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COMPARATIVE ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS WITH RESPECT TO VARYING MEASURES

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Comparative asymptotics for orthogonal polynomials with respect to varying measures

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Abstract

We study comparative asymptotics for orthogonal polynomials with respect to varying measures. These results are used to give comparative asymptotics of families of orthogonal polynomials with respect to a fixed measure.

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1 Introduction

1.1 Motivation

One of the most important results in the theory of orthogonal polynomials is the Szegő asymptotic formula (see Theorem 12.1.1, page 297 of [22] or Theorem C here). This formula can be understood as a relation between two families of orthogonal polynomials (comparative asymptotics). In this direction many extensions of the Szegő theory have been published (see [12]), [17], [18], and [21]).

The purpose of this paper is to describe comparative asymptotic formulas for orthogonal polynomials with respect to varying measures. Some results of this kind can be seen in the paper [12]; this paper also shows that the orthogonal polynomials with respect to varying measures are a powerful tool in solving problems where a fixed measure and orthogonality in the usual sense are involved. Some other applications can be seen in [2], [4], [5], [9], [12], [16], [20], and [23]; in these papers, some problems are transferred to varying measures on the unit circle. Sequences of polynomials which are orthogonal with respect to varying measures arise naturally in the study of the convergence of sequences of rational functions which interpolate a given analytic function along a table of interpolation points (see [8] and [10]).

Two kind of results have been used in the proofs given here. Namely, some algebraic results, the main of which is the relationship between families of orthogonal polynomials whose measures are related by a rational modification of the type $|z-\zeta|^2$, $|\zeta| \le 1$ (see Theorem 7 in this respect, see also [7] and [21]). Moreover, some results from the theory of polynomials orthogonal with respect to varying measures are used.

1.2 Some notations and main results

Let M denote the set of positive Borel measures on $[0, 2\pi)$ with an infinite set of points in its support. For each $\mu \in M$, let $\varphi_m(z) := \varphi_m(\mu; z), m = 0, 1 \ldots$, be the orthonormal polynomials associated with μ:

$$\varphi_m(z) = \kappa_m z^m + \text{lower degree terms}; \quad \kappa_m = \kappa_m(\mu) > 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_k(z)\overline{\varphi_m(z)}d\mu(\theta) = \begin{cases} 0, & k \neq m \\ 1, & k = m \end{cases} ; \quad z = e^{i\theta}.$$

All the integrals are taken on $[0, 2\pi)$; therefore, from now on we do not write the integration interval.

We will always write with capital letter, $\Phi_m(\mu; z) = \Phi_m(z)$, the respective monic orthogonal polynomial.

Let $\{\omega_{n,j} : j = 1, \ldots, n\}_{n=1}^{\infty}$ be a triangular array of complex numbers on the unit disk, i.e. $|\omega_{n,j}| \leq 1$. Set $W_n(z) = \prod_{j=1}^{n}(z - \omega_{n,j})$, $n \geq 1$. For each $\mu \in M$ such that $\int \frac{d\mu}{|W_n(z)|^2} < \infty$ ($z = e^{i\theta}$, $n \in \mathbb{N}$), we consider the sequence of measures $\{d\mu_n := \frac{d\mu}{|W_n(z)|^2}\}_{n\in\mathbb{N}}$ in M. For $n \in \mathbb{N}$, let $\{\varphi_{n,m}(z) = \varphi_m(\mu_n; z)\}_{m=0}^{\infty}$ be the sequence of orthonormal polynomials associated with μ_n, $\varphi_{n,m}(z) = \alpha_{n,m}z^m + \ldots$, and $\alpha_{n,m} > 0$. Of course, if $\omega_{n,j} = 0$, $j = 1, 2, \cdots, n$, then $|W_n(e^{i\theta})| = 1$, $\theta \in [0, 2\pi)$, and the orthogonal polynomials with respect to these varying measures become the orthogonal polynomials with respect to the fixed measure μ.

Moreover, when $W_n(z) = \varphi_n(z)$, using the Geronimus identity (see [6], pages 198 and 199)

$$\int \frac{z^j}{|\varphi_n(z)|^2} d\theta = \int z^j \, d\mu, \quad j = 0, \pm 1 \ldots, \pm n, \; z = e^{i\theta}, \tag{1}$$

it follows that $\varphi_{n,m}(z) = \varphi_m(z)$, $m = 0, \ldots, n$.

The following definition was introduced by G. López (see [14]).

Definition 1. *Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\mu, \{W_n\}, k)$ is admissible on $\{z \in \mathbb{C} : |z| = 1\}$ if:*

(i) $\mu' > 0$ *almost everywhere;*

(ii) $\int d\mu_n < \infty$, $\quad n \in \mathbb{N}$;

(iii) *In the case of $k < 0$, there exist $j_1, \ldots, j_{-k}$ such that*

$$\int \prod_{i=1}^{-k} |z - w_{n,j_i}|^{-2} \, d\mu \leq M < \infty, \quad n \geq -k;$$

(iv) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = \infty$.

