ASYMPTOTIC BEHAVIOUR OF ORTHOGONAL POLYNOMIALS RELATIVE TO MEASURES WITH MASS POINTS

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Abstract. *General expressions are found for the orthonormal polynomials and the kernels relative to measures on the real line of the form* $d\mu + M\delta_c$, *in terms of those of the measures* $d\mu$ *and* $(x-c)^2 d\mu$. *In particular, these relations allow to obtain several bounds for the polynomials and kernels relative to a generalized Jacobi weight with a finite number of mass points.*

§0. Introduction.

Let μ be a positive measure on $\mathbb{R}$ with infinitely many points of increase and such that all the moments

$$\int_{\mathbb{R}} x^n d\mu \qquad (n = 0, 1, \dots)$$

exist. Then, there exists a unique sequence $\{P_n\}_{n\ge 0}$ of orthogonal polynomials

$$P_n(x) = k_n x^n + \dots, \qquad k_n > 0$$

such that

$$\int_{\mathbb{R}} P_n P_m d\mu = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

As usual, $\{K_n(x,y)\}_{n\ge 0}$ denotes the sequence of kernels associated to μ, that is,

$$K_n(x,y) = \sum_{j=0}^{n} P_j(x) P_j(y).$$

An interesting problem in the theory of orthogonal polynomials is that of finding asymptotic estimates for $\{P_n\}$, their leading coefficients $\{k_n\}$, the sequence $\{K_n(x,x)\}$ ($x \in \operatorname{supp}\mu$), etc. (see, for example, [8] and [4] for Jacobi polynomials, [2] and [6] for generalized Jacobi polynomials, [1] and [5] for Laguerre and Hermite; general results can be found in [7], [6], etc.).

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Let M be a positive constant and let δ_c denote a Dirac measure on a point $c \in \mathbb{R}$, that is,

$$\int_{\mathbb{R}} f\delta_c = f(c)$$

for every function f. Then, associated to the measure $d\nu = d\mu + M\delta_c$ there exists a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials. We are interested in finding expressions which relate the sequences $\{Q_n\}$ and $\{P_n\}$ in order to deduce estimates for $\{Q_n\}$ whenever they are known for $\{P_n\}$. We will prove some relations which also involve the polynomials orthonormal with respect to the measure $(x-c)^2 d\mu(x)$. The following notation will be used from now on:

$$d\mu^c = (x-c)^2 d\mu(x);$$

$\{P_n^c\}$ is the sequence of orthonormal polynomials relative to $d\mu^c$;

$$P_n^c(x) = k_n^c x^n + \ldots, \qquad k_n^c > 0;$$

$\{K_n^c(x,y)\}$ is the sequence of kernels relative to $d\mu^c$.

§1. General results.

The following lemma gives a first relation among the sequences $\{K_n\}$, $\{P_n\}$ and $\{P_n^c\}$:

Lemma 1. *With the above notation,*

$$K_n(x,c) = \frac{k_n}{k_n^c} P_n(c) P_n^c(x) - \frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c) P_{n-1}^c(x) \qquad \forall n \geq 1.$$

Proof. We can put

$$K_n(x,c) = \sum_{j=0}^{n} \alpha_j P_j^c(x),$$

with

$$\alpha_j = \int_{\mathbb{R}} K_n(x,c) P_j^c(x)(x-c)^2 d\mu(x).$$

Therefore, we only need to show that

a) $\alpha_n = \frac{k_n}{k_n^c} P_n(c)$; b) $\alpha_{n-1} = -\frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c)$; c) $\alpha_j = 0$, $j = 0, 1, \ldots, n-2$.

