Pseudo-uniform convexity in H^p and some extremal problems on Sobolev spaces.

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Abstract

We extend Newman and Keldysh theorems to the behavior of sequences of functions in $H^p(\mu)$ which explain geometric properties of discs in these spaces. Through Keldysh's theorem we obtain asymptotic results for extremal polynomials in Sobolev spaces.

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1 Introduction and main results

In this paper we extend Newman and Keldysh theorems to $H^p(\mu)$ with $0 < p < \infty$. These results are very useful for obtaining convergence in norm. It is known that H^p, $0 < p \leq 1$, are not locally uniformly convex spaces and they are used for checking that Hahn-Banach theorem fails in a non-locally convex space with "reasonable" properties. That H^p would seem destined to be of further interest in the future can be guessed from the fact that the most common "singularities" in analysis, such as those given by rational functions, or carried on analytic subvarieties, or representable by Fourier integral ("Lagrangian") distributions, are all of them locally in H^p, for some $p < 1$.

We will use these theorems for proving a result about asymptotics of extremal Sobolev polynomials. Sobolev orthogonal polynomials have been receiving considerable attention in the last two decades, as a natural consequence of the great importance of Sobolev spaces. They are also connected with spectral theory for ordinary differential equations, matrix orthogonal polynomials, and higher order recurrence relations. They appear also in a natural way in some problems of approximation theory where the derivatives are considered. Two updated surveys on Sobolev orthogonal polynomials are presented in [12] and [15] (look at the references therein). Asymptotics for Sobolev orthogonal polynomials have been described among others in [1], [10], [11], [13], and [14].

We begin with the extensions of Newman and Keldysh theorems that will be proved in Section 3, but first we set some notations. Let m be the normalized Lebesgue measure on $[0, 2\pi)$ and let μ be a positive Borel measure on $[0, 2\pi)$ satisfying Szegő's condition, i.e. $\mu \in \mathbf{S} \Leftrightarrow \log \mu'(\theta) \in L^1$, where $\mu'(\theta)$ denotes the Radon-Nikodym derivate of μ with respect to m. $H^p(\mu)$ is defined as the $L^p(\mu)$ closure of the polynomials in $e^{i\theta}$ and $\|f\|_{p,\mu}$ denotes as usually $\left(\int |f(e^{i\theta})|^p \mathrm{d}\mu(\theta)\right)^{1/p}$ for $f \in L^p(\mu)$. For a sake of simplicity we will write H^p instead of $H^p(m)$. Let $D_p(\mu, z)$ be the Szegő function

$$D_p(\mu, z) = \exp\left\{\frac{1}{p}\int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} \log \mu'(\theta)\, \mathrm{d}m(\theta)\right\}, \quad \zeta = e^{i\theta} \tag{1}$$

and

$$K_p(\mu, z) = \begin{cases} \frac{D_p(\mu,0)}{D_p(\mu,z)}, & \text{if } z \in \ \mathrm{supp}\mu_a, \\ 0, & \text{if } z \in \ \mathrm{supp}\mu_s, \end{cases} \tag{2}$$

where μ_a and μ_s are the absolutely continuous and singular parts, respectively, of μ with respect to m. Let $L^p_s(\mu) = \{f \in L^p(\mu) : f = 0, \mu_a - a.e.\}$ and $L^p_a(\mu) = \{f \in L^p(\mu) : f = 0, \mu_s - a.e.\}$ be the absolutely continuous and singular subspaces, respectively. Similarly, we define $H^p_s(\mu)$ and $H^p_a(\mu)$. Set $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{E} = \{z : |z| > 1\}$.

Theorem 1. *Assume that $\mu \in \boldsymbol{S}$. If f_n and f are in $H^p_a(\mu)$, $0 < p < \infty$, such that*

i) $\lim_{n\to\infty} \|f_n\|_{p,\mu} = \|f\|_{p,\mu}$,

ii) $\lim_{n\to\infty} f_n(z) = f(z)$ *holds uniformly on each compact subset of* $\mathbb{D}$,

then

$$\lim_{n\to\infty} \|f_n - f\|_{p,\mu} = 0. \tag{3}$$

Remark 1. Theorem 1 was proved by Newman (see [16]) for the cases $p = 1$ and $\mu = m$, the Lebesgue measure. This theorem gives an alternative look for the uniform convexity of the $H^p(\mu)$ spaces.