We consider k_1, $k_2 \in \mathbb{N}$. Let $\{\zeta_{n,j}^{(1)} : j = 1, \ldots, k_1\}_{n=1}^{\infty}$ and $\{\zeta_{n,j}^{(2)} : j = 1, \ldots, k_2\}_{n=1}^{\infty}$ be two arrays of complex numbers on the unit disk. Set $V_n^{(i)}(z) = \prod_{j=1}^{k_i}(z - \zeta_{n,j}^{(i)})$, $\quad i = 1, 2$. For each

$\mu \in M$ such that $\int \left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n < \infty$ and $n \in \mathbb{N}$, let $\{\psi_{n,m}(z) = \varphi_m(\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n; z)\}_{m=0}^{\infty}$ be the sequence of polynomials orthonormal with respect to $\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n$. We denote by $\{\beta_{n,m}\}_{m=0}^{\infty}$ their sequence of positive leading coefficients.

Theorem 2. *Let $\mu \in M$. We consider a triangular array of complex numbers on the unit disk $\{\omega_{n,j} : j = 1, \dots, n\}_{n=1}^{\infty}$ and set $W_n(z) = \prod_{j=1}^{n}(z - \omega_{n,j})$, $n \geq 1$. If $(\mu, \{W_n\}, -k_2)$ is admissible, then*

$$\lim_{n\to\infty} \frac{\psi_{n,n+k_1}(z)}{\varphi_{n,n+k_2}(z)} - z^{2(k_1-k_2)} \frac{V_n^{(2)}(z)}{V_n^{(1)}(z)} = 0,$$

uniformly on each compact subset of $|z| > 1$. Besides

$$\lim_{n\to\infty} \frac{\beta_{n,n+k_1}}{\alpha_{n,n+k_2}} = 1.$$

Remark 3. *The factor $\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2$ is equal to*

$$\frac{z^{k_2-k_1} V_n^{(1)}(z) \prod_{j=1}^{k_1}(1 - \overline{\zeta_{n,j}^{(1)}} z)}{V_n^{(2)}(z) \prod_{j=1}^{k_2}(1 - \overline{\zeta_{n,j}^{(2)}} z)}, \quad |z| = 1,$$

thus Theorem 2 shows that zeros and poles out of the unit circle of the analytic extension of the weight do not play any role (see [13], Theorem 6).

Using similar techniques, a comparative asymptotics for families of polynomials orthogonal with respect to varying measures and with respect to a fixed measure, is obtained.

Theorem 4. *Let μ, $\sigma \in M$ such that $\lim_{n\to\infty} \Phi_n(\sigma, 0) = 0$ and $\log \mu' \in L_1$. If $\{\psi_{n,m}(z)\}_{m=0}^{\infty}$ denotes the sequence of orthonormal polynomials with respect to $\frac{|V_n^{(1)}(z)|^2 d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}$, then*

$$\lim_{n\to\infty} \frac{\psi_{n,n}(z)\overline{D(\mu'; \overline{z}^{-1})}}{\varphi_n(\sigma; z)} - \frac{V_n^{(1)}(z)}{V_n^{(2)}(z)} = 0,$$

uniformly on each compact subset of $|z| > 1$ and

$$\lim_{n\to\infty} \frac{\beta_{n,n}}{\kappa_n(\sigma)} = \exp\left\{-\frac{1}{4\pi}\int \log \mu'(\theta)\right\}$$

As usual,

$$D(\mu'; \zeta) = \exp\left(\frac{1}{4\pi}\int \log \mu'(\theta) \frac{z+\zeta}{z-\zeta} d\theta\right), \; z = e^{i\theta}, \; |\zeta| < 1,$$

is the Szegő function for μ'.

The outline of this paper is as follows. In Section 2 we give some auxiliary results. Theorems 2 and 4 are proved in Section 3. Theorems 7 and 8 seem to be interesting on their own.

2 Auxiliary results

1 If $p(z)$ is a polynomial of degree n, the reverse polynomial is usually defined as $p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Of course, if $p(z) \neq 0$ for $|z| = 1$, then

$$\left|\frac{p^*(z)}{p(z)}\right| = 1, \quad |z| = 1. \tag{2}$$

Let $\{\sigma_n\}$ be a sequence of measures in M. It is very well known that the monic polynomials associated to each σ_n satisfy the following recurrence relations:

$$\Phi_{m+1}(\sigma_n; z) = z\Phi_m(\sigma_n; z) + \Phi_{m+1}(\sigma_n; 0)\Phi_m^*(\sigma_n; z); \tag{3}$$

$$\Phi_{m+1}^*(\sigma_n; z) = \Phi_m^*(\sigma_n; z) + \overline{\Phi_{m+1}(\sigma_n; 0)}\, z\Phi_m(\sigma_n; z).$$

and for the leading coefficients we have

$$\frac{\kappa_m(\sigma_n)^2}{\kappa_{m+1}(\sigma_n)^2} = 1 - |\Phi_{m+1}(\sigma_n; 0)|^2. \tag{4}$$

These formulae lead us to another known result:

Theorem A. *Let $l \in \mathbb{Z}$. The following conditions are equivalent:*

$$\lim_{n\to\infty} \Phi_{n+l+1}(\sigma_n; 0) = 0;$$

$$\lim_{n\to\infty} \frac{\kappa_{n+l+1}(\sigma_n)}{\kappa_{n+l}(\sigma_n)} = 1;$$

$$\lim_{n\to\infty} \frac{\varphi_{n+l+1}(\sigma_n; z)}{\varphi_{n+l}(\sigma_n; z)} = \lim_{n\to\infty} \frac{\Phi_{n+l+1}(\sigma_n; z)}{\Phi_{n+l}(\sigma_n; z)} = z, \quad |z| \geq 1,$$

where the convergence in the last limits are uniform on each compact subset of the prescribed region.