Part a) can be obtained by a direct observation of the leading coefficients. As to c), it is an easy consequence of a well-known property of the kernels K_n: if R_n is a polynomial of degree at most n, then

$$\int_{\mathbb{R}} K_n(x,c) R_n(x) d\mu(x) = R_n(c).$$

In order to prove equation b), we use Christoffel-Darboux formula (see [8], for example) and the orthonormality of $\{P_n\}$ with respect to $d\mu$:

$$\begin{aligned}
\alpha_{n-1} &= \int_{\mathbb{R}} K_n(x,c) P_{n-1}^c(x)(x-c)^2 d\mu(x) = \int_{\mathbb{R}} [K_n(x,c)(x-c)][P_{n-1}^c(x)(x-c)] d\mu(x) \\
&= \int_{\mathbb{R}} \frac{k_n}{k_{n+1}} [P_n(c)P_{n+1}(x) - P_{n+1}(c)P_n(x)][P_{n-1}^c(x)(x-c)] d\mu(x) \\
&= -\frac{k_n}{k_{n+1}} P_{n+1}(c) \int_{\mathbb{R}} P_n(x)[P_{n-1}^c(x)(x-c)] d\mu(x) = -\frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c),
\end{aligned}$$

and the lemma is proved.

In order to find bounds for the orthogonal polynomials and the kernels, it is important to know the size of the coefficients which appear in the formulae we are going to deal with. Frequently, we will restrict ourselves to the case of measures of compact support and absolutely continuous part almost everywhere positive.

Lemma 2. *Assume* supp $d\mu = [-1,1]$, $\mu' > 0$ *a.e.. Let* $c \in [-1,1]$. *Then*

$$\lim_{n\to\infty} \frac{k_n}{k_n^c} = \frac{1}{2}, \qquad \lim_{n\to\infty} \frac{k_{n-1}^c}{k_{n+1}} = \frac{1}{2}, \qquad \lim_{n\to\infty} \frac{k_n}{k_{n-1}^c} = 1.$$

Proof. The first limit is a consequence of a result of Máté, Nevai and Totik (see [3], theorem 11), from which it follows

$$\lim_{n\to\infty} \frac{k_n}{k_n^c} = \exp(-\frac{1}{4\pi} \int_0^{2\pi} \log(\cos t - a)^2 dt).$$

It is not difficult to see that the integral equals $-4\pi \log 2$, $\forall a \in [-1,1]$.

The other limits can be obtained from the first one, since from our hypothesis it follows

$$\lim_{n\to\infty} \frac{k_n}{k_{n+1}} = \frac{1}{2}$$

(see [7], pag. 212 or [3], theorem 10). The lemma is proved.

Lemma 3. *Let* $d\mu$ *be a measure on* $\mathbb{R}$ *and* $n \geq 1$. *With the above notation,*

$$\int_{\mathbb{R}} P_{n-1}^c(x)(x-c) d\mu(x) = -\frac{k_{n-1}^c}{k_n} \frac{P_n(c)}{K_{n-1}(c,c)}$$

Proof. We can write

$$P_{n-1}^c(x)(x-c) = \sum_{j=0}^{n} \alpha_j P_j(x).$$

By a direct observation of leading coefficients, $\alpha_n = \frac{k^c_{n-1}}{k_n}$. If $j = 1, 2, \ldots, n-1$, the orthonormality properties of $\{P_n\}$ and $\{P^c_n\}$ yield

$$\begin{aligned}\alpha_j &= \int_{\mathbb{R}} P^c_{n-1}(x)(x-c)P_j(x)d\mu(x) \\ &= \int_{\mathbb{R}} P^c_{n-1}(x)\frac{P_j(x)-P_j(c)}{x-c}(x-c)^2 d\mu(x) + \int_{\mathbb{R}} P^c_{n-1}(x)P_j(c)(x-c)d\mu(x) \\ &= P_j(c)\int_{\mathbb{R}} P^c_{n-1}(x)(x-c)d\mu(x).\end{aligned}$$

For α_0 we obtain the same expression, because $P_0(x)$ is a constant:

$$\alpha_0 = \int_{\mathbb{R}} P^c_{n-1}(x)(x-c)P_0(x)d\mu(x) = P_0(c)\int_{\mathbb{R}} P^c_{n-1}(x)(x-c)d\mu(x).$$

Therefore

$$\begin{aligned}P^c_{n-1}(x)(x-c) &= \frac{k^c_{n-1}}{k_n}P_n(x) + \sum_{j=0}^{n-1} P_j(c)[\int_{\mathbb{R}} P^c_{n-1}(u)(u-c)d\mu(u)]P_j(x) \\ &= \frac{k^c_{n-1}}{k_n}P_n(x) + K_{n-1}(x,c)\int_{\mathbb{R}} P^c_{n-1}(u)(u-c)d\mu(u).\end{aligned}$$

The lemma follows immediately taking $x = c$ in this equality.

