpha > -1$. Then the generalized Hermite polynomials

$$H_n^\alpha(x) = e^{x^2/2} h_n^\alpha(x), \qquad n \in \mathbb{N},$$

form an orthonormal basis in $L^2(d\gamma_\alpha)$ of eigenfunctions of U_α, being $U_\alpha H_n^\alpha = 2nH_n^\alpha$. This leads to a natural in this situation self-adjoint extension U_α^{cls} of the difference-differential operator U_α.

The semigroup $\mathbf{B}_t^\alpha = \exp(-tU_\alpha^{\mathrm{cls}})$ has an integral representation

$$\mathbf{B}_t^\alpha f(x) = \frac{1}{2}\int_{\mathbb{R}} \mathbf{B}_t^\alpha(x,y) f(y)\, d\gamma_\alpha(y), \qquad x \in \mathbb{R}, \quad t > 0, \tag{4.1}$$

with

$$\mathbf{B}_t^\alpha(x,y) = B_t^\alpha(x,y) + e^{-2t} xy B_t^{\alpha+1}(x,y),$$

where

$$B_t^\alpha(x,y) = e^{2(\alpha+1)t} e^{(x^2+y^2)/2} G_t^\alpha(x,y).$$

The kernel $\mathbf{B}_t^\alpha(x,y)$ and the integral representation (4.1) have analogous properties as their counterparts in Section 2.1. In particular, $\{\mathbf{B}_t^\alpha\}$ is a Markovian symmetric diffusion semigroup for $\alpha \geq -1/2$. We also have

$$e^{-2t}|xy| B_t^{\alpha+1}(x,y) \leq C_\alpha B_t^\alpha(x,y), \qquad x, y \in \mathbb{R}, \quad t > 0, \tag{4.2}$$

with the optimal $C_\alpha = 1$ when $\alpha \geq -1/2$ and $C_\alpha > 1$ for $\alpha \in (-1, -1/2)$.

Passing to the exotic situation, we now let $\alpha < 0$ and consider the 'reflected' system

$$\widetilde{H}_n^\alpha(x) = e^{x^2/2} \widetilde{h}_n^\alpha(x), \qquad n \in \mathbb{N},$$

which forms an orthonormal basis in $L^2(d\gamma_\alpha)$. Moreover,

$$U_\alpha \widetilde{H}_n^\alpha = \big(2n - 4\alpha - 2 + (-1)^n 2\big)\widetilde{H}_n^\alpha, \qquad n \in \mathbb{N}.$$

Thus the exotic self-adjoint extension U_α^{exo} of U_α is defined in the natural way.

The exotic semigroup $\widetilde{\mathbf{B}}_t^\alpha = \exp(-tU_\alpha^{\mathrm{exo}})$ has the integral representation

$$\widetilde{\mathbf{B}}_t^\alpha f(x) = \frac{1}{2}\int_{\mathbb{R}} \widetilde{\mathbf{B}}_t^\alpha(x,y) f(y)\, d\gamma_\alpha(y), \qquad x \in \mathbb{R}, \quad t > 0, \tag{4.3}$$

with

$$\widetilde{\mathbf{B}}_t^\alpha(x,y) = \widetilde{B}_t^\alpha(x,y) + e^{-2t} xy \widetilde{B}_t^{\alpha+1}(x,y),$$

being

$$\widetilde{B}_t^\alpha(x,y) = e^{4\alpha t}|xy|^{-2\alpha} B_t^{-\alpha}(x,y).$$

Again, the kernel $\widetilde{\mathbf{B}}_t^\alpha(x,y)$ and the representation (4.3) have analogous properties as their counterparts from Section 2.1, including the pencil phenomenon. Further, we have the bound

$$\widetilde{B}_t^\alpha(x,y) \le C_\alpha e^{-2t}|xy|\widetilde{B}_t^{\alpha+1}(x,y), \qquad x,y\in\mathbb{R}, \quad t>0, \tag{4.4}$$

where the optimal $C_\alpha = 1$ for $\alpha \le -1/2$ and $C_\alpha > 1$ if $\alpha \in (-1/2,0)$. Furthermore, one also has

$$e^{-2t}|xy|\widetilde{B}_t^{\alpha+1}(x,y) \le C_\alpha B_t^\alpha(x,y), \qquad x,y\in\mathbb{R}, \quad t>0, \quad \alpha\in(-1,-1/2], \tag{4.5}$$

with $C_\alpha = 1$ and equality for $\alpha = -1/2$.

