Aliquot sequences starting with a number under 10000

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Abstract

In this paper we describe some advances in the knowledge of the behavior of aliquot sequences starting with a number less than 10000. We give the URL address of the web pages with the free number theory packages that we have used to compute the aliquot sequences up to their present status.

Introduction and results

For a positive integer n, let $\sigma(n)$ denote the sum of its divisors (including 1 and n), and $s(n) = \sigma(n) - n$ the sum of its divisors without n. A perfect number is a number n such that $s(n) = n$, and an amicable pair of numbers is a pair (n, m) satisfying $s(n) = m$, $s(m) = n$. In a similar way, cycles of numbers $(a_1, a_2, \dots, a_l)$ such that $s(a_i) = a_{i+1}$ for $1 \le i \le l-1$ and $s(a_l) = a_1$ are known as aliquot cycles or sociable numbers.

The function s generates an aliquot sequence $\{s^k(n)\}_{k=0}^{\infty}$ by taking $s^0(n) = n$ and $s^{k+1}(n) = s(s^k(n))$. For each one of these sequences, there are four possibilities:

(i) it terminates at 1 (being the previous term a prime number),

(ii) it reaches a perfect number,

(iii) it reaches an amicable pair or a cycle,

(iv) it is unbounded.

The Catalan-Dickson conjecture [2, 4] says that (iv) does not actually happen. But other researchers disagree with this conjecture and think that there are unbounded sequences; in fact, the alternative conjecture from Guy-Selfridge [7] (see also [5], [6]) states that there are many sequences that go to infinity, perhaps almost all of those that start at an even number (i.e., the proportion of even integers n such that $\{s^k(n)\}_{k=0}^{\infty}$ is bounded tends to zero).

This alternative conjecture is based upon the existence of several patterns that, whenever they appear in the factor decomposition of n, they appear again (with high order of probability) in the decomposition of $s(n)$. These patterns are called *drivers* or *guides* (see [7] for details; also, see an example later in this paper).

Let us explain how to compute a aliquot sequence. It is enough to describe how to compute $\sigma(n)$ for any positive integer n. Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be its decomposition in prime numbers. We claim that

$$\sigma(p_1^{a_1} \cdots p_k^{a_k}) = (1 + p_1 + \cdots + p_1^{a_1}) \cdots (1 + p_k + \cdots + p_k^{a_k}). \qquad (1)$$

Indeed, if we expand the expression on the right, there appear, as summands, all the divisors of $p_1^{a_1} \cdots p_k^{a_k}$. Moreover, $1 + p + \cdots + p^a = \frac{p^{a+1}-1}{p-1}$ so we also have the following formula to compute $\sigma(n)$:

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}. \tag{2}$$

Now, let us suposse $n = 2^3 \cdot 3 \cdot 5 \cdot p \cdot q$, with p and q prime numbers, $p \neq q$, $p, q > 5$ (the case with more prime factors is similar). By (1), $s(n) = (1{+}2{+}4{+}8) \cdot 4 \cdot 6 \cdot (p{+}1) \cdot (q{+}1) - 2^3 \cdot 3 \cdot 5 \cdot p \cdot q = 15 \cdot 2^2 \cdot 2 \cdot 3 \cdot (p{+}1) \cdot (q{+}1) - 2^3 \cdot 3 \cdot 5 \cdot p \cdot q = 2^3 \cdot 3 \cdot 5 \cdot [3 \cdot (p{+}1) \cdot (q{+}1) - p \cdot q] = 2^3 \cdot 3 \cdot 5 \cdot m$, with m an odd number that is not divisible by 3. Then, we get again $2^3 \cdot 3 \cdot 5$ with 2 and 3 raised to the same power as before. In this way, we say that $2^3 \cdot 3 \cdot 5$ is a driver. However, the power of 5 can, eventually, change; this may cause the breaking of the driver at a later moment.

The smallest n for which there was ever doubt about the end of $\{s^k(n)\}_{k=0}^{\infty}$ was 138, but Lehmer showed that the sequence terminates at $s^{177}(138) = 1$. Currently, the first number whose behavior is not known is 276. Many researchers have investigated the behavior of this sequence, but it is not yet known if its end exists.

