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NOTES ON PROPER HOMOTOPY THEORIES ASSOCIATED WITH COMPACT PL-MANIFOLDS

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Abstract: The purpose of this note is to present a bigraded sequence of functors associated with compact PL-manifolds. These functors are invariants of proper homotopy type, and distinguish topological spaces which have the same homotopy type. On the category of compact topological spaces these functors agree with the Hurewicz homotopy groups, hence the principal interest of these functors is for noncompact spaces.

Consider the category of compact PL-manifolds and the bordism theory between these manifolds. We are thinking of closed PL-manifolds M of dimension n-m, PL-imbedded into S^n (M can be the empty set), and we can ask about the different ways that S^n-M can be mapped by a proper map into a topological space. We will consider a proper map as a continuous map $f:X \longrightarrow Y$ such that for any compact closed subset K of Y, $f^{-1}(K)$ is compact subset of X. If we work in the category TOP_* of topological spaces with base point and proper maps preserving base point, we can give to the set of proper maps $f:(S^n-M, *) \longrightarrow (X, x_0)$ an equivalence relation such that the quotient set has a group structure. Concretely:

A pointed (n,m)-sphere is a pair $(S,*)$, where $S = S^n$-M is complement in S^n of a closed (n-m)-manifold M, PL-imbedded in $S^n - \{*\}$. A singular (n,m)-sphere of a pointed space (X,x_0) is a pair (S,f), where S is a pointed (n,m)-sphere and $f:(S,*) \longrightarrow (X,x_0)$ is a pointed proper map. A homotopy relation between two pointed (n,m)-spheres $S = S^n - M$ and $S' = S^n - M'$ is the complement H in $S^n \times I$ of an (n+1-m)-manifold P PL-imbedded in $(S^n \times I) - (* \times I)$ such that

$$P \cap (S^n \times \partial I) = \partial P \ , \quad (\partial P) \cap (S^n \times 0) = M \quad , \quad (\partial P) \cap (S^n \times 1) = M'$$

A homotopy relation between two singular (n,m)-spheres (S,f) and (S',f') of a pointed space (X,x_0) is a pair (H, φ), where H is a homotopy relation between the pointed spheres S and S', and $\varphi : (H, * \times I) \longrightarrow (X,x_0)$ is a proper map such that $\varphi/S = f$ and $\varphi/S' = f'$. The above homotopy relations are equivalence relations and a proper (n,m)-sphere of (X,x_0) will be a class of singular (n,m)-spheres.

The comultiplication of spheres induces a multiplication in the set $\pi_n^m(X,x_0)$ of proper (n,m)-spheres of (X,x_0). With this multiplication $\pi_n^m(X,x_0)$ admits a group structure for $n \geq 1$, which is abelian for $n \geq 2$. These sets define functors from TOP_* to the category of sets, groups or abelian groups if $n=0$, $n=1$ or $n \geq 2$, respectively, and they are invariant of the proper homotopy type.

Note that if $f:(S^n - M, *) \longrightarrow (X,x_0)$ is a proper map, then X admits a proper map $g:J \longrightarrow X$, where J is the interval $[0,\infty)$. Hence $M = \emptyset$ if X does not admit applications of this class. Therefore the groups $\pi_n^m(X,x_0)$ agree with the Hurewicz groups $\pi_n(X,x_0)$ on spaces which do not admit a proper map $g:J \longrightarrow X$; particularly on the compact spaces. Note also that if $n \leq m-2$, $\pi_n^m(X,x_0) = \pi_n(X,x_0)$. In [2] we have considered an end of a topological space X, as a class of proper maps $g:J \longrightarrow X$ under the proper homotopy relation. $F(X)$ will represent the set of ends of X.

There are similar concepts of singular (n,m)-balls of a pointed pair (X,A,x_0) and homotopy relations between these balls. The set $\pi_n^m(X,A,x_0)$ of proper (n,m)-balls admits a structure of group for $n \geq 2$ which is abelian for $n \geq 3$, and these sets define functors which are invariant of proper homotopy type. As in the case of Hurewicz groups, if (X,A,x_0) is a pointed pair, where A is a closed subset of X, there is an exact sequence

$$\dots \longrightarrow \pi_{n+1}^m(X,A,x_0) \longrightarrow \pi_n^m(A,x_0) \longrightarrow \pi_n^m(X,x_0) \longrightarrow \pi_n^m(X,A,x_0) \longrightarrow \dots$$

The following theorem permits us to calculate the functor π_1^1.

<u>Theorem</u>. Let X be a path connected space and $x_0 \in X$. Then $\pi_1^1(X,x_0)$ is the free product of $\pi_1^2(X,x_0)$ with the group presented by $| c_{\alpha\beta} ; c_{\alpha\beta} \cdot c_{\beta\alpha} = 1 |$, where $c_{\alpha\beta}$ is a generator element associated with each pair (α, β) of $F(X) \times F(X)$.