The reproducing kernel $K_m(\sigma_n; z, y)$ associated with the measure σ_n is defined in the usual way by

$$K_m(\sigma_n; z, y) = \sum_{j=0}^{m} \overline{\varphi_j(\sigma_n; y)}\varphi_j(\sigma_n; z).$$

The following Christoffel-Darboux formulae are also well known. If $z\bar{y} \neq 1$

$$K_m(\sigma_n; z, y) = \frac{\overline{\varphi_{m+1}^*(\sigma_n; y)}\varphi_{m+1}^*(\sigma_n; z) - \overline{\varphi_{m+1}(\sigma_n; y)}\varphi_{m+1}(\sigma_n; z)}{1 - z\bar{y}}; \tag{5}$$

$$K_m(\sigma_n; z, y) = \frac{\overline{\varphi_m^*(\sigma_n; y)}\varphi_m^*(\sigma_n; z) - z\bar{y}\overline{\varphi_m(\sigma_n; y)}\varphi_m(\sigma_n; z)}{1 - z\bar{y}}, \tag{6}$$

Besides

$$K_m(\sigma_n; z, 0) = \kappa_m(\sigma_n)\varphi_m^*(\sigma_n; z).$$

The $m-$kernel is characterized by the reproducing properties:

Lemma 5. **(1)** *Let $p(z)$ be a polynomial of degree at most m, then*

$$\frac{1}{2\pi}\int \overline{K_m(\sigma_n; z, y)}p(z)d\sigma_n = p(y). \tag{7}$$

(2) *If $p(z) = a_{m+1}z^{m+1}$ + lower degree terms, then*

$$\frac{1}{2\pi}\int \overline{K_m(\sigma_n; z, y)}p(z)d\sigma_n = p(y) - \frac{a_{m+1}}{\kappa_{m+1}(\sigma_n)}\varphi_{m+1}(\sigma_n; y). \tag{8}$$

Since

$$\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2}; z) - \frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)}(z-\zeta)\varphi_m(\sigma_n; z)$$

is a polynomial of degree m and has the reproducing property (7), we obtain (see also [7], Corollary 2):

Corollary 6. *If $|\zeta| \leq 1$ and such that $\int \frac{d\sigma_n}{|z-\zeta|^2} < \infty$, then*

$$\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2}; z) = \frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)}(z-\zeta)\varphi_m(\sigma_n; z) + \frac{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2}; \zeta)}{K_m(\frac{d\sigma_n}{|z-\zeta|^2}; \zeta, \zeta)}K_m(\frac{d\sigma_n}{|z-\zeta|^2}; z, \zeta), \ n \geq 0. \tag{9}$$

In ([13], Theorem 3) G. López proved the following result:

Theorem B. *Let (μ, $\{W_n\}$, l) be admissible on $\{z \in \mathbb{C} : |z| = 1\}$. Then $\lim_{n\to\infty} \Phi_{n,n+m+1}(0) = 0$ for all $m \geq l$.*

Also we shall need the following two theorems:

Theorem C. (see [15], Theorem 2; this theorem has been improved in [3]) *Let $\sigma \in M$ such that $\int \log \sigma'(\theta)\, d\theta > -\infty$. Let $\{\omega_{n,j} : j = 1, \ldots, n\}_{n=1}^{\infty}$ be a triangular array of complex numbers on the unit disk such that $\lim_{n\to\infty} \sum_{j=1}^{n}(1 - |w_{n,j}|) = \infty$. Let $\varphi_{n,m}(z) = \varphi_m(\frac{d\sigma}{|W_n(z)|^2}; z)$ be the orthogonal polynomials with respect to $\frac{d\sigma}{|W_n(z)|^2}$. Then*

$$\lim_{n\to\infty} \frac{\varphi_{n,n}(z)}{W_n(z)} = \left(\overline{D(\sigma'; 1/\overline{z})}\right)^{-1},$$

uniformly on each compact subset of $|z| > 1$.

Theorem D. (see [1]) *Let* $\{z_{n,j}\}_{j=1}^{n}$ *denote the zeros of the orthogonal polynomial* $\varphi_n(z)$. *The following statements are equivalent:*

(a) $\lim_{n\to\infty} \Phi_n(0) = 0$.

(b) $\lim_{n\to\infty} \sum_{j=1}^{n}(1 - |z_{n,j}|) = \infty$.