In the next section, we are going to prove that if $f \in H^p(\mu)$, then there exist unique functions $\tilde{f}, f_s$ such that $f = K_p\tilde{f} + f_s$, $\tilde{f} \in H^p$, and $f_s \in L^p_s(\mu)$. With these notations we set

Theorem 2. *Let $\{z_i\}_{i=1,\dots,\Lambda}$ be a set of points in $\mathbb{D}$ where Λ can be finite or infinite, $\mu \in \boldsymbol{S}$, and $\{f_n\} \subset H^p(\mu)$, $0 < p < \infty$, such that*

i) $\lim\limits_{n\to\infty} \tilde{f}_n(0) = 1$;

ii) $\lim\limits_{n\to\infty} \tilde{f}_n(z_i) = 0, \quad i = 1, 2, \dots$;

iii) $\sum\limits_{i=1}^{\Lambda}(1 - |z_i|) < +\infty$;

iv) $\lim\limits_{n\to\infty} \|f_n\|_{p,\mu} = \dfrac{D_p(\mu, 0)}{\prod_{i=1}^{\Lambda} |z_i|^p}$.

Then

a) $\lim\limits_{n\to\infty} \tilde{f}_n(z) = \prod\limits_{i=1}^{\Lambda} \dfrac{z - z_i}{\bar{z}_i z - 1} \dfrac{\bar{z}_i}{|z_i|^2}$ *holds uniformly on each compact subset of* $\mathbb{D}$.

b) $\lim\limits_{n\to\infty} \left\| f_n - \prod\limits_{i=1}^{\Lambda} \dfrac{z - z_i}{\bar{z}_i z - 1} \dfrac{\bar{z}_i}{|z_i|^2} K_p(z) \right\|_{p,\mu} = 0.$

Remark 2. For the cases $\mu = m$ and $\Lambda = \emptyset$, Theorem 2 was setting by Keldysh (see [8]).

Remark 3. Newman and Keldysh theorems do not hold in H^∞ as the following example shows. Set $f_n(z) = \frac{nz+n-1}{n+(n-1)z}$. It is easy to check that $f_n \in H^\infty$, $\|f_n\|_\infty = 1$, and $\lim\limits_{n\to\infty} f_n = 1$ uniformly on each compact subset of $\mathbb{D}$, but $\|f_n - 1\|_\infty \not\to 0$.

The other results that will be proved in Section 4 give us strong asymptotics for extremal polynomials, $\{P_n\}_{n=0,1,\dots}$, solving the following extremal problem:

$$\tau_{n,\mu_0,\dots,\mu_k,p} = \inf\left\{ \sum_{j=0}^{k} \|Q^{(j)}\|_{p,\mu_j} : Q(z) = z^n + \dots \right\}, \tag{4}$$

where $0 < p < \infty$ and $\mu_0, \dots, \mu_k$ are positive Borel measures on $[0, 2\pi)$, with $\mu_0 \not\equiv 0$. When $p = 2$ the polynomials $\{P_n\}_{n=0,1,\dots}$ are usually said to be Sobolev orthogonal polynomials. The special case $k = 1$ has been studied by many authors (see, for instance [12], [14], [20]). For $k = 0$ we have the classical orthogonality.

Theorem 3. *The following statements are equivalent.*

(i) $\mu_k \in \boldsymbol{S}$*;*

(ii) $\limsup_{n\to\infty} \frac{\tau_{n,\mu_0,\dots,\mu_k,p}}{n^k} > 0$;

(iii) *There exists a function* $\Delta \in H_a^p(\mu_k)$ *with* $\Delta(0) = 1$ *such that*

$$\lim_{n\to\infty} \int \left| \frac{P_n^{(k)}(z)}{n^k\, z^{n-k}} - \overline{\Delta(1/\overline{z})} \right|^p \mu_k'(\theta) dm(\theta) = 0, \quad z = e^{i\theta}.$$

Moreover, if **(i)** *holds then*

$$\lim_{n\to\infty} \frac{\tau_{n,\mu_0,\dots,\mu_k,p}}{n^k} = D_p(\mu_k, 0),$$

the function $\Delta(z)$ *is equal to* $K_p(\mu_k, z)$ *in* **(iii)**, *and*

$$\lim_{n\to\infty} \frac{P_n^{(k)}(z)}{n^k\, z^{n-k}} = \overline{K_p(\mu_k, 1/\overline{z})} \tag{5}$$

holds uniformly on each compact subset of $\mathbb{E}$.

When $k = 0$ the extremal problem (4) was studied by Geronimus (see [5]) who stated more precisely that Theorem 3 holds for $k = 0$. The following theorem is similar to Theorem 2 of [14] where it was observed that there exist asymptotically extremal polynomials for all j, $0 \le j \le k$, and $p = 2$.