We can now obtain an expression for the polynomials orthonormal with respect to the measure $d\mu + M\delta_c$ in terms of the polynomials $\{P_n\}$ and $\{P^c_n\}$.

Proposition 4. *Let $d\mu$ be a measure on $\mathbb{R}$, $c \in \mathbb{R}$, $M > 0$. Let $\{Q_n\}_{n\geq 0}$ be the polynomials orthonormal with respect to $d\mu + M\delta_c$. Then, for each $n \in \mathbb{N}$ there exist two constants $A_n, B_n \in (0,1)$ such that*

$$Q_n(x) = A_n P_n(x) + B_n(x-c)P^c_{n-1}(x) \tag{1}$$

Furthermore, if supp $d\mu = [-1,1]$, $\mu' > 0$ *a.e. and* $c \in [-1,1]$, *then*

$$\lim_{n\to\infty} A_n K_{n-1}(c,c) = \frac{1}{\lambda(c)+M}$$

and

$$\lim_{n\to\infty} B_n = \frac{M}{\lambda(c)+M},$$

where

$$\lambda(c) = \lim_{n\to\infty}\frac{1}{K_n(c,c)}.$$

Proof. We will find firstly a constant C_n such that $P_n(x) + C_n(x-c)P^c_{n-1}(x)$ is orthogonal to the polynomials of degree at most $n-1$ with respect to the measure $d\mu + M\delta_c$. We only need to obtain

$$\int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)](x-c)^j [d\mu(x) + M\delta_c(x)] = 0, \quad j = 0, 1, \dots, n-1. \quad (2)$$

Let $j \geq 1$. Then

$$\begin{aligned}
&\int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)](x-c)^j [d\mu(x) + M\delta_c(x)] \\
&= \int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)](x-c)^j d\mu(x) \\
&= \int_{\mathbb{R}} P_n(x)(x-c)^j d\mu(x) + C_n \int_{\mathbb{R}} P^c_{n-1}(x)(x-c)^{j-1} d\mu^c(x) = 0.
\end{aligned}$$

Therefore, all we have to do is to find a constant C_n for which (2) is verified with $j = 0$. In this case, we can calculate the integral in (2):

$$\begin{aligned}
&\int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)][d\mu(x) + M\delta_c(x)] \\
&= MP_n(c) + \int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)] d\mu(x) \\
&= MP_n(c) + C_n \int_{\mathbb{R}} (x-c)P^c_{n-1}(x) d\mu(x) = P_n(c)[M - C_n \frac{k^c_{n-1}}{k_n} \frac{1}{K_{n-1}(c,c)}],
\end{aligned}$$

according to lemma 3. If we take

$$C_n = M \frac{k_n}{k^c_{n-1}} K_{n-1}(c,c),$$

then $P_n(x) + C_n(x-c)P^c_{n-1}(x)$ is orthogonal to every polynomial of degree at most $n-1$. As $C_n > 0$, it is a polynomial of degree n and leading coefficient positive. Thus, we will obtain the orthonormal polynomial Q_n by dividing it by its $L^2(d\mu + M\delta_c)$–norm.

$$\begin{aligned}
&\|P_n(x) + C_n(x-c)P^c_{n-1}(x)\|^2_{L^2(d\mu+M\delta_c)} \\
&= MP_n(c)^2 + \int_{\mathbb{R}} [P_n(x) + C_n(x-c)P^c_{n-1}(x)]^2 d\mu(x) \\
&= MP_n(c)^2 + \int_{\mathbb{R}} P_n(x)^2 d\mu(x) + C_n^2 \int_{\mathbb{R}} P^c_{n-1}(x)^2 (x-c)^2 d\mu(x) \\
&\qquad + 2C_n \int_{\mathbb{R}} P_n(x)(x-c)P^c_{n-1}(x) d\mu(x) \\
&= MP_n(c)^2 + 1 + C_n^2 + 2C_n \frac{k^c_{n-1}}{k_n}.
\end{aligned}$$

If we denote

$$D_n = [MP_n(c)^2 + 1 + C_n^2 + 2C_n \frac{k_{n-1}^c}{k_n}]^{1/2},$$

then we have

$$Q_n(x) = \frac{1}{D_n} P_n(x) + \frac{C_n}{D_n}(x-c)P_{n-1}^c(x),$$

that is, equation (1) with $A_n = 1/D_n$ and $B_n = C_n/D_n$. From its definition, it is clear that $D_n > 1$ and $D_n > C_n$, so $A_n, B_n \in (0,1)$.

