ON A THEOREM OF ERDŐS CONCERNING ADDITIVE FUNCTIONS

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ABSTRACT. Erdős proved that every increasing additive function must be a constant multiple of the logarithmic function. We prove a weaker result that assumes that the function is completely additive. In particular, what this paper does show is how wide the gulf is between additive and completely additive functions: proving the result for completely additive functions is very easy, but Erdős's proof for merely additive functions required a formidable effort.

An *additive* function is defined as a real-valued arithmetic function $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(nm) = f(n) + f(m)$ for all pairs of coprime integers n and m. If $f(nm) = f(n) + f(m)$ for all $n, m \in \mathbb{N}$, then we say that f is a *completely additive* function.

In [1, Theorem XI, p. 17], Erdős states that if f is an additive function such that $f(n+1) \geqslant f(n)$ for all $n \in \mathbb{N}$, then $f(n) = C \log n$ for a constant $C \in \mathbb{R}$. Without the hypothesis $f(n+1) \geqslant f(n)$, this is not true in general, as is shown, for instance, by the completely additive function $\Omega(n)$ defined by $\Omega(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = a_1 + a_2 + \cdots + a_k$, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime decomposition of n. Erdős's Theorem is a deep and interesting result, but its proof is rather complicated; see [3, p. 133] and [2, § 8.33 and 8.34, p. 265 and ff.] for adequate comments and additional references, including a proof (due to Moser and Lambek) that simplifies the original proof of Erdős.

In this note, we pose a weaker result, but with a very elementary proof. Also, we will show a nice consequence.

Theorem 1. *Let f be a completely additive function such that $f(n+1) \geqslant f(n)$ for all $n \in \mathbb{N}$. Then, there is a real constant $C \geqslant 0$ such that $f(n) = C \log n$ for all $n \in \mathbb{N}$.*

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Proof. We claim that, if f and g are two functions satisfying the conditions, they must satisfy

$$f(n)g(2) = f(2)g(n) \quad \forall n \in \mathbb{N}. \tag{1}$$

In particular, this would be true for $g(n) = \log n$. Then, our claim implies that $f(n) = C \log n$ with $C = f(2)/\log 2$ and so the theorem is proved. Thus, we only need to check (1).

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let us take $l \in \mathbb{N}$ such that $2^{l-1} \leqslant n^k < 2^l$. Then $(l-1)f(2) \leqslant kf(n) \leqslant lf(2)$. The same inequality is true for the function g; we can write it as $-lg(2) \leqslant -kg(n) \leqslant -(l-1)g(2)$. Multiplying the first expression by $g(2)$, the second by $f(2)$, adding them, and dividing by k, we get

$$-\frac{1}{k} f(2)g(2) \leqslant f(n)g(2) - f(2)g(n) \leqslant \frac{1}{k} f(2)g(2).$$

Since this happens for every $k \in \mathbb{N}$, equation (1) follows and the proof is complete. □

As a consequence, let us establish the following result:

Theorem 2. *Let* $f\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ *be an increasing and completely additive function. Then,* f *is the zero function.*

Proof. Let us suppose that f is not the zero function. By Theorem 1, there is a real constant $C > 0$ such that $f(n) = C \log n$ for all $n \in \mathbb{N}$. If f takes integer values, then $f(n)/f(m) = (\log n)/(\log m)$ for all integers $n, m \geqslant 2$. Therefore, $(\log n)/(\log m) = a/b$ with $a, b \in \mathbb{N}$. This implies that $n^b = m^a$. But this is impossible if n and m have a non common prime factor. □

Remark. By using Erdős's Theorem, we see that the same proof serves to establish the corresponding result for additive functions.

For completeness, let us show that, without using Theorem 1, another proof of the Theorem 2 can be given. Let us begin with

Lemma. *Let* f *be an increasing and completely additive function. Then,* $f(1) = 0$*. Moreover, if* f *is not the zero function, it satisfies* $f(n) > 0$ *for all* $n > 1$*, and* f *is strictly increasing.*

Proof. We have $f(n) = f(1 \cdot n) = f(1) + f(n)$ and so $f(1) = 0$. If f is not the zero function, there exists $a \in \mathbb{N}$ such that $f(a) \neq 0$. Since $a \geqslant 1$, it follows that $f(a) \geqslant f(1) = 0$; hence, $f(a) > 0$. Now, given $n > 1$, there exists k such that $n^k > a$; then $kf(n) = f(n^k) \geqslant f(a) > 0$, and we have $f(n) > 0$.

Finally, let us suppose that $f(n) = f(m)$ with $n < m$. This is not possible if $n = 1$, because $f(1) = 0$ and $f(m) > 0$; thus, we may assume that $1 < n < m$. Let us take k large enough such that $n^{k+1} < m^k$ (it suffices to take $k > (\log n)/(\log m - \log n)$). Then

$$f(n^k) = kf(n) = kf(m) = f(m^k).$$

Consequently, $f(r) = f(n^k) = f(m^k)$ for every r such that $n^k \leqslant r \leqslant m^k$. In particular, $f(n^{k+1}) = f(n^k)$, and so $(k+1)f(n) = kf(n)$, which is impossible, because $f(n) > 0$. □

Using this lemma we get the following.

Second proof of Theorem 2. For every $n \in \mathbb{N}$, there exists an integer k such that $2^{k+1} - 2^k = 2^k > n$. Let us take n intermediate numbers r_i between 2^k and 2^{k+1}; that is,

$$2^k < r_1 < r_2 < \cdots < r_n < 2^{k+1}.$$

Now, let us suppose that f is not the zero function. By the lemma, f is strictly increasing which implies that

$$\begin{aligned} kf(2) = f(2^k) < f(r_1) < f(r_2) < \cdots < f(r_n) < f(2^{k+1}) \\ = (k+1)f(2) = kf(2) + f(2). \end{aligned}$$

Then, by the pigeonhole principle, we have $f(2) > n$. But this happens for every $n \in \mathbb{N}$, which is absurd. □

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