2 Throughout this paragraph let us consider $\zeta \in \{|z| \leq 1\}$ and we set σ_n as in the previous paragraph. We also assume that $\int \frac{d\sigma_n(\theta)}{|z-\zeta|^2} < \infty$, $z = e^{i\theta}$. Rakhmanov proved the following formula in [21], page 157:

$$(z-\zeta)(z-\zeta^*)\Phi_m(|z-\zeta|^2 d\sigma_n; z) = \Phi_{m+2}(\sigma_n; z) + c_{n,m}\,\Phi_{m+1}(\sigma_n; z) + d_{n,m}\,\Phi_m^*(\sigma_n; z), \qquad (10)$$

where

$$d_{n,m} = \zeta^*\,\Phi_{m+1}(\sigma_n; 0)\frac{\kappa_m^2(\sigma_n)}{\kappa_m^2(|z-\zeta|^2\sigma_n)},$$

$$c_{n,m} = -\frac{\Phi_{m+2}(\sigma_n;\zeta^*)}{\Phi_{m+1}(\sigma_n;\zeta^*)} - d_{n,m}\frac{\Phi_m^*(\sigma_n;\zeta^*)}{\Phi_{m+1}(\sigma_n;\zeta^*)},$$

and $\zeta^* = 1/\overline{\zeta}$.

Some other algebraic relations that we will also need are going to be obtained in what follows. The next one can be seen as an inverse formula of (10) or as extension of the recurrence relation (3):

Theorem 7. *For* $m \geq 0$, *the following equality holds:*

$$\Phi_{m+1}\Big(\frac{d\sigma_n}{|z-\zeta|^2}; z\Big) = \left(z - \frac{\kappa_m^2(\sigma_n)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}\zeta\right)\Phi_m(\sigma_n; z) + \left(\Phi_{m+1}\Big(\frac{d\sigma_n}{|z-\zeta|^2}; 0\Big) + \zeta\frac{\kappa_m^2(\sigma_n)\Phi_m(\sigma_n;0)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}\right)\Phi_m^*(\sigma_n; z). \qquad (11)$$

Proof. Let us consider the polynomial

$$\Pi_m(z) = \varphi_{m+1}\Big(\frac{d\sigma_n}{|z-\zeta|^2}; z\Big) - \frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)}\left(z - \frac{\kappa_m^2(\sigma_n)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}\zeta\right)\varphi_m(\sigma_n; z),$$

it is obvious that the degree of Π_m is at most m. Moreover, from the orthonormality relation, simple computations give us

$$\int \Pi_m^*(z)\,\overline{z^k}\,d\sigma_n = \int z^{m-k}(z-\zeta)(\frac{1}{z}-\overline{\zeta})\overline{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};z)}\,\frac{d\sigma_n}{|z-\zeta|^2}$$
$$-\frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)}\int z^{m-k}\left(\frac{1}{z}-\frac{\kappa_m^2(\sigma_n)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}\overline{\zeta}\right)\overline{\varphi_m(\sigma_n;z)}\,d\sigma_n$$
$$=0, \quad k=0,\ldots,m-1.$$

Thus the proof is completed since (11) is true at $z=0$. ∎

Moreover, in this framework the respective extension of (4) is:

Theorem 8. *The leading coefficients satisfy*

$$\frac{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m^2(\sigma_n)} = 1 + \frac{|\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)|^2}{K_m(\frac{d\sigma_n}{|z-\zeta|^2};\zeta,\zeta)}, \qquad m\geq 0 \tag{12}$$

Proof. ¿From the orthonormality we have

$$\frac{1}{2\pi}\int \varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};z)\overline{(z-\zeta)\varphi_m(\sigma_n;z)}\frac{d\sigma_n}{|z-\zeta|^2} = \frac{\kappa_m^2(\sigma_n)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})};$$
$$\frac{1}{2\pi}\int (z-\zeta)\varphi_m(\sigma_n;z)\overline{(z-\zeta)\varphi_m(\sigma_n;z)}\frac{d\sigma_n}{|z-\zeta|^2} = \frac{1}{2\pi}\int |\varphi_m(\sigma_n;z)|^2 d\sigma_n = 1;$$

taking into account (8) we obtain

$$\frac{1}{2\pi}\int K_m(\frac{d\sigma_n}{|z-\zeta|^2};z,\zeta)\overline{(z-\zeta)\varphi_m(\sigma_n;z)}\frac{d\sigma_n}{|z-\zeta|^2} = -\frac{\kappa_m(\sigma_n)}{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}\overline{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}.$$

Therefore, we prove (12) multiplying both sides of (9) by $\overline{(z-\zeta)\varphi_m(\sigma_n;z)}$ and integrating with respect to $\frac{d\sigma_n}{|z-\zeta|^2}$. ∎

Since $0\leq K_m(\frac{d\sigma_n}{|z-\zeta|^2};\zeta,\zeta)$ from the previous theorem we have:

Corollary 9. *For each $m\geq 0$,*

$$1\leq \frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)}.$$

Corollary 10. *Let us suppose that $|\zeta|\leq r<1$. Then*

1. $\lim_{m\to\infty}\frac{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m(\sigma_n)} = 1 \iff \lim_{m\to\infty}\frac{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}{\varphi_{m+1}^*(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)} = 0;$
2. $\lim_{m\to\infty}\frac{\kappa_m(\sigma_n)}{\kappa_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2})} = 0 \iff \lim_{m\to\infty}\left|\frac{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}{\varphi_{m+1}^*(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}\right| = 1.$