Proof. There is a monomorphic transformation

$$\eta : \pi_1^2(X,x_0) \longrightarrow \pi_1^1(X,x_0)$$

defined by $\eta[f]_1^2 = [f]_1^1$, where $f:(S^1,*) \longrightarrow (X,x_0)$ is a proper map. It is not difficult to see that η is well defined and monomorphic using the following facts: (1) The complement of a point in $\mathbb{R}^2$ is homeomorphic to the complement of a closed PL-disk imbedded in $\mathbb{R}^2$. (2) If X has an end, then there exists a proper map $f:\mathbb{R}^2 \longrightarrow X$. (3) Given a finite family of disjoin simple closed PL-curves in $\mathbb{R}^2$, then the exterior of the "more external curves" is homeomorphic to the complement of finitely many points in $\mathbb{R}^2$. Because X is path connected, $\pi_1^1(X,x_0)$ has a set of generators formed by elements that admit a representative map $f:(S^1-M,*) \longrightarrow (X,x_0)$ with $M = \emptyset$ or $M = \{x\}$, where x is a point of $S^1-\{*\}$. Note that if $M = \emptyset$, then $[f]_1^1 \in \eta\,\pi_1^2(X,x_0)$. Now for each $\alpha \in F(X)$ we can choose a representative $f_\alpha : J \longrightarrow X$ such that $f_\alpha(0) = x_0$. Let $S^1-\{x\}$ the quotient of

$[0,1/2) \cup (1/2,1]$ by the identification $0 = 1$. Then, we define $g_{\alpha\beta} : S^1 - \{*\} \longrightarrow X$ by

$$g_{\alpha\beta}(t) = f_\alpha(2t) \quad \text{if} \quad 0 \leq t < 1/2,$$
$$g_{\alpha\beta}(t) = f_\beta(2-2t) \quad \text{if} \quad 1/2 < t \leq 1.$$

Let $c_{\alpha\beta} = [g_{\alpha\beta}]_1^1 \in \pi_1^1(X,x_0)$. We are going to prove that

$$L = \eta\, \pi_1^2(X,x_0) \cup \{c_{\alpha\beta} \;;\; \alpha,\beta \in F(X)\}$$

is a set of generators of $\pi_1^1(X,x_0)$. It suffices to show that an element of $\pi_1^1(X,x_0)$ represented by a proper map $f:(S^1-\{x\}, *) \longrightarrow (X,x_0)$ is product of elements of L. This is true because the restrictions $f/[0,1/2)$ and $f/(1/2, 1]$ can be considered as two ends α, β of X. Then there exists $u,v:(S^1,*) \longrightarrow (X,x_0)$ such that $[f]_1^1 = [u^{-1}.c_{\alpha\beta}.v]_1^1$.

Note that $c_{\alpha\beta} \neq 1$ and $c_{\alpha\beta}.c_{\beta\alpha} = 1$. Finally, suppose that $x_1,\ldots,x_r \in L$ with $x_1\ldots x_r = 1$. By the definition of homotopy relation it can be proved inductively that this relation is a consequence of relations of $\eta\,\pi_1^2(X,x_0)$ and relations of type $c_{\alpha\beta}.c_{\beta\alpha} = 1$.

Examples. (1) Because $\pi_1^2(X,x_0)$ is a quotient of $\pi_1(X,x_0)$, $\pi_1^2(\mathbb{R}^n) = 0$ for $n \geq 1$. It es easy to see that $F(\mathbb{R}) = \{-\infty, +\infty\}$ and $F(\mathbb{R}^n) = \{\infty\}$ for $n \geq 2$. Then $\pi_1^1(\mathbb{R})$ is the free product $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and $\pi_1^1(\mathbb{R}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.

(2) Let M be the open Moebius band. Then $\pi_1^1(M) = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

(3) Let $M_{1n} = (D^2 \times S^1) - K_{1n}$ be the complement of the torus knot of type $(1,n)$ imbedded in $\partial D^2 \times S^1$. Then $\pi_1^1(M_{1n}) = (\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Hence, if $m \neq n$, M_{1m} and M_{1n} are two manifolds which are of the same homotopy type but they are not of the same proper homotopy type.

Notes. (1) In the above theorem the product is free because if we have an element $x \in \pi_1^1(X,x_0)$ which has two representative maps

$f:(S^1,*) \longrightarrow (X,x_0)$ and $f':(S^1-M,*) \longrightarrow (X,x_0)$ with $M \neq \emptyset$, then $x = 1$. But this situation is not generalizable to the group $\pi_2^2(X,x_0)$. We are going to construct an 1-connected space X such that a non zero element $x \in \pi_2^2(X,x_0)$ has two representative maps $f:(S^2,*) \longrightarrow (X,x_0)$ and $f':(S^2-M,*) \longrightarrow (X,x_0)$ with $M \neq \emptyset$. Consider $Y = (S^2 \times I) - K$, where K is an unknotted path PL-imbedded in $S^2 \times I$ with $K \cap (S^2 \times \partial I) = \partial K$ and $\partial K \subset S^2 \times 1$. Now we can kill the generator of $\pi_1(Y)$ glueing adequately the boundary of a disk D^2 into $(S^2 \times 1) - \partial K$. It is easy to prove that the new space X has the following property: The nonzero element $x \in \pi_2^2(X)$ represented by the inclusion $f:S^2 \times 0 \longrightarrow X$ has a representative map $f': S^2-M \longrightarrow X$ with $M \neq \emptyset$.

(2) See [1] for other results on the groups $\pi_n^m(X,x_0)$ and some generalizations of these theories.

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