Theorem 4. *If the measures* $\mu_l \in \boldsymbol{S}$, $j \le l \le k$, *then for all* l, $j \le l \le k$,

$$\lim_{n\to\infty} \frac{P_n^{(l)}(z)}{n^l z^{n-l}} = \overline{K_p(\mu_k, 1/\overline{z})}, \tag{6}$$

holds uniformly on each compact subset of $\mathbb{E}$.

The framework of this paper is in Section 2 we describe some known properties of the H^p-extremal Szegő function and the necessary results for proving the three main results. Finally in Sections 3 and 4 we prove all the theorems presented in Section 1.

2 Auxiliary Results

Given $\mu \in \mathbf{S}$ the corresponding Szegő function satisfies the following properties:

1 $D_p(\mu, z)$ is analytic on $\mathbb{D}$, more precisely, $D_p(\mu, z) \in H^p$;

2 $D_p(\mu, z) \neq 0$ in $\mathbb{D}$, and $D_p(\mu, 0) > 0$;

3 $|D_p(\mu, e^{i\theta})|^p = \mu'(\theta)$ a. e. on $[0, 2\pi)$.

The function $D_p(\mu, .)$ is not uniquely determined by the conditions $1 - 3$. To this aim it is also required that $D_p(\mu, .)$ must be an outer function (see page 277 in [22] or page 118 in [17]).

We shall need the following well known result (looking for an easy reading we include the proof).

Lemma 1. *There is a unique solution for the extremal problem*

$$\inf \{\|\Phi D_p(\mu, \cdot)\|_p : \Phi \in K_p(\mu, \cdot)H^p,\ \Phi(0) = 1\}$$

given by $\Phi(z) = K_p(\mu, z)$ *and the infimum is* $D_p(\mu, 0)$.

Proof. First, let assume $p = 2$. Let $\Phi \in K_2(\mu, \cdot)H^2$ be such that $\Phi(0) = 1$, then $|\Phi(z)D_2(\mu, z)|^2$ is a subharmonic function on $\mathbb{D}$, so $D_2(\mu, 0)^2 = |\Phi(0)D_2(\mu, 0)|^2 \leq \|\Phi D_p(\mu, \cdot)\|_2^2$. Moreover, we have that K_2 belongs to $K_2(\mu, \cdot)H^2$, its value at $z = 0$ is 1, and $\|K_2 D_2(\mu, \cdot)\|_2^2 = D_2(\mu, 0)^2$. The uniqueness follows immediately from the parallelogram law. If $p \neq 2$ we reduce these cases to $p = 2$ because of each function $f \in K_p(\mu, \cdot)H^p$ has a decomposition $f(z) = B(z)[h(z)]^{2/p}$, where B is the Blaschke product associated with the zeros of f and $h \in K_2(\mu, \cdot)H^2$ with $\|f\|_p^p = \|h\|_2^2$ (see [9], p. 96). $\square$

It is very well known that the density of the space of polynomials in $L^p(\mu)$ can be also characterized in terms of the Szegő condition for μ. $H^p(\mu) = L^p(\mu)$ if and only if $\mu \notin \mathbf{S}$ (it is an inmediatly consecuence of Geronimus and Weierstrass theorems and the fact that continuous functions are dense in $L^p(\mu)$). For $p = 2$

and Sobolev's norms look at [19]. It is even the characterization of $H^p(\mu)$ for $\mu \in \mathbf{S}$. The following theorem is very well known but we have only found references for $1 \leq p \leq \infty$ (see page 29 in [7], page 22 in [18]).

Theorem 5. *If we assume $\mu \in \boldsymbol{S}$, then $H^p(\mu) = K_p H^p \oplus L^p_s(\mu)$.*

Proof. Set $g \in H^p(\mu)$. Then $g \in L^p(\mu)$ and $g = g_1 + g_2$ with $g_1 \in L^p_a(\mu)$ and $g_2 \in L^p_s(\mu)$. We must prove that either $g_1 \in K_p H^p$ or that $\dfrac{g_1}{K_p} \in H^p$. It is enough to prove that there exists $\{j_n\}$, $j_n \in H^p$, such that $\|j_n - \dfrac{g_1}{K_p}\|_p \longrightarrow 0$. Indeed, since $g \in H^p(\mu)$ we can get a sequence of polynomials $\{h_n\}$ such that, $\|h_n - g\|_{p,\mu} \longrightarrow 0$. Hence, with $z = e^{i\theta}$,

$$\int \left| \frac{h_n(z)}{K_p(z)} - \frac{g_1(z)}{K_p(z)} \right|^p \mathrm{d}m(\theta) = \int |h_n(z) - g_1(z)|^p \frac{\mu'(\theta)}{|D_p(\mu,0)|^p} \mathrm{d}m(\theta) =$$