Proof. Both results are obtained easily if we write (12) in the form

$$\frac{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})}{\kappa_m^2(\sigma_n)} = \frac{K_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta,\zeta)}{K_m(\frac{d\sigma_n}{|z-\zeta|^2};\zeta,\zeta)}$$

and we use the Christoffel-Darboux formulas (5) and (6) to deduce

$$\frac{\kappa_m^2(\sigma_n)}{\kappa_{m+1}^2(\frac{d\sigma_n}{|z-\zeta|^2})} = \frac{1-\left|\frac{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}{\varphi_{m+1}^*(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}\right|^2}{1-|\zeta|^2\left|\frac{\varphi_{m+1}(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}{\varphi_{m+1}^*(\frac{d\sigma_n}{|z-\zeta|^2};\zeta)}\right|^2}.$$

■

3 During this paragraph let us consider $\{\sigma_n\}$ a sequence of measures in M and $\{\zeta_n\}$ a sequence of points in the unit disk such that $\int \frac{d\sigma_n}{|z-\zeta_n|^2} < \infty$.

Lemma 11. *Let $l \in \mathbb{Z}$. If $\lim_{n\to\infty} \Phi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};0) = 0$, then we have*

$$\lim_{n\in\Lambda} \frac{1}{2\pi}\int \left|\frac{\varphi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l}(\sigma_n;z)}\right|^2 d\theta = \lim_{n\in\Lambda}(1+|\zeta_n|^2),\ z=e^{i\theta},$$

where this equality means that if Λ is a sequence of indexes such that one of the limits exists, then the other one also exists and both are equal.

Proof. The main tool in proving this lemma is to use together Theorem A and (1), twice. Some additional calculus allow us to write:

$$\begin{aligned}
\lim_{n\in\Lambda} \frac{1}{2\pi}\int \left|\frac{\varphi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l}(\sigma_n;z)}\right|^2 d\theta &= \lim_{n\in\Lambda} \frac{1}{2\pi}\int \left|\frac{\varphi_{n+l}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l}(\sigma_n;z)}\right|^2 d\theta \\
&= \lim_{n\in\Lambda} \frac{1}{2\pi}\int \left|(z-\zeta_n)\varphi_{n+l}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)\right|^2 \frac{d\sigma_n}{|z-\zeta_n|^2} \\
&= \lim_{n\in\Lambda} \frac{1}{2\pi}\int \left|(z-\zeta_n)\frac{\varphi_{n+l}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}\right|^2 d\theta \\
&= \lim_{n\in\Lambda}(1+|\zeta_n|^2).
\end{aligned}$$

■

Corollary 12. *If $\lim_{n\to\infty} \Phi_{n+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};0) = 0$, then*

$$\limsup_{n\to\infty} \frac{\kappa_{n+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}{\kappa_n(\sigma_n)} \le \sqrt{2}.$$

Proof. The inequality follows from

$$\frac{\kappa_{n+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}{\kappa_n(\sigma_n)} \leq \frac{1}{2\pi}\int \left|\frac{\varphi^*_{n+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi^*_n(\sigma_n;z)}\right|^2 d\theta$$
$$= \frac{1}{2\pi}\int \left|\frac{\varphi_{n+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_n(\sigma_n;z)}\right|^2 d\theta,$$

lemma above, and $|\zeta_n| \leq 1$.

■

Lemma 13. *Let $l \in \mathbb{Z}$. If*

$$\lim_{n\to\infty} \Phi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};0) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \Phi_{n+l}(\sigma_n;0) = 0,$$

then we have

$$\lim_{n\to\infty} \frac{\kappa_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}{\kappa_{n+l}(\sigma_n)} = 1 \tag{13}$$

and

$$\lim_{n\to\infty} \left\{\frac{\varphi_{n+l}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l}(\sigma_n;z)} - \frac{z-\zeta_n}{z}\right\} = 0, \tag{14}$$

uniformly on each compact subset of $|z| \geq 1$.

Proof. Using (2), Corollary 9, and the hypothesis in Theorem 7 we obtain

$$\lim_{n\to\infty}\left\{\frac{\varphi_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l}(\sigma_n;z)} - \left(\frac{\kappa_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}{\kappa_{n+l}(\sigma_n)}z - \frac{\kappa_{n+l}(\sigma_n)}{\kappa_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}\zeta_n\right)\right\} = 0, \tag{15}$$

uniformly on each compact subset of $|z| \geq 1$. As a result of (15) we have that (14) follows from (13) and Theorem A. The rest of the proof is devoted to prove (13). Let Λ be a sequence of indexes such that $\lim_{n\in\Lambda} \frac{\kappa_{n+l+1}(\frac{d\sigma_n}{|z-\zeta_n|^2})}{\kappa_{n+l}(\sigma_n)}$ exists and let L be its value. Now let us consider that $\limsup_{n\in\Lambda}|\zeta_n| = |\zeta|$ and take a subsequence of Λ (to avoid excessive notations we also denote this subsequence of indexes by Λ) such that $\lim_{n\in\Lambda}|\zeta_n| = |\zeta|$. Because of Lemma 11 and (15), a simple computation give us the equation $1+|\zeta|^2 = L^2 + L^{-2}|\zeta|^2$; since $L \geq 1$ and $|\zeta| \leq 1$ the unique solution of this equation is $L = 1$. ■