For the second part, let us assume supp $d\mu = [-1,1]$, $\mu' > 0$ a.e. and $c \in [-1,1]$. From the above definitions for A_n, C_n and D_n, we have

$$\frac{1}{A_n K_{n-1}(c,c)} = [M \frac{P_n(c)^2}{K_{n-1}(c,c)^2} + \frac{1}{K_{n-1}(c,c)^2} + M^2 (\frac{k_n}{k_{n-1}^c})^2 + \frac{2M}{K_{n-1}(c,c)}]^{1/2}. \qquad (3)$$

It can be shown (see [6], theorem 3, pag. 26 and [7], pag. 212 or [3], theorem 10) that

$$\lim_{n\to\infty} \frac{P_n(x)^2}{K_{n-1}(x,x)} = 0 \qquad \forall x \in [-1,1].$$

Since $K_{n-1}(c,c) \geq P_0^2$ this also implies

$$\lim_{n\to\infty} \frac{P_n(c)^2}{K_{n-1}(c,c)^2} = 0.$$

From this and lemma 2 we obtain

$$\lim_{n\to\infty} \frac{1}{A_n K_{n-1}(c,c)} = \lambda(c) + M.$$

Finally,

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} A_n M \frac{k_n}{k_{n-1}^c} K_{n-1}(c,c) = \frac{M}{\lambda(c)+M}$$

and the proposition is completely proved.

Remark. If $P_n(c) = 0$, it is easy to show directly that $Q_n = P_n$. This is not in contradiction with our proposition, since in this case it can also be proved that $P_n(x) = (x-c)P_{n-1}^c$ and $A_n + B_n = 1$.

We can also find some relations which involve the kernels.

Proposition 5. *Let $d\mu$ be a measure on $\mathbb{R}$, $c \in \mathbb{R}$ and $M > 0$. Let $\{L_n\}_{n\geq 0}$ be the kernels relative to $d\mu + M\delta_c$. Then $\forall n \in \mathbb{N}$*

$$L_n(x,y) = \frac{1}{1+MK_n(c,c)} K_n(x,y) + \frac{MK_n(c,c)}{1+MK_n(c,c)}(x-c)(y-c)K_{n-1}^c(x,y)$$

Proof. If $y \in \mathbb{R}$, it is a well-known fact that the kernels $\{K_n^c\}$ verify

$$\int_{\mathbb{R}} R_n(x) K_n^c(x,y)(x-c)^2 d\mu(x) = R_n(y)$$

for every polynomial R_n of degree at most n. Actually, this property characterizes the kernels relative to any measure.

If we write

$$(x-c)(y-c)K_{n-1}^c(x,y) = \sum_{j=0}^{n} \alpha_j(y) P_j(x), \tag{4}$$

then it is easy to show that $\forall j \geq 1$

$$\alpha_j(y) = (y-c)\int_{\mathbb{R}} K_{n-1}^c(x,y)\frac{P_j(x)-P_j(c)}{x-c}(x-c)^2 d\mu(x)$$

$$+(y-c)P_j(c)\int_{\mathbb{R}} K_{n-1}^c(x,y)(x-c)d\mu(x).$$

By the above property, we obtain

$$\alpha_j(y) = P_j(y) - P_j(c) + (y-c)P_j(c)\int_{\mathbb{R}} K_{n-1}^c(x,y)(x-c)d\mu(x)$$

and it is immediate to see that α_0 also verifies this formula.