4.2. **Multi-dimensional [DOU, $\mathbb{Z}_2^d$] situation and the maximal theorem.** Let $d \ge 1$, $\alpha \in \mathbb{R}^d$ and consider the product measure

$$\gamma_\alpha = \gamma_{\alpha_1} \otimes \ldots \otimes \gamma_{\alpha_d}$$

in $\mathbb{R}^d$. Assume that $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1,\ldots,d\}$.

For $n \in \mathbb{N}^d$ define

$$\mathfrak{H}_n^{\alpha,\mathcal{E}} = \bigotimes_{i=1}^d \begin{cases} \widetilde{H}_{n_i}^{\alpha_i}, & i\in\mathcal{E}, \\ H_{n_i}^{\alpha_i}, & i\in\mathcal{E}^c. \end{cases}$$

The system $\{\mathfrak{H}_n^{\alpha,\mathcal{E}} : n\in\mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d, d\gamma_\alpha)$ of eigenfunctions of $U_\alpha = \Delta_\alpha + 2x\cdot\nabla$. We consider the natural in this situation self-adjoint extension $U_{\alpha,\mathcal{E}}$ of U_α acting initially on $C_c^2((\mathbb{R}\backslash\{0\})^d)$.

The semigroup $\mathbb{B}_t^{\alpha,\mathcal{E}} = \exp(-tU_{\alpha,\mathcal{E}})$ has the integral representation

$$\mathbb{B}_t^{\alpha,\mathcal{E}} f(x) = \frac{1}{2^d}\int_{\mathbb{R}^d} \mathbb{B}_t^{\alpha,\mathcal{E}}(x,y) f(y)\, d\gamma_\alpha(y), \qquad x\in\mathbb{R}^d, \quad t>0,$$

where the kernel is

$$\mathbb{B}_t^{\alpha,\mathcal{E}}(x,y) = \prod_{i\in\mathcal{E}} \widetilde{\mathbf{B}}_t^{\alpha_i}(x_i,y_i) \prod_{i\in\mathcal{E}^c} \mathbf{B}_t^{\alpha_i}(x_i,y_i), \qquad x,y\in\mathbb{R}^d, \quad t>0.$$

Basic properties of the representation and the kernel are completely analogous to those for their counterparts in Section 2.2.

For the maximal operator

$$\mathbb{B}_*^{\alpha,\mathcal{E}} f = \sup_{t>0} \big|\mathbb{B}_t^{\alpha,\mathcal{E}} f\big|$$

we prove the following result, which in the non-exotic case $\mathcal{E} = \emptyset$ is already known, see [5, 19, 20].

Theorem 4.1. *Let $d \ge 1$ and $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1,\ldots,d\}$. Assume that $m_{\mathcal{E}}(\alpha) \le -1/2$. Then $\mathbb{B}_*^{\alpha,\mathcal{E}}$ is bounded on $L^p(\mathbb{R}^d, d\gamma_\alpha)$ for $1 < p \le \infty$, and satisfies the weak type $(1,1)$ estimate*

$$\gamma_\alpha\big\{x\in\mathbb{R}^d : \mathbb{B}_*^{\alpha,\mathcal{E}} f(x) > \lambda\big\} \le \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(x)|\, d\gamma_\alpha(x)$$

with a constant C independent of $\lambda > 0$ and $f \in L^1(\mathbb{R}^d, d\gamma_\alpha)$.