Now if we use Theorem A we obtain:

Corollary 14. *Under conditions of lemma above and moreover if* $\lim_{n\to\infty} \Phi_{n+l+1}(\sigma_n;0) = 0$ *then*

$$\lim_{n\to\infty} \left\{ \frac{\varphi_{n+l}(\frac{d\sigma_n}{|z-\zeta_n|^2};z)}{\varphi_{n+l+1}(\sigma_n;z)} - \frac{z-\zeta_n}{z^2} \right\} = 0,$$

uniformly on each compact subset of $|z| \geq 1$.

Lemma 15. *If* $\lim_{n\to\infty} \Phi_{n+l+1}(\sigma_n;0) = 0$, *for* $l = 0,1$, *then we have*

$$\lim_{n\to\infty} \Phi_{n+l}(|z-\zeta_n|^2 d\sigma_n;0) = 0, \quad \textit{for } l = 0,1; \tag{16}$$

$$\lim_{n\to\infty} \frac{\kappa_n(\sigma_n)}{\kappa_{n+l}(|z-\zeta_n|^2 d\sigma_n)} = 1, \quad \textit{for } l = 0,1; \tag{17}$$

and

$$\lim_{n\to\infty} \left\{ \frac{\varphi_{n+1}(|z-\zeta_n|^2 d\sigma_n;z)}{\varphi_n(\sigma_n;z)} - \frac{z^2}{z-\zeta_n} \right\} = 0, \tag{18}$$

uniformly on each compact subset of $|z| > 1$.

Proof. Given the hypothesis, using (13) and Theorem A (replacing in it σ_n by $|z-\zeta_n|^2 d\sigma_n$), we have that (16) and (17) are equivalent. Moreover, it is enough to prove (16) or (17) for $l = 0$ or for $l = 1$.

Because of Theorem A, hypothesis $\lim_{n\to\infty} \Phi_{n+l+1}(\sigma_n;0) = 0$ for $l = 0,1$ are respectively equivalent to

$$\lim_{n\to\infty} \frac{\Phi_{n+l+1}(\sigma_n;z)}{\Phi_{n+l}(\sigma_n;z)} = z, \quad \text{for } l = 0,1, \tag{19}$$

uniformly on each compact subset of $|z| \geq 1$ and

$$\lim_{n\to\infty} \frac{\kappa_{n+l}(\sigma_n)}{\kappa_{n+l+1}(\sigma_n)} = 1. \tag{20}$$

Besides, replacing ζ by ζ_n and m by n in (10) and dividing both sides of this new equation by $\Phi_n(\sigma_n;z)$ we obtain

$$(z-\zeta_n)(z-\zeta_n^*)\frac{\Phi_n(|z-\zeta_n|^2 d\sigma_n;z)}{\Phi_n(\sigma_n;z)} = \frac{\Phi_{n+2}(\sigma_n;z)}{\Phi_n(\sigma_n;z)} + c_{n,n}\frac{\Phi_{n+1}(\sigma_n;z)}{\Phi_n(\sigma_n;z)} + d_{n,n}\frac{\Phi_n^*(\sigma_n;z)}{\Phi_n(\sigma_n;z)}, \tag{21}$$

then taking limits when $n \to \infty$ and using hypothesis, (19), and (20) we deduce

$$\lim_{n\to\infty} \frac{\Phi_n(|z-\zeta_n|^2 d\sigma_n;z)}{\Phi_n(\sigma_n;z)} - \frac{z}{z-\zeta_n} = 0, \tag{22}$$

uniformly on each compact subset of $|z| > 1$. Thus, if we prove (17) or equivalently (16), then from (22), hypothesis of lemma and Theorem 11 we obtain (18).

Now we consider two cases. First, if $|\zeta_n| < r < 1$ then the equality in (22) is also true for $|z| = 1$. Thus, (17) follows immediately from the definition $\kappa_n(\sigma_n)^{-2} = \frac{1}{2\pi}\int |\Phi_n(\sigma_n; z)|^2\, d\sigma_n$ (similarly for the measure $|z-\zeta_n|^2\, d\sigma_n$) and (22).