But there are many other sequences whose end is in doubt. For instance, there are five main sequences starting in a number smaller than 1000 whose behavior is unknown (Lehmer five: 276, 552, 564, 660 and 966). Before the year 1980, there were fourteen main sequences starting between 1000 and 2000 (Godwin fourteen), but at the present time only twelve remain unknown (Godwin showed that the 1848 sequence terminates, and so did Dickerman for the 1248 one).

In 1994, A. W. P. Guy and R. K. Guy published an article [5] with a table showing the status of the sequences starting with numbers less than or equal to 7044. In the same year, the book [6], from R. K. Guy, updates the information of the article. In this book, a extensive bibliography on this subject can be found.

Since then, the authors have been checking the aliquot sequences for numbers under 10000. For some starting values, we have shown for the first time that the sequence terminates. In December 22, 1996, we found the record for the maximum of a terminating sequence: the one starting at 4170 converges to 1 after 869 iterations, getting a maximum of 84 decimal digits at iteration 289. This result was published in [1].

Later, in October 1999, W. Bosma broke this record: he found that the aliquot sequence starting with 44922 terminates after 1689 iterations (at 1), reaching a maximum of 85 digits at step 1167. In December 3, 1999, he broke again the record by finding that the sequence starting at 43230 finished: it terminates (at 1) after 4357 iterations, reaching a maximum of 91 digits at step 967.

We have continued analyzing the behavior of the aliquot sequences starting with a number $n < 10000$. In Table 1 we summarize our work with such sequences; it is an update of the one that appears in [1]. In the table we show the main sequences whose status is yet unknown (by *main* we mean that all the other sequences with unknown end merge with any sequence in the table; in this case, we only show in the table the sequence that starts with the smaller number). We also include the number of decimal digits of the last term

$s^k(n)$ reached for each sequence and the guide in that stage. Actually, all the sequences are in driver (a driver is more stable than a guide), except the ones marked with $^{(*)}$.

In [1], any of the sequences of the table had been pursued up to, at least, 75 digits. Now, we have reached more that 92 digits for all of them; 100 digits for many of them. The sequences corresponding 276, 552, 564, 660, 966, 1074, 1134 are due to W. Creyaufmüller, P. Zimmermann and J. Howell. Finally, let us note that we have reached at least a hundred digits for the Godwin sequences.

The method

According to (1) or (2), to compute an aliquot sequence, we have to decompose m in factors to calculate $s(m)$. This is hard when the number m is big. And we are dealing with numbers more than a hundred of digits long, so powerful algorithms are necessary. In particular, we have used the elliptic curve method (ECM) and the multipolynomial quadratic sieve (MPQS). We have checked the primality of the factors with the Adleman-Pomerance-Rumely (APR) primality test.

All our work has been done by using free packages available on internet. We have run the programs on many computers from the authors and some colleages, and their respective institutions.

We have used the following packages, that are available at their corresponding web pages (or anonymous `ftp` sites):

- UBASIC, `ftp://rkmath.rikkyo.ac.jp/pub/ubibm/`
- PARI-GP, `http://www.parigp-home.de/`
- KANT-KASH, `http://www.math.TU-Berlin.DE/algebra/`
- MIRACL, `http://indigo.ie/~mscott/`

Other people and the web

There are some other people working in aliquot sequences: Wolfgang Creyaufmüller is computing aliquot sequences starting with a number between 10^5 and 10^6 up to a minimum of 60 digits; also, he has written the book [3]. Paul Zimmermann is computing iterates of the sequences starting in the numbers 276, 552, 564, 660, 966, 1074, 1134 and 204828; also, he is working with aliquot sequences starting between 50000 and 100000. Jim Howell is working in the aliquot sequence starting in 966. Finally, Wieb Bosma is computing with aliquot sequences starting between 10000 and 50000.