The proof reduces to analysis of an auxiliary maximal operator, acting on functions on $\mathbb{R}_+^d$,

$$\mathbb{S}_t^{\alpha,\mathcal{E}} \colon f(x) \mapsto \sup_{t>0} \int_{\mathbb{R}_+^d} \mathbb{S}_t^{\alpha,\mathcal{E}}(x,y) f(y)\, d\gamma_\alpha(y),$$

where the kernel is given by

$$\mathbb{S}_t^{\alpha,\mathcal{E}}(x,y) = \prod_{i\in\mathcal{E}} e^{-2t} x_i y_i \widetilde{B}_t^{\alpha_i+1}(x_i,y_i) \prod_{i\in\mathcal{E}^c} B_t^{\alpha_i}(x_i,y_i), \qquad x,y\in\mathbb{R}_+^d, \quad t>0.$$

Taking into account (4.2), (4.4) and the symmetries involved, we see that in order to prove Theorem 4.1 it suffices to show the following.

Lemma 4.2. *Let $d, \alpha, \mathcal{E}$ be as in Theorem 4.1. Then $\mathbb{S}_*^{\alpha,\mathcal{E}}$ is bounded on $L^p(\mathbb{R}_+^d, d\gamma_\alpha)$ for $1 < p \le \infty$ and from $L^1(\mathbb{R}_+^d, d\gamma_\alpha)$ to weak $L^1(\mathbb{R}_+^d, d\gamma_\alpha)$.*

When $\mathcal{E} = \emptyset$ this result is well-known since up to a simple change of variables $x_i \mapsto x_i^2$ and $y_i \mapsto y_i^2$ it corresponds to analogous mapping properties of the maximal operator of the Laguerre polynomial semigroup; see [8] for details and further references.

Proving Lemma 4.2 we make analogous reductions to those introduced to prove Lemma 2.5 in Section 2.2; we also use the same notation. Thus we consider $f \ge 0$, may assume $m_{\mathcal{E}}(\alpha) \le -1$ (see (4.5)) and $\mathcal{E} \neq \emptyset$, $\mathcal{E} = \{1, \ldots, d'\}$ for some $1 \le d' \le d$; we also drop $\mathcal{E}$ from the notation. The kernel then can be assumed to have the form

$$\mathbb{S}_t^{\alpha}(x,y) = e^{4\langle \alpha' \rangle t + 2d't} (x'y')^{-2\alpha' - \mathbf{1}'} B_t^{-\alpha' - \mathbf{1}'}(x', y') B_t^{\alpha''}(x'', y''), \qquad x, y \in \mathbb{R}^d_+, \quad t > 0,$$

where $B_t^{-\alpha'-\mathbf{1}'}(x', y')$ and $B_t^{\alpha''}(x'', y'')$ are d'- and d''-dimensional kernels that are simply products of one-dimensional Laguerre-type kernels $B_t^{\beta_i}(x_i, y_i)$.

In view of interpolation, it is enough to prove that $\mathbb{S}_*^{\alpha}$ is bounded from $L^1(\mathbb{R}^d_+, d\gamma_\alpha)$ to weak $L^1(\mathbb{R}^d_+, d\gamma_\alpha)$ and bounded on $L^\infty(\mathbb{R}^d_+)$. As for the latter, since the d''-dimensional semigroup defined via $B_t^{\alpha''}(x'', y'')$ is conservative, the L^∞-boundedness follows readily from the following one-dimensional result.

Lemma 4.3. *Let $\beta \le -1/2$. Then*

$$e^{2(2\beta+1)t} \int_0^\infty (xy)^{-2\beta-1} B_t^{-\beta-1}(x,y)\, d\gamma_\beta(y) \le C < \infty$$

with C independent of $x > 0$ and $t > 0$.

Proof. The expression on the left-hand side of the inequality, denote it by $\mathcal{J}$, can be expressed in terms of a function $H_{\eta,\gamma}$ from [13, p. 440]. Indeed, after some computations (see [13, Lemma 2.2]) one gets

$$\mathcal{J} = H_{1/2, -\beta}\left(\frac{e^{-2t} x^2}{2 \sinh 2t} \right).$$

Then [13, Lemma 2.3] gives the desired conclusion. □

To proceed with the weak type $(1,1)$ bound for $\mathbb{S}_*^{\alpha}$, we invoke the kernel

$$K_s^{\alpha}(x,y) = s^{-d/2} \prod_{i=1}^d \left(x_i + \sqrt{s} \right)^{-2\alpha_i - 1} \exp\left(|x|^2 - \frac{|(1+s)x - (1-s)y|^2}{8s} \right)$$

from [8, Section 2.2] that controls $B_t^{\alpha}(x,y)$ after a suitable 'time' transformation. In dimension one, one has

$$B_{t(s)}^{\alpha}(x,y) \lesssim K_s^{\alpha}(x,y), \qquad x, y > 0, \quad t(s) = \frac{1}{2} \log \frac{1+s}{1-s}, \quad s \in (0,1).$$