From this formula and (4) it follows

$$(x-c)(y-c)K_{n-1}^c(x,y)$$

$$= K_n(x,y) - K_n(x,c) + (y-c)K_n(x,c)\int_{\mathbb{R}} K_{n-1}^c(u,y)(u-c)d\mu(u)$$

If we let $x = c$, we obtain

$$(y-c)\int_{\mathbb{R}} K_{n-1}^c(x,y)(x-c)d\mu(x) = 1 - \frac{K_n(c,y)}{K_n(c,c)},$$

and, replacing this equation into the previous one,

$$\frac{1}{1+MK_n(c,c)}K_n(x,y) + \frac{MK_n(c,c)}{1+MK_n(c,c)}(x-c)(y-c)K_{n-1}^c(x,y)$$

$$= K_n(x,y) - \frac{MK_n(c,y)}{1+MK_n(c,c)}K_n(x,c).$$

Therefore, it will be enough to prove that

$$\int_{\mathbb{R}} [K_n(x,y) - \frac{MK_n(c,y)}{1+MK_n(c,c)}K_n(x,c)]R_n(x)[d\mu(x) + M\delta_c(x)] = R_n(y)$$

whenever R_n is a polynomial of degree at most n. This an easy consequence of the fact that the kernels $\{K_n\}$ verify the analogous property with respect to the measure $d\mu$. The proposition is proved.

§2. Generalized Jacobi weights with mass points.

Let ω be a generalized Jacobi weight, that is:

$$\omega(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^{N} |x-t_i|^{\gamma_i} \ , \ x \in [-1,1]$$

where:

a) α,β,$\gamma_i > -1$, $t_i \in (-1,1)$, $t_i \neq t_j \ \forall i \neq j$;

b) h is a positive, continuous function on $[-1,1]$ and $\omega(h,\delta)\delta^{-1} \in L^1(0,2)$, $\omega(h,\delta)$ being the modulus of continuity of h.

If we define

$$d(x,n) = (1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2},$$

then the polynomials $\{P_n\}$ orthonormal with respect to the measure $w(x)dx$ on the interval $[-1,1]$ verify the estimate

$$|P_n(x)| \leq Cd(x,n) \qquad \forall x \in [-1,1], \ \forall n \geq 1, \tag{5}$$

where C is a constant independent of n and x (see [2]). In the sequel C will denote a constant independent of n and x, but possibly different in each occurrence.

As to the kernels, it can be shown (see [6], pag. 120 and pag. 4) that

$$K_n(x,x) \sim n(1-x+n^{-2})^{-(2\alpha+1)/2}(1+x+n^{-2})^{-(2\beta+1)/2}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i} \tag{6}$$

uniformly in $|x| \leq 1$, $n \geq 1$, where by $f \sim g$ in a domain D we mean that there exist some positive constants C_1 and C_2 such that $C_1 f(y) \leq g(y) \leq C_2 f(y) \ \forall y \in D$.

Our aim is to prove similar bounds for the polynomials and the kernels relative to a measure which consists of a generalized Jacobi weight and a finite number of mass points on the interval $[-1,1]$. So, let $k \in \mathbb{N}$, $a_i \in [-1,1]$ and $M_i > 0$, $i = 1,\ldots,k$. We will denote

$$d\nu = w(x)dx + \sum_{i=1}^{k} M_i\delta_{a_i}$$

on the interval $[-1,1]$. By $\{Q_n\}$ and $\{L_n\}$ we mean, respectively, their orthonormal polynomials and kernels. Without loss of generality we can assume $a_i \in \{1,-1,t_1,\ldots,t_N\}$, since in the definition of w we can allow some of the exponents to be 0. Furthermore, for every $t \in [-1,1]$ we can speak of its exponent in w, referring to the exponent of the factor $|x-t|^\gamma$ in w. Obviously, there are only finitely many points with an exponent different from 0.

With this notation, we can deduce some bounds from the results of the previous section.

Proposition 6. *There exists a constant C such that $\forall n \geq 1$, $\forall x \in [-1,1]$*

$$|Q_n(x)| \leq C(1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2},$$

$$|L_n(x,x)| \leq C(1-x+n^{-2})^{-(2\alpha+1)/2}(1+x+n^{-2})^{-(2\beta+1)/2}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i}.$$

Proof. a) We are going to prove the bound for Q_n by induction on the number k of mass points. If $k=0$, the measure is a generalized Jacobi weight and we already know the formula (5). Let $k>0$ and assume the property holds for $k-1$ mass points.