$$\int |h_n(z) - g(z)|^p \frac{\mu'(\theta)}{|D_p(\mu,0)|^p} \mathrm{d}m(\theta) \leq \left\| \frac{h_n - g}{D_p(\mu,0)} \right\|^p_{p,\mu} \longrightarrow 0,$$

and $j_n = \dfrac{h_n}{K_p} \in H^p$ for each n. The uniqueness of the representation follows immediately from the fact $H^p_a(\mu) \cap L^p_s(\mu) = 0$. Hence, we have proved one of the inclusions. We are going to see that $K_p H^p \subset H^p(\mu)$. Consider $f = K_p \tilde{f}$ with $\tilde{f} \in H^p$. Then there exist polynomials h_n such that

$$\|\tilde{f} - h_n\|_p \to 0 \Longrightarrow \|K_p \tilde{f} - K_p h_n\|_{p,\mu} \to 0,$$

and because of $K_p h_n \in H^p(\mu)$, we get $f \in H^p(\mu)$.

Now set $f \in L^p_s(\mu)$. As $\mu_s \notin \mathbf{S}$, there exists polynomials Q_n such that

$$\|f - Q_n\|_{\mu_s,p} \to 0. \tag{7}$$

Moreover, because of $\dfrac{Q_n}{K_p} \in H^p$, we can find a sequence of polynomials $\{h_n\}$ that satisfy

$$\left\| \frac{Q_n}{K_p} - h_n \right\|_p \to 0 \Longleftrightarrow \|Q_n - K_p h_n\|_{\mu_a,p} \to 0. \tag{8}$$

Combining (7) and (8) with

$$\|f - Q_n + K_p h_n\|_{\mu,p} = \|f - Q_n\|_{\mu_s,p} + \|Q_n - K_p h_n\|_{\mu_a,p},$$

the proof is concluded. $\square$

The last auxiliary result that we need is

Lemma 2. *(see [3], p. 21) Let $\varphi_n, \varphi \in L^p$, $0 < p < \infty$. If $\varphi_n(x) \to \varphi(x)$ a.e. and $\|\varphi_n\|_p \to \|\varphi\|_p$, then $\|\varphi_n - \varphi\|_p \to 0$.*

3 Proof of the Theorems 1 and 2

Proof of Theorem 1

Proof. First, we consider the case $\mu = m$ the Lebesgue measure.

a) It is easy to see that the theorem holds for $1 < p < \infty$. Indeed, from the asumptions **i)** and **ii)** of the Theorem we obtain $\lim_{n\to\infty} \|(f_n + f)/2\|_p = \|f\|_p$, so, because of the uniform convexity of L^p, $1 < p < \infty$, (or Clarkson inequalities, see [2], p. 3) we get $\|f_n - f\|_p \to 0$.

b) Let us consider $0 < p \leq 1$. We can suppose that f is not identically 0 otherwise the proof of the theorem is trivial, since $\|f_n\|_p \longrightarrow \|f\|_p = 0$.

c) From Lemma 2, if **i)** holds and for any $\Gamma \subset \mathbb{N}$ there exists $\Gamma' \subset \Gamma$ such that $\lim\limits_n f_n(z) = f(z)$, *a.e.*, $n \in \Gamma'$, then we have (3).

d) If $f_n(z) \neq 0$ holds for $n \geq N_0$ and $z \in \mathbb{D}$, then using Hurwitz's theorem also $f(z) \neq 0$ holds in $\mathbb{D}$. So we can fix a branch of $w^{p/2}$ such that $h_n = f_n^{p/2}$ and $h = f^{p/2}$ with $h_n, h \in H^2$, $\|h_n\|_2^2 = \|f_n\|_p^p$, $\|h\|_2^2 = \|f\|_p^p$, $\lim h_n = h$ uniformly on each compact subset of $\mathbb{D}$, and $\lim \|h_n\|_2 = \|h\|_2$. This means i) and ii) hold for h_n and h in H^2. Therefore, from a) we have (3). Then according to the Riesz theorem, there exists $\Gamma \subset \mathbb{N}$ such that $\lim\limits_{n\in\Gamma} h_n(z) = h(z)$, *a.e.*, hence $\lim\limits_{n\in\Gamma} f_n(z) = f(z)$, *a.e.* and from c) we obtain (3).

e) If f_n has some zeros in $\mathbb{D}$, then $f_n = B_n h_n$ where h_n is a zero-free function in H^p, $\|h_n\|_p = \|f_n\|_p$, and B_n is a Blaschke product so $B_n \in H^\infty$. Then $\{B_n\}$ and $\{h_n\}$ are uniformly bounded in each compact subset of $\mathbb{D}$. Hence, from the Montel theorem there exists a subsequence