Then $\mathbb{S}_*^{\alpha}$ is controlled by the maximal operator

$$\mathcal{K}_*^{\alpha} f(x) = \sup_{0<s<1} \int_0^\infty \mathcal{K}_s^{\alpha}(x,y) f(y)\, d\gamma_\alpha(y), \qquad x \in \mathbb{R}^d_+, \quad 0 \le f \in L^1(\mathbb{R}^d_+, d\gamma_\alpha),$$

where

$$\mathcal{K}_s^{\alpha}(x,y) = (1-s)^{-2\langle \alpha' \rangle - d'} (x'y')^{-2\alpha' - \mathbf{1}'} K_s^{-\alpha' - \mathbf{1}'}(x', y') K_s^{\alpha''}(x'', y'').$$

Our task then reduces to showing the following.

Lemma 4.4. *Let $d \ge 1$ and $\alpha \in (-\infty, -1]^{d'} \times (-1, \infty)^{d-d'}$ for some $1 \le d' \le d$. Then $\mathcal{K}_*^{\alpha}$ satisfies*

$$\gamma_\alpha\big\{ x \in \mathbb{R}^d_+ : \mathcal{K}_*^{\alpha} f(x) > \lambda \big\} \le \frac{C}{\lambda} \int_{\mathbb{R}^d_+} f(x)\, d\gamma_\alpha(x), \qquad \lambda > 0, \quad 0 \le f \in L^1(\mathbb{R}^d_+, d\gamma_\alpha),$$

with a constant C independent of λ and f.

The proof of this, taking into account the product structure of $\mathcal{K}_s^{\alpha}(x,y)$ and [8, Theorem 2.6], boils down to showing Lemmas 4.5–4.8 below that are suitable modifications of [8, Lemmas 3.3–3.6].

Lemma 4.5. *For any $d \ge 1$, $1 \le d' \le d$ and each $\alpha \in (-\infty, -1]^{d'} \times (-1, \infty)^{d-d'}$ the maximal operator*

$$\mathcal{K}_{*,1}^{\alpha} f(x) = \sup_{0<s<1} \int_{\mathbb{R}^d_+} \chi_{\{x_i/2 \le y_i \le 2x_i \ for\ i=1,\ldots,d'\}} \mathcal{K}_s^{\alpha}(x,y) f(y)\, d\gamma_\alpha(y), \qquad f \ge 0,$$

is of weak type $(1,1)$ with respect to γ_α on $\mathbb{R}^d_+$.

Proof. Observe that in the definition of $\mathcal{K}_s^\alpha(x,y)$ neglecting the factor $(1-s)^{-2\langle\alpha'\rangle-d'}$ makes the kernel even bigger. Then one can proceed as in the proof of [8, Lemma 3.3], with the aid of [8, Theorem 2.6], see also the proof of Lemma 2.7. □

Lemma 4.6. *For each $\beta \le -1$ the one-dimensional operator*

$$N_1^\beta f(x) = \int_0^\infty \chi_{\{y<x/2 \text{ or } y>2x\}} \Big[(xy)^{-2\beta-1} \sup_{0<s\le 1/4} (1-s)^{-2\beta-1} K_s^{-\beta-1}(x,y)\Big] f(y)\, d\gamma_\beta(y), \qquad f \ge 0,$$

is of strong type $(1,1)$ with respect to γ_β on $\mathbb{R}_+$.