Let $\{P_n\}$ be the orthonormal polynomials with respect to the measure

$$d\mu = w(x)dx + \sum_{i=1}^{k-1} M_i\delta_{a_i}$$

so that, according to the notation we used in section 1, $\{P_n^{a_k}\}$ are the polynomials orthonormal with respect to

$$(x-a_k)^2 d\mu(x) = (x-a_k)^2 w(x)dx + \sum_{i=1}^{k-1}(a_i-a_k)^2 M_i\delta_{a_i}.$$

Since $d\nu = d\mu + M_k\delta_{a_k}$, from proposition 4 it follows

$$Q_n(x) = A_nP_n(x) + B_n(x-a_k)P_{n-1}^{a_k},$$

with $A_n, B_n \in (0,1)$. Taking into account that both $d\mu$ and $(x-a_k)^2d\mu(x)$ are generalized Jacobi weights with $k-1$ mass points, they satisfy the boundedness in the statement. Now, it is easy to see that Q_n satisfies that boundedness.

Therefore, part a) is proved. As to b), proposition 5 yields

$$L_n(x,x) = C_nK_n(x,x) + (1-C_n)(x-a_k)^2K_{n-1}^{a_k}(x,x)$$

with $C_n \in (0,1)$. Similar arguments and formula (6) lead to this bound and the proposition is completely proved.

The previous result establishes only upper bounds, which sometimes is not enough. In some applications (for example, in the study of the convergence of the Fourier series) it is necessary to estimate more exactly the rate of growth of $L_n(x,x)$, at least at some points. In the case of a generalized Jacobi weight, with no point masses, we have even uniform estimates (formula (6)). This estimates cannot hold when the measure has mass points, since there the kernels $L_n(x,x)$ are bounded (see [6], pag. 4, for example). However, we can obtain such estimates for each point with no mass.

Proposition 7. *a) Let $t \in (-1,1)$, $t \neq a_i$ $(i = 1, \ldots, N)$ and let γ be its exponent in w. Then $L_n(t,t) \sim n^{1+\gamma}$.*
b) Suppose $a_i \neq 1$ $(i = 1, \ldots, N)$. Then $L_n(1,1) \sim n^{2\alpha+2}$.
c) Suppose $a_i \neq -1$ $(i = 1, \ldots, N)$. Then $L_n(-1,-1) \sim n^{2\beta+2}$.

Proof. By induction, making use of similar arguments to that of part b) of proposition 6.

As an application of the results of section 1, some bounds for $L_n(x, a_i)$ can also be obtained. These bounds can be used to study the convergence of the Fourier series with respect to this type of measure.

Proposition 8. *a) Let $1 \leq i \leq k$ and suppose $a_i \neq \pm 1$. Then there exists a constant C such that $\forall x \in [-1,1]$ and $\forall n \geq 1$*

$$|L_n(x,a_i)| \leq C(1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4} \prod_{t_j \neq a_i} (|x-t_j| + n^{-1})^{-\gamma_j/2}.$$

b) If 1 is a mass point, then there exists a constant C such that $\forall x \in [-1,1]$ and $\forall n \geq 1$

$$|L_n(x,1)| \leq C(1+x+n^{-2})^{-(2\beta+1)/4} \prod_{i=1}^{N} (|x-t_i| + n^{-1})^{-\gamma_i/2}.$$

c) If -1 is a mass point, then there exists a constant C such that $\forall x \in [-1,1]$ and $\forall n \geq 1$

$$|L_n(x,-1)| \leq C(1-x+n^{-2})^{-(2\alpha+1)/4} \prod_{i=1}^{N} (|x-t_i| + n^{-1})^{-\gamma_i/2}.$$

Proof. a) Assume $1 \leq i \leq k$ and $a_i \neq \pm 1$. Let γ be the exponent of $|x - a_i|$ en w. If we denote

$$d\mu = w(x)dx + \sum_{j=1, j\neq i}^{k} M_j \delta_{a_j},$$

then $d\nu = d\mu + M_i \delta_{a_i}$. Let $\{P_n\}$ and $\{K_n\}$ be the orthonormal polynomials and the kernels relative to $d\mu$ and k_n the leading coefficient of P_n. Analogously, $\{P_n^{a_i}\}$, $\{K_n^{a_i}\}$ and $\{k_n^{a_i}\}$ with respect to $(x-a_i)^2 d\mu$.