$\Gamma_1 \subset \mathbb{N}$ such that $\lim_{n\in\Gamma_1} h_n(z) = h(z),\, h \in H^p,$ and $\lim_{n\in\Gamma_1} B_n(z) = B(z)$ hold uniformly on each compact subset of $\mathbb{D}$. Moreover, $\|h\|_p \le \limsup \|h_n\|_p = \limsup \|f_n\|_p = \|f\|_p$, while $\|f\|_p \le \|h\|_p \|B\|_\infty$ and $\|B\|_\infty \le \limsup \|B_n\|_\infty = 1$. Thus, $\|hB\|_p = \|h\|_p$, and as a consequence $|B(e^{i\theta})| = 1$, a.e.. So $\lim_n \|h_n\|_p = \|h\|_p$ and $\lim_n \|B_n\|_2 = \|B\|_2$. Then using d) and a), there exists $\Gamma_2 \subset \Gamma_1$ such that $\lim_{n\in\Gamma_2} h_n(z) = h(z)$, *a.e.* and $\lim_{n\in\Gamma_2} B_n(z) = B(z)$, *a.e.*, respectively, hence $\lim_{n\in\Gamma_2} f_n(z) = f(z)$, *a.e.* and from c) the theorem is proved.

Now, in order to complete the proof it is enough to see that the functions $\tilde{f}_n$ and $\tilde{f}$ hold the assumption of theorem with Lebesgue measure.

□

Proof of Theorem 2

Proof. The sketch of the proof is the following. First, we will prove the theorem for Lebesgue measure and $\Lambda = \emptyset$ in two steps: $p = 2$ and $p \neq 2$. Second, we consider a general $\mu \in \mathbf{S}$ and again $\Lambda = \emptyset$. Finally, we prove the general case.

A) Set $\mu = m$, $p = 2$, and $\Lambda = \emptyset$. Notice that in this case $D_p(\mu, z) \equiv 1$. From the monotonicity of the means and triangular inequality, we get $|f_n(0) + 1| \le \|f_n + 1\|_2 \le \|f_n\|_2 + \|1\|_2$. Hence $\lim_{n\to\infty} \|f_n + 1\|_2 = 2$. Now, using the parallelogram law we obtain $\lim_{n\to\infty} \|f_n - 1\|_2 = 0$, this is **b)**.
The statement **a)** follows immediately from Cauchy formula and Hőlder inequality.

B) Now let us consider $p \neq 2$ and again $\mu = m$, and $\Lambda = \emptyset$. Using the factorization theorem for H^p, we get that there exist $B_n \in H^\infty$, more precisely, Blaschke products, and $h_n \in H^2$, such that

$$f_n(z) = B_n(z) h_n(z)^{2/p} = \frac{B_n(z)}{B_n(0)} (B_n(0)^{p/2} h_n(z))^{2/p} \quad \text{and} \quad \|f_n\|_p^p = \|h_n\|_2^2.$$

We are going to see that $\bar{h}_n(z) = B_n(0)^{p/2} h_n(z) \in H^2$ holds the conditions studied in A).

$$1 = \lim_{n\to\infty} |f_n(0)|^{p/2} = \lim_{n\to\infty} |B_n(0)^{p/2} h_n(0)| \le \lim_{n\to\infty} \|B_n(0)^{p/2} h_n\|_2 =$$

$$\lim_{n\to\infty} |B_n(0)|^{p/2} \|h_n\|_2 = \lim_{n\to\infty} |B_n(0)|^{p/2} \le 1,$$

because $|B_n(z)| = 1$ if $|z| = 1$ and from the maximun principle the inequality follows.
Hence

$$\lim_{n\to\infty} |B_n(0)|^{p/2} = 1, \tag{9}$$

and we obtain $\lim\limits_{n\to\infty} \|\bar{h}_n\|_2 = 1$. Then, from the previous case, we have **a)** and **b)** for $\bar{h}_n$. The same holds for $\left\{\bar{B}_n(z) = \dfrac{B_n(z)}{B_n(0)}\right\}$, $\bar{B}_n \in H^\infty \subset H^2$. Hence $\{\bar{B}_n\}$ holds **a)** and **b)**. Then, we have **a)** for f_n.
It remains to see that **b)** holds. Since **b)** holds for $\bar{h}_n$ and $\bar{B}_n$, there exists $\{n_j\} \subset \Gamma$ such that

$$\lim_j \bar{h}_{n_j}(z) = 1, \; a.e. \quad \text{and} \quad \lim_j \bar{B}_{n_j}(z) = 1, \; a.e..$$

Using Lemma 2 the proof of this case is completed.