Proof. The relevant kernel is comparable with

$$N_1(x,y) = \chi_{\{y<x/2 \text{ or } y>2x\}} (xy)^{-2\beta-1} \sup_{0<s\le 1/4} s^{-1/2}(x+\sqrt{s})^{2\beta+1} \exp\left(x^2 - \frac{|(1+s)x-(1-s)y|^2}{8s}\right).$$

Proceeding as in the proof of [8, Lemma 3.4], with the aid of the triangle inequality and [8, Lemma 2.3], we get

$$N_1(x,y) \lesssim (x\wedge y)^{-2\beta-1}(x\vee y)^{-1} e^{x^2}.$$

Then it is enough to verify that $\int_0^\infty (x\wedge y)^{-2\beta-1}(x\vee y)^{-1}e^{x^2}\, d\gamma_\beta(x) \lesssim 1$, $y>0$, which is immediate. □

Lemma 4.7. *For each $\beta \le -1$ and $\delta > -1$ the one-dimensional operators*

$$N_2^\beta f(x) = \chi_{(0,1)}(x) \int_0^\infty \Big[(xy)^{-2\beta-1} \sup_{1/4<s<1} (1-s)^{-2\beta-1} K_s^{-\beta-1}(x,y)\Big] f(y)\, d\gamma_\beta(y), \qquad f \ge 0,$$

$$N_3^\delta f(x) = \chi_{(0,1)}(x) \int_0^\infty \Big[\sup_{1/4<s<1} K_s^\delta(x,y)\Big] f(y)\, d\gamma_\delta(y), \qquad f \ge 0,$$

are of strong type $(1,1)$ with respect to γ_β and γ_δ on $\mathbb{R}_+$, respectively.

Proof. The strong type $(1,1)$ of N_3^δ is a part of [8, Lemma 3.5]. To treat N_2^δ, we follow the proof of [8, Lemma 3.5]. The relevant kernel is controlled by

$$N_2(x,y) = \chi_{(0,1)}(x)(xy)^{-2\beta-1} \sup_{1/4<s<1} (1-s)^{-2\beta-1} \exp\Big(-\frac{1}{16}|(1-s)y|^2\Big).$$

This implies $N_2(x,y) \lesssim \chi_{(0,1)}(x) x^{-2\beta-1}$. Since $\int_0^1 x^{-2\beta-1}\, d\gamma_\beta(x) < \infty$, the conclusion follows. □

Lemma 4.8. *For any $d \ge 1$, $1 \le d' \le d$ and each $\alpha \in (-\infty,-1]^{d'} \times (-1,\infty)^{d-d'}$ the operator*

$$\mathcal{K}_{*,2}^\alpha f(x) = \chi_{\mathbb{R}_1^d}(x) \int_{\mathbb{R}_+^d} \sup_{1/4<s<1} \mathcal{K}_s^\alpha(x,y) f(y)\, d\gamma_\alpha(y), \qquad f \ge 0,$$

is of weak type $(1,1)$ with respect to γ_α on $\mathbb{R}_+^d$; here $\mathbb{R}_1^d := [1,\infty)^d$.

Proof. It is enough to show that $\sup_{1/4<s<1} \mathcal{K}_s^\alpha(x,y)$ is controlled by the right-hand side of [8, (21)]. This will give the desired conclusion, in view of the proof of [8, Lemma 3.6].

We proceed as in the preliminary part of the just mentioned proof. Taking into account that $1/4 < s < 1$ and then using [8, Lemma 3.7] with suitably chosen parameters we get

$$\begin{aligned}
&\mathcal{K}_s^\alpha(x,y) \\
&\lesssim (1-s)^{-2\langle\alpha'\rangle-d'}(x'y')^{-2\alpha'-\mathbf{1}'}(x')^{2\alpha'+\mathbf{1}'}(x'')^{-2\alpha''-\mathbf{1}''} \exp\left(|x|^2 - \frac{|(1+s)x-(1-s)y|^2}{8}\right) \\
&\lesssim x^{-2\alpha-\mathbf{1}} \exp\left(|x|^2 - \frac{|(1+s)x-(1-s)y|^2}{16}\right)
\end{aligned}$$

uniformly in $1/4 < s < 1$, $x \in \mathbb{R}_1^d$ and $y \in \mathbb{R}_+^d$. This bound already implies what is needed, see the details in the proof of [8, Lemma 3.6]. □

This finishes proving Theorem 4.1.