If we write

$$\Psi_n(x) = (1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4} \prod_{t_j \neq a_i} (|x-t_j| + n^{-1})^{-\gamma_j/2}$$

what we have to proof is $|L_n(x, a_i)| \leq C\Psi_n(x)$. Now, from proposition 5 and lemma 1 we obtain

$$L_n(x,a_i) = \frac{k_n}{k_n^{a_i}} \frac{P_n(a_i)}{1 + M_i K_n(a_i,a_i)} P_n^{a_i}(x) - \frac{k_{n-1}^{a_i}}{k_{n+1}} \frac{P_{n+1}(a_i)}{1 + M_i K_n(a_i,a_i)} P_{n-1}^{a_i}(x).$$

We only need to estimate the right hand side. From proposition 6 we get

$$\begin{aligned} |P_n(a_i)| &\le Cn^{\gamma/2}; \\ |P_{n+1}(a_i)| &\le Cn^{\gamma/2}; \\ |P_n^{a_i}(x)| &\le C(|x-a_i|+n^{-1})^{-(\gamma+2)/2}\Psi_n(x); \\ |P_{n-1}^{a_i}(x)| &\le C(|x-a_i|+n^{-1})^{-(\gamma+2)/2}\Psi_n(x). \end{aligned}$$

Since a_i is not a mass point for $d\mu$, proposition 7 yields

$$K_n(a_i,a_i) \sim n^{1+\gamma}.$$

Finally, by lemma 2

$$\lim_{n\to\infty} \frac{k_n}{k_n^{a_i}} = \frac{1}{2}; \qquad \lim_{n\to\infty} \frac{k_{n-1}^{a_i}}{k_{n+1}} = \frac{1}{2}.$$

It is now easy to deduce

$$|L_n(x,a_i)| \le Cn^{-1-\gamma/2}(|x-a_i|+n^{-1})^{-1-\gamma/2}\Psi_n(x) \le C\Psi_n(x).$$

b) Assume 1 is a mass point. We define now

$$d\mu = w(x)dx + \sum_{i=1, a_i\neq 1}^{k} M_i\delta_{a_i},$$

so $d\nu = d\mu + M\delta_1$, $M>0$. If, according to our usual notation, $\{P_n\}$, $\{K_n\}$ and $\{k_n\}$ refer to $d\mu$ and $\{R_n\}$ are the orthonormal polynomials relative to the measure $(1-x)d\mu$, $\{r_n\}$ being their leading coefficients, it is not difficult to show that

$$K_n(x,1) = \frac{k_n}{r_n}P_n(1)R_n(x)$$

(only standard properties of $K_n(x,1)$ are needed). Thus, proposition 5 leads to

$$L_n(x,1) = \frac{k_n}{r_n}\frac{P_n(1)}{1+MK_n(1,1)}R_n(x).$$

We proceed now analogously to part a), since $d\mu$ and $(1-x)d\mu$ are generalized Jacobi weights with masses at points different from 1. Notice that, by Hölder's inequaltiy

$$\begin{aligned} \frac{k_n}{r_n} &= \int_{-1}^{1} R_n(x)P_n(x)(1-x)d\mu(x) \\ &\le (\int_{-1}^{1} R_n(x)^2(1-x)^2 d\mu(x))^{1/2}(\int_{-1}^{1} P_n(x)^2 d\mu(x))^{1/2} \\ &\le \sqrt{2}(\int_{-1}^{1} R_n(x)^2(1-x) d\mu(x))^{1/2}(\int_{-1}^{1} P_n(x)^2 d\mu(x))^{1/2} = \sqrt{2}. \end{aligned}$$

Part c) is similar to b). Thus, the result is proved.

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