One of the authors maintains the following web related with aliquot sequences:

`http://www.unirioja.es/dptos/dmc/jvarona/aliquot.html`

In this page, you can find links to the aliquot web pages of W. Creyaufmüller, P. Zimmermann, J. Howell and W. Bosma. Also, the sequences themselves can be found in the `ftp` server

Table 1: Aliquot sequences whose end is in doubt

n	276	552	564	660	966	1074	1134	1464	1476
k	1132	818	2970	439	481	1543	2242	1897	1055
digits	108	118	102	110	108	103	124	101	106
guide	$2\cdot 3$	$2^5\cdot 3\cdot 7$	$2^2\cdot 7$	$2^{2(*)}$	$2\cdot 3$	$2^2\cdot 7$	$2^5\cdot 3\cdot 7$	$2^2\cdot 7$	$2^3\cdot 3\cdot 5$
n	1488	1512	1560	1578	1632	1734	1920	1992	2232
k	824	1632	1336	1109	713	1393	1992	985	390
digits	103	101	101	104	102	101	108	102	102
guide	$2^2\cdot 7$	$2^2\cdot 7$	$2^5\cdot 3\cdot 7$	$2^{2(*)}$	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^2\cdot 7$	$2^3\cdot 3\cdot 5$	$2^{6(*)}$
n	2340	2360	2484	2514	2664	2712	2982	3270	3366
k	471	974	796	2836	761	1347	810	417	1062
digits	99	95	97	95	100	95	97	98	100
guide	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^3\cdot 3$	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^2\cdot 7$	$2^4\cdot 31$	$2^5\cdot 3\cdot 7$	$2^{3(*)}$
n	3408	3432	3564	3630	3678	3774	3876	3906	4116
k	840	933	779	1240	1201	1193	830	679	1192
digits	95	103	100	100	98	98	96	94	105
guide	$2^3\cdot 3\cdot 5$	$2^3\cdot 3\cdot 5$	$2^3\cdot 3$	$2^{2(*)}$	$2^2\cdot 7$	$2^7\cdot 3^{(*)}$	$2^2\cdot 7$	$2^2\cdot 7$	$2^3\cdot 3$
n	4224	4290	4350	4380	4788	4800	4842	5148	5208
k	519	913	1165	965	2152	1135	473	1545	1710
digits	98	92	97	100	105	101	98	95	96
guide	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^2\cdot 7$	$2^2\cdot 7$	$2^3\cdot 3\cdot 5$	$2^4\cdot 31$	$2^2\cdot 7$	$2\cdot 3$	$2^3\cdot 3\cdot 5$
n	5250	5352	5400	5448	5736	5748	5778	6160	6396
k	1564	683	2696	1185	1093	1045	742	1630	1234
digits	100	93	93	96	100	98	95	96	92
guide	$2^{4(*)}$	$2^2\cdot 7$	$2^2\cdot 7$	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^2\cdot 7$	$2^4\cdot 31$	$2^{2(*)}$	$2^3\cdot 3\cdot 5$
n	6552	6680	6822	6832	6984	7044	7392	7560	7890
k	893	1815	1177	885	1764	1113	498	846	891
digits	93	94	97	104	96	102	96	97	99
guide	$2^3\cdot 3$	$2^4\cdot 31$	$2^4\cdot 31$	$2^3\cdot 3$	$2^4\cdot 31$	$2^4\cdot 31$	$2^3\cdot 3\cdot 5$	$2^3\cdot 5^{(*)}$	$2^2\cdot 7$
n	7920	8040	8154	8184	8288	8352	8760	8844	8904
k	951	2205	647	1241	849	1291	2145	1184	963
digits	95	94	96	102	103	96	94	101	95
guide	$2^6\cdot 127$	$2^3\cdot 3\cdot 5$	$2^2\cdot 7$	$2^2\cdot 7$	$2^{8(*)}$	$2^2\cdot 7$	$2^3\cdot 3\cdot 5$	$2^4\cdot 31$	$2^2\cdot 7$
n	9120	9282	9336	9378	9436	9462	9480	9588	9684
k	532	505	608	2111	545	447	945	1848	617
digits	92	95	97	92	94	97	93	103	92
guide	$2^3\cdot 3$	$2^3\cdot 3$	$2^6\cdot 127$	$2^2\cdot 7$	$2\cdot 3$	$2^3\cdot 3\cdot 5$	$2\cdot 3$	$2^5\cdot 3\cdot 7$	$2^5\cdot 3\cdot 7$
n	9708	9852							
k	671	669							
digits	95	105							
guide	$2^2\cdot 7$	$2^2\cdot 7$							

`ftp://mat.unirioja.es/pub/aliquot/00xxxx`

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