Second, if $\limsup |\zeta_n| = 1$ then evaluating at $z = 0$ in (21) we obtain

$$\zeta_n \zeta_n^* \Phi_n(|z-\zeta_n|^2 d\sigma_n; 0) = \Phi_{n+2}(\sigma_n; 0) + c_n \Phi_{n+1}(\sigma_n; 0) + d_n,$$

thus, in this case using again hypothesis of lemma, (19), and (20) we obtain (16). ■

4 In this paragraph we proof some auxiliary lemmas that we shall need in the proof of Theorem 4.

Lemma 16. *Let* $l \in \mathbb{Z}$. *Let* μ, $\sigma \in M$ *such that* $\lim_{n\to\infty} \Phi_n(\sigma; 0) = 0$. *Let* $V_n^{(2)}(z)$ *be as above. Then we achieve*

$$\left| \frac{V_n^{(2)}(z)\varphi_n(\sigma; z)}{\varphi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)} \right|^2 d\theta \underset{n\to\infty}{\overset{*}{\to}} \mu$$

Proof. ¿From Theorem A and hypothesis we have

$$\lim_{n\to\infty} \int z^j \left| \frac{V_n^{(2)}(z)\varphi_n(\sigma; z)}{\varphi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)} \right|^2 d\theta = \lim_{n\to\infty} \int z^j \left| \frac{V_n^{(2)}(z)\varphi_{n-k_2-j}(\sigma; z)}{\varphi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)} \right|^2 d\theta$$

for $j \in \mathbb{Z}$. Using then the Geronimus identity (1) we obtain

$$\lim_{n\to\infty} \int z^j \left| \frac{V_n^{(2)}(z)\varphi_n(\sigma; z)}{\varphi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)} \right|^2 d\theta = \lim_{n\to\infty} \int z^j \left| \frac{V_n^{(2)}(z)\varphi_{n-k_2-j}(\sigma; z) d\mu(\theta)}{V_n^{(2)}(z)\varphi_n(\sigma; z)} \right|^2 d\mu.$$

The proof is completed using again Theorem A and the well known result that the trigonometric polynomials are dense in the space of 2π-periodic continuous function on $\mathbb{R}$. ■

¿From this result, using the same proofs as for Lemma 2 in [11] (see also Theorem 3 in [19]) and Theorem 2 in [19] respectively), the following result is obtained:

Lemma 17. *Let* $l \in \mathbb{Z}$. *Let* μ, $\sigma \in M$ *such that* $\lim_{n\to\infty} \Phi_n(\sigma, 0) = 0$, *then*

$$\lim_{n\to\infty} \int \left| \left| \frac{\varphi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)}{\varphi_{n+l+1}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; z)} \right|^2 - 1 \right| d\theta = 0;$$

$$\lim_{n\to\infty} \Phi_{n+l}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma; z)|^2}; 0) = 0.$$

3 Proof of main results

1 Proof of Theorem 2

Obviously, we have

$$\frac{\psi_{n,n+k_1}(z)}{\varphi_{n,n+k_2}(z)}=\frac{\varphi_{n+k_1}(\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n;z)}{\varphi_{n+k_2}(\mu_n;z)}$$

$$=\frac{\varphi_{n+k_1}(\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n;z)}{\varphi_{n+k_1-1}(\left|\frac{V_n^{(1)}(z)/(z-\zeta_{n,1}^{(1)})}{V_n^{(2)}(z)}\right|^2 d\mu_n;z)}\cdots\frac{\varphi_{n+1}(\left|\frac{z-\zeta_{n,k_1}^{(1)}}{V_n^{(2)}(z)}\right|^2 d\mu_n;z)}{\varphi_n(\frac{d\mu_n}{|V_n^{(2)}(z)|^2};z)}$$

$$\frac{\varphi_n(\frac{d\mu_n}{|V_n^{(2)}(z)|^2};z)}{\varphi_{n+1}(\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu_n}{|V_n^{(2)}(z)|^2};z)}\cdots\frac{\varphi_{n+k_2-1}(\frac{d\mu_n}{|z-\zeta_{n,k_2}^{(2)}|^2};z)}{\varphi_{n+k_2}(\mu_n;z)}.$$

Because of $(\mu,\{W_n\},-k_2)$ is admissible by Theorem B we achieve

$$\lim_{n\to\infty}\Phi_{n+m+1}(\frac{d\mu_n}{|V_n^{(2)}(z)|^2};0)=\lim_{n\to\infty}\Phi_{n+m}(\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu_n}{|V_n^{(2)}(z)|^2};0)=\dots$$

$$=\lim_{n\to\infty}\Phi_{n+m-k_2+2}(\frac{d\mu_n}{|z-\zeta_{n,k_2}^{(2)}|^2};0)=\lim_{n\to\infty}\Phi_{n+m-k_2+1}(\mu_n;0)=0, \tag{23}$$

for all $m\geq 0$.

On the one hand, using Corollary 14 for $d\sigma_n=\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu_n}{|V_n^{(2)}(z)|^2}$ and $\zeta_n=\zeta_{n,1}^{(2)}$, we obtain

$$\lim_{n\to\infty}\frac{\varphi_n(\frac{d\mu_n}{|V_n^{(2)}(z)|^2};z)}{\varphi_{n+1}(\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu_n}{|V_n^{(2)}(z)|^2};z)}-\frac{z-\zeta_{n,1}^{(2)}}{z^2}=0,$$

and with analogous reasoning we have

$$\lim_{n\to\infty}\frac{\varphi_{n+1}(\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu_n}{|V_n^{(2)}(z)|^2};z)}{\varphi_{n+2}(|(z-\zeta_{n,1}^{(2)})(z-\zeta_{n,2}^{(2)})|^2\frac{d\mu_n}{|V_n^{(2)}(z)|^2};z)}-\frac{z-\zeta_{n,1}^{(2)}}{z^2}=\dots$$

$$=\lim_{n\to\infty}\frac{\varphi_{n+k_2-1}(\frac{d\mu_n}{|(z-\zeta_{n,k_2}^{(2)})|^2};z)}{\varphi_{n+k_2}(\mu_n;z)}-\frac{z-\zeta_{n,k_2}^{(2)}}{z^2}=0.$$