4.3. **Alternative $[\mathbf{DOU}, \mathbb{Z}_2^d]$ context.** We briefly consider a variant of the Dunkl Ornstein-Uhlenbeck context where the standard gradient in U_α is replaced by the Dunkl gradient; this situation was studied e.g. in [5]. Thus now our operator is

$$U_\alpha^\nabla = \Delta_\alpha + 2x \cdot \nabla_\alpha.$$

In dimension one, for $\alpha > -1$ we have

$$U_\alpha^\nabla H_n^\alpha = \left\{ \begin{array}{ll} 2n, & n \text{ even} \\ 2n + 4\alpha + 2, & n \text{ odd} \end{array} \right\} \times H_n^\alpha$$

and for $\alpha < 0$

$$U_\alpha^\nabla \widetilde{H}_n^\alpha = \left\{ \begin{array}{ll} 2n - 4\alpha, & n \text{ even} \\ 2n - 2, & n \text{ odd} \end{array} \right\} \times \widetilde{H}_n^\alpha,$$

so proceeding as in Section 4.1 one can define classical and exotic self-adjoint extensions of U_α^∇. Then the one-dimensional classical and exotic semigroups have integral representations, analogous to those from Section 4.1, with the kernels

$$\begin{aligned} \mathbf{B}_t^{\alpha,\nabla}(x,y) &= B_t^\alpha(x,y) + e^{-4(\alpha+1)t} xy B_t^{\alpha+1}(x,y), \\ \widetilde{\mathbf{B}}_t^{\alpha,\nabla}(x,y) &= \widetilde{B}_t^\alpha(x,y) + e^{-4(\alpha+1)t} xy \widetilde{B}_t^{\alpha+1}(x,y), \end{aligned}$$

respectively, where $B_t^\alpha(x,y)$ and $\widetilde{B}_t^\alpha(x,y)$ are as before.

In the multi-dimensional situation, assuming that $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1,\ldots,d\}$, $\mathfrak{H}_n^{\alpha,\mathcal{E}}$ are eigenfunctions of U_α^∇, so proceeding as in Section 4.2 we arrive at the semigroup $\{\mathbb{B}_t^{\alpha,\nabla,\mathcal{E}}\}$ represented as

$$\mathbb{B}_t^{\alpha,\nabla,\mathcal{E}} f(x) = \frac{1}{2^d} \int_{\mathbb{R}^d} \mathbb{B}_t^{\alpha,\nabla,\mathcal{E}}(x,y) f(y)\, d\gamma_\alpha(y), \qquad x \in \mathbb{R}^d, \quad t > 0,$$

with

$$\mathbb{B}_t^{\alpha,\nabla,\mathcal{E}}(x,y) = \prod_{i\in\mathcal{E}} \widetilde{\mathbf{B}}_t^{\alpha_i,\nabla}(x_i,y_i) \prod_{i\in\mathcal{E}^c} \mathbf{B}_t^{\alpha_i,\nabla}(x_i,y_i), \qquad x,y \in \mathbb{R}^d, \quad t > 0.$$

For the restricted maximal operator

$$\mathbb{B}_{*,T}^{\alpha,\nabla,\mathcal{E}} f = \sup_{0<t<T} \left| \mathbb{B}_t^{\alpha,\nabla,\mathcal{E}} \right|$$

we can readily conclude the following result.

Theorem 4.9. *Let $d \geq 1$ and $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1,\ldots,d\}$. Assume that $m_{\mathcal{E}}(\alpha) \leq -1/2$. Then for any fixed $0 < T < \infty$, $\mathbb{B}_{*,T}^{\alpha,\nabla,\mathcal{E}}$ is bounded on $L^p(\mathbb{R}^d, d\gamma_\alpha)$ for $1 < p \leq \infty$, and from $L^1(\mathbb{R}^d, d\gamma_\alpha)$ to weak $L^1(\mathbb{R}^d, d\gamma_\alpha)$.*

Proof. This result is implicitly contained in the proof of Theorem 4.1, due to domination of even and odd parts of the one-dimensional component kernels $\mathbf{B}_t^{\alpha,\nabla}(x,y)$ and $\widetilde{\mathbf{B}}_t^{\alpha,\nabla}(x,y)$ by their counterparts from Section 4.2. $\square$

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