C) In this step we consider a general $\mu \in \mathbf{S}$ and again $\Lambda = \emptyset$. The main idea is to apply the previous argument to $\tilde{f}_n$. In fact,

$$\lim_{n\to\infty} \|f_n\|^p_{p,\mu} = \lim_{n\to\infty} \|K_p \tilde{f}_n\|^p_{p,\mu_a} + \|f_{n,s}\|^p_{p,\mu_s} = D_p(\mu, 0)^p,$$

and this yields

$$\limsup_{n\to\infty} \|K_p \tilde{f}_n\|^p_{p,\mu_a} \le D_p(\mu, 0)^p.$$

Then

$$\limsup_{n\to\infty} \int |K_p(e^{i\theta}) \tilde{f}_n(e^{i\theta})|^p \mu'(\theta) \mathrm{d}m(\theta) =$$

$$D_p(\mu, 0)^p \limsup_{n\to\infty} \int |\tilde{f}_n(e^{i\theta})|^p \mathrm{d}m(\theta) \le D_p(\mu, 0)^p,$$

hence $\limsup_{n\to\infty} \|\tilde{f}_n\|_p^p \leq 1$. From the case analyzed above, we get $\lim_{n\to\infty} \tilde{f}_n(z) = 1$ uniformly on compact subsets of $\mathbb{D}$ and $\lim_{n\to\infty} \|\tilde{f}_n - 1\|_p = 0$. Therefore $\lim_{n\to\infty} \|\tilde{f}_n\|_p = 1$ and $\lim_{n\to\infty} \|K_p\tilde{f}_n\|_{p,\mu_a} = D_p(\mu, 0)$. Thus $\lim_{n\to\infty} \|f_{n,s}\|_{p,\mu_s} = 0$ and as a consequence

$$\lim_{n\to\infty} \|f_n - K_p\|_{p,\mu}^p = \lim_{n\to\infty} \left(\|f_n - K_p\|_{p,\mu_a}^p + \|f_n - K_p\|_{p,\mu_s}^p\right) =$$

$$\lim_{n\to\infty} \left(\|K_p\tilde{f}_n - K_p\|_{p,\mu_a}^p + \|f_{n,s}\|_{p,\mu_s}^p\right) =$$

$$\lim_{n\to\infty} \left(D_p(\mu, 0)^p \|\tilde{f}_n - 1\|_p^p + \|f_{n,s}\|_{p,\mu_s}^p\right) = 0.$$

D) Finally, we prove the general case.

Let $f_n = K_p\tilde{f}_n + f_{n,s}$ with $\tilde{f}_n \in H^p$ and $\|f_n\|_{\mu,p} \geq \|K_p\tilde{f}_n\|_{\mu,p}$. Then $\limsup_{n\to\infty} \|\tilde{f}_n\|_p \leq \dfrac{1}{\prod |z_i|^p}$ from **iv)**. We are going to see that all convergent subsequence of $\tilde{f}_n$ converges to the same limit uniformly on each compact subset of $\mathbb{D}$.

Let $\tilde{f}$ be a limit function. From **ii)**, $\tilde{f}(z_i) = 0$, $\tilde{f} \in H^p$ and $\|\tilde{f}\|_p \leq \dfrac{1}{\prod |z_i|^p}$,

$$\tilde{f}(z) = \prod \frac{z - z_i}{z\bar{z}_i - 1} \frac{\bar{z}_i}{|z_i|^2} \prod \frac{z - w_i}{z\bar{w}_i - 1} \frac{\bar{w}_i}{|w_i|^2} h(z)$$

where $\{z_i\}$, $\{w_i\}$ are zeros of $\tilde{f}$, h is a zero-free function in H^p, $h(0) = 1$, and $\|\tilde{f}\|_p = \prod \dfrac{1}{|z_i|^p} \prod \dfrac{1}{|w_i|^p} \|h\|_p$. So $\|h\|_p \leq 1$ and, as a consequence, $h \equiv 1$. Therefore, the set $\{w_i\}$ is empty and $\tilde{f}(z) = \prod \dfrac{z - z_i}{z\bar{z}_i - 1} \dfrac{\bar{z}_i}{|z_i|^2}$. Moreover, we have $\lim_{n\to\infty} \tilde{f}_n(z) = \tilde{f}(z)$ uniformly on each compact subset of $\mathbb{D}$ and $\|\tilde{f}_n\|_p \leq \|\tilde{f}\|_p$. Then from Theorem 1, we obtain $\lim_{n\to\infty} \|\tilde{f}_n - \tilde{f}\|_p = 0$ and this is the same that $\lim_{n\to\infty} \|K_p\tilde{f}_n - K_p\tilde{f}\|_{p,\mu_a} = 0$. In particular, $\lim_{n\to\infty} \|K_p\tilde{f}_n\|_{p,\mu_a} = \|K_p\tilde{f}\|_{p,\mu_a} = \dfrac{1}{\prod |z_i|^p} D_p(\mu, 0)$, and then $\lim_{n\to\infty} \|f_n\|_{p,\mu_s} = 0$. Therefore, we obtain **b)**.