On the other hand, as we said before, since $(\mu,\{W_n\},-k_2)$ is admissible, using Theorem B, we obtain

$$\lim_{n\to\infty}\Phi_{n+m+1}(\frac{d\mu_n}{|V_n^{(2)}(z)|^2};0)=0,$$

for all $m \geq 0$, so by (16) in Lemma 15 we have

$$\lim_{n\to\infty} \Phi_{n+m}(\left|\frac{z-\zeta_{n,k_1}^{(1)}}{V_n^{(2)}}\right|^2 d\mu_n; 0) = \dots$$

$$= \lim_{n\to\infty} \Phi_{n+m-k_1+1}(\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n; 0) = 0,$$

for all $m \geq 0$.

Thus, applying (17) in Lemma 15, we have

$$\lim_{n\to\infty} \frac{\varphi_{n+k_1}(\left|\frac{V_n^{(1)}(z)}{V_n^{(2)}(z)}\right|^2 d\mu_n; z)}{\varphi_{n+k_1-1}(\left|\frac{V_n^{(1)}(z)/(z-\zeta_{n,1}^{(1)})}{V_n^{(2)}(z)}\right|^2 d\mu_n; z)} - \frac{z^2}{z-\zeta_{n,1}^{(1)}} = \dots$$

$$= \lim_{n\to\infty} \frac{\varphi_{n+1}(\left|\frac{(z-\zeta_{n,k_1}^{(1)})}{V_n^{(2)}(z)}\right|^2 d\mu_n; z)}{\varphi_n(\frac{d\mu_n}{|V_n^{(2)}(z)|^2}; z)} - \frac{z^2}{z-\zeta_{n,k_1}^{(1)}} = 0.$$

Therefore, the last equalities prove Theorem 2.

Remark 18. *Under the conditions of Theorem 2, if $l_1, l_2 \in (\mathbb{N} \cup \{0\})$, $l_1 \geq k_1$, $l_2 \geq k_2$ we have that*

$$\lim_{n\to\infty} \frac{\psi_{n,n+l_1}(z)}{\varphi_{n,n-l_2}(z)} - \frac{z^{2(l_1-l_2)} V_n^{(1)}(z)}{V_n^{(2)}(z)} = 0,$$

uniformly on each compact subset of $|z| > 1$.

2 Proof of Theorem 4

The main step in the proof of Theorem 2 is (23) and this follows from admissibility hypothesis of $(\sigma, \{W_n\}, -k_2)$. But in the case under study, since σ satisfies the Szegő condition, we have obviously that $\lim_{n\to\infty} \Phi_n(\sigma; 0) = 0$. Therefore, from Lemma 17 we obtain

$$\lim_{n\to\infty} \Phi_{n+m}(\frac{d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; 0) = \lim_{n\to\infty} \Phi_{n+m}(\frac{|z-\zeta_{n,1}^{(2)}|^2 d\mu}{|V_n^{(2)}(z)\varphi_n(\sigma;z)|^2}; 0) = \dots$$

$$= \lim_{n\to\infty} \Phi_{n+m}(\frac{d\mu}{|(z-\zeta_{n,k_2}^{(2)})\varphi_n(\sigma;z)|^2}; 0) = \lim_{n\to\infty} \Phi_{n+m}(\frac{d\mu}{|\varphi_n(\sigma;z)|^2}; 0) = 0,$$

for all $m \in \mathbb{Z}$. Thus, following step by step the proof of Theorem 2 we achieve

$$\lim_{n\to\infty} \frac{\psi_{n,n}(z)}{\varphi_n(\frac{d\mu}{|\varphi_n(\sigma;z)|^2}; z)} - \frac{V_n^{(1)}(z)}{V_n^{(2)}(z)} = 0,$$

uniformly in each compact subset of $|z| > 1$, and the proof of Theorem 4 is finished combining Theorems C and D.

Remark 19. *As it was observed, under assumption of Theorem 4,* $\varphi_n(\frac{d\mu}{|\varphi_n(\sigma;z)|^2};z)$ *has relative asymptotic behaviour with respect to* $\varphi_n(\sigma;z)$*. Moreover, since*

$$\frac{d\theta}{|\varphi_n(\sigma;z)|^2} \xrightarrow[n\to\infty]{*} \sigma$$

we think that $\varphi_n(\frac{d\mu}{|\varphi_n(\sigma;z)|^2};z)$ *could be also a good "bridge" for obtain comparative asymptotics of* $\varphi_n(\sigma;z)$ *and* $\varphi_n(\mu'\,d\sigma;z)$*.*

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