□

4 Asymptotics for extremal Sobolev polynomials

Proof of Theorem 3

Proof. **(i)⇒(ii)**

We are going to prove that $\lim_{n\to\infty} \frac{\tau_{n,\,\mu_0,\dots,\mu_k,\,p}}{n^k} = D_p(\mu_k, 0) > 0$. Obviously $\tau_{n,\,\mu_0,\dots,\mu_k,\,p} \geq \|P_n^{(k)}\|_{p,\,\mu_k} \geq (n)_k \tau_{n-k,\,\mu_k,\,p}$, with $(n)_k = n(n-1)\dots(n-k+1)$. Since μ_k satisfies the Szegő's condition, as a consequence of the Geronimus theorem we obtain $\lim_{n\to\infty} \tau_{n-k,\,\mu_k,\,p} = D_p(\mu_k, 0)$. Thus

$$\liminf_{n\to\infty} \frac{\tau_{n,\,\mu_0,\dots,\mu_k,\,p}}{n^k} \geq \liminf_{n\to\infty} \frac{(n)_k \tau_{n-k,\,\mu_k,\,p}}{n^k} = \liminf_{n\to\infty} \tau_{n-k,\,\mu_k,\,p} = D_p(\mu_k, 0).$$

Now we prove

$$\limsup_{n\to\infty} \frac{\tau_{n,\,\mu_0,\dots,\mu_k,\,p}}{n^k} \leq D_p(\mu_k, 0). \tag{10}$$

Let Q_n be the monic polynomial of degree n minimizing the norm $\|\cdot\|_{p,\mu_k}$; since we are on $\mathbb{T}$ and using Minkowski's inequality, we get

$$\tau_{n+m,\{\mu_j\},p} \leq \sum_{j=0}^{k} \|(z^m Q_n)^{(j)}\|_{p,\mu_j} = \sum_{j=0}^{k} \left\| \sum_{i=0}^{j} \binom{j}{i} (z^m)^{(i)} (Q_n)^{(j-i)} \right\|_{p,\mu_j}$$

$$\leq (m)_k \|Q_n\|_{p,\mu_k} + f(n) o(m^k).$$

Dividing these inequalities through by m^k and taking limits (first, $m \to \infty$, and then $n \to \infty$) we obtain (10).

(ii)⇒(i)

Set $k = 1$ and assume that μ_1 does not satisfy the Szegő's condition. Then from the Geronimus theorem $\lim_{n\to\infty} \tau_{n,\,\mu_1,\,p} = 0$. For a fixed $\epsilon > 0$, there exists $n_0(\epsilon)$ such that for $n \geq n_0(\epsilon)$ the set

$$\{Q : Q(z) = z^n + \cdots,\ \|Q\|_{p,\,\mu_1} \leq \epsilon\}$$

is non empty. For each $n \geq n_0$ we consider the extremal problem

$$\alpha_{n,\,\mu_0,\,\mu_1,\,p}(\epsilon) = \inf\{\|Q\|_{p,\,\mu_0} + \|Q'\|_{p,\,\mu_1} : \\ Q(z) = z^n + \cdots,\ \|Q\|_{p,\,\mu_1} \leq \epsilon\}.$$

It is obvious $\tau_{n,\,\mu_0,\,\mu_1,\,p} \leq \alpha_{n,\,\mu_0,\,\mu_1,\,p}(\epsilon)$ and through the same argument as before for n large enough we have $\alpha_{n+n_0,\,\mu_0,\,\mu_1,\,p}(\epsilon) \leq \alpha_{n_0,\,\mu_0,\,\mu_1,\,p}(\epsilon) + n\epsilon$. Hence, we have $\limsup\limits_{n\to\infty} \dfrac{\tau_{n,\,\mu_0,\,\mu_1,\,p}}{n} \leq \limsup\limits_{n\to\infty} \dfrac{\alpha_{n,\,\mu_0,\,\mu_1,\,p}(\epsilon)}{n} \leq \epsilon$, and this is a contradiction.

Now by induction we obtain the general case.

(i)⇒(iii)

If Q is a polynomial of degree n, then $Q^*(z) = z^n\overline{Q(\frac{1}{\bar{z}})}$ and if $|z| = 1$, then $|Q(z)| = |Q^*(z)|$. So

$$\tau_{n,\,\mu_0,\dots,\mu_k,\,p} \geq \|(P_n^{(k)})^*\|_{p,\,\mu_k} \geq n^{(k)} D_p(\mu_k, 0).$$

Hence

$$\lim_{n\to\infty} \left\| \frac{(P_n^{(k)})^*}{n^k} \right\|_{p,\,\mu_k} = D_p(\mu_k, 0).$$

Therefore, the sequence of functions $\{\dfrac{(P_n^{(k)}(z))^*}{n^k}\}$ holds the hypothesis of Theorem 2, and hence iii) is proved.

(iii)⇒(ii)

From **(iii)** we have

$$\lim_{n\to\infty} \int \left| \frac{P_n^{(k)}(z)}{n^k z^{n-k}} \right|^p \mu_k'(\theta)\mathrm{d}m(\theta) = \int \left| \overline{\Delta(\frac{1}{\bar{z}})} \right|^p \mu_k'(\theta)\mathrm{d}m(\theta) > 0.$$

On the other hand

$$\tau_{n,\,\mu_0,\dots,\mu_k,\,p} \geq \|(P_n^{(k)})^*\|_{p,\,\mu_k} \geq \left(\int \left| \frac{P_n^{(k)}(z)}{z^{n-k}} \right|^p \mu_k'(\theta)\mathrm{d}m(\theta) \right)^{1/p}.$$

Hence

$$\lim_{n\to\infty} \frac{\tau_{n,\,\mu_0,\dots,\mu_k,\,p}}{n^k} \geq \left(\int \left| \frac{P_n^{(k)}(z)}{n^k z^{n-k}} \right|^p \mu_k'(\theta)\mathrm{d}m(\theta) \right)^{1/p} > 0.$$

□

Proof of Theorem 4

Proof. Let us consider $l = k-1$ and assume $\mu_{k-1}, \mu_k \in \mathbf{S}$. By definition $\dfrac{\tau_{n,\,\mu_0,\dots,\mu_k,\,p}}{n^k} \geq \dfrac{\|(P_n^{(k-1)})^*\|_{p,\,\mu_{k-1}}}{n^k} + \dfrac{\|(P_n^{(k)})^*\|_{p,\,\mu_k}}{n^k}$. Because of $\mu_k \in \mathbf{S}$, from (ii) in Theorem 3 we get $\lim_{n\to\infty} \left\| \dfrac{P_n^{(k-1)}}{n^k z^{n-k+1}} \right\|_{p,\,\mu_{k-1}} = 0$. Hence, using $\mu_{k-1} \in \mathbf{S}$, the Cauchy integral formula, and Hőlder inequality, we obtain $\lim_{n\to\infty} \dfrac{P_n^{(k-1)}(z)}{n^k z^{n-k+1}} = 0$, uniformly on each compact subset of $\mathbb{E}$. Thus $\lim_{n\to\infty} \left(\dfrac{P_n^{(k-1)}(z)}{n^k z^{n-k+1}} \right)' = 0$. Taking into account $\dfrac{(n-k+1)P_n^{(k-1)}(z)}{n^k z^{n-k+2}} = \dfrac{P_n^{(k)}(z)}{n^k z^{n-k+1}} - \left(\dfrac{P_n^{(k-1)}(z)}{n^k z^{n-k+1}} \right)'$, for $l = k-1$ (6) follows if it holds for $l = k$,

$$\lim_{n\to\infty} \frac{P_n^{(k-1)}(z)}{n^{k-1} z^{n-k+1}} = \lim_{n\to\infty} \frac{(n-k+1)P_n^{(k-1)}(z)}{n^k z^{n-k+1}} =$$

$$\lim_{n\to\infty} \frac{P_n^{(k)}(z)}{n^k z^{n-k}} - z \left(\frac{P_n^{(k-1)}(z)}{n^k z^{n-k+1}} \right)' = \frac{D_p(\mu_k, 0)}{\overline{D_p(\mu_k, 1/\bar{z})}}.$$

Repeating this reasoning, we obtain the corresponding results for all l, with $j \leq l \leq k$. □

Other extremal problems can be considered. For example, let $0 < p_0,\, p_1, \dots, p_k < \infty$ and $\mu_0,\, \mu_1, \dots, \mu_k$ be positive Borel measures in $[0, 2\pi)$, set

$$\inf\{\sum_{j=0}^{k} \|Q^{(j)}\|_{p_j,\,\mu_j} : Q(z) = z^n + \cdots\}, \text{ or}$$

$$\inf\{(\sum_{j=0}^{k} \|Q^{(j)}\|_{p,\,\mu_j})^{1/p} : Q(z) = z^n + \cdots\}.$$

Of course, similar asymptotic results for the corresponding extremal polynomials can be proved.

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