# Mean and Weak Convergence of Some Orthogonal Fourier Expansions by Using $A_p$ Theory

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## Introduction

Let $d\mu$ be a finite positive Borel measure on $\mathbb{R}$ such that $\operatorname{supp}(d\mu)$ is an infinite set and let $p_n(d\mu)$ denote the corresponding orthonormal polynomials. For $f \in L^1(d\mu)$, $S_n f$ stands for the $n$th partial sum of the orthogonal Fourier expansion of $f$ in $\{p_n(d\mu)\}_{n=0}^{\infty}$, that is,

$$S_n(f,x) = \sum_{k=0}^{n} a_k p_k(x), \quad a_k = \widehat{f}(k) = \int_{\mathbb{R}} f p_k \, d\mu.$$

The study of the convergence of $S_n f$ in $L^p(d\mu)$ ($p \neq 2$) has been discussed for several classes of orthogonal polynomials (c.f. Askey-Wainger [1], Badkov [2–4], Muckenhoupt [9–11], Newman-Rudin [13], Pollard [14–16], Wing [19]). For instance, in the case of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ which are orthogonal in $[-1,1]$ with respect to the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta \geq -1/2$, Pollard proved that $|1/p - 1/2| < \min\{1/(4\alpha+4), 1/(4\beta+4)\}$ is a sufficient condition for the uniform boundedness $\|S_n f\|_{p,w} \leq C\|f\|_{p,w}$, which is equivalent to the convergence in $L^p(w)$, $1 < p < \infty$. Newman and Rudin showed that the previous condition is also necessary and later Muckenhoupt extended these results to $\alpha, \beta > -1$.

On the other hand, Máté, Nevai and Totik [8] obtained, in a general way, necessary conditions for the mean convergence of Fourier expansions:

**Theorem (Máté-Nevai-Totik).** *Let $d\mu$ be such that* $\operatorname{supp}(d\mu) = [-1,1]$, *$\mu' > 0$ almost everywhere, $U$ and $V$ nonnegative Borel measurable functions such that neither of them vanishes almost everywhere in $[-1,1]$ and $V$ is finite*

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*on a set with positive Lebesgue mea-sure. If $S_n$ is uniformly bounded from $L^p(V^p\,d\mu)$ into $L^p(U^p\,d\mu)$, then*

*(i)* $U^p \in L^1(d\mu)$, $V^{-q} \in L^1(d\mu)$, $q = p/(p-1)$,

*(ii)* $\displaystyle\int_{-1}^{1} U(x)^p \mu'(x)^{1-p/2}(1-x^2)^{-p/4}\,dx < \infty$,

*(iii)* $\displaystyle\int_{-1}^{1} V(x)^{-q} \mu'(x)^{1-q/2}(1-x^2)^{-q/4}\,dx < \infty$.

## Mean convergence

The main subject in this paper is the study of the mean and weak boundedness of the orthogonal Fourier expansion, in some particular cases, by using $A_p$-theory, which plays a central role in the weighted norm inequalities for the Hardy-Littlewood maximal operator and the Hilbert transform.

We start with the mean convergence recalling some definitions:

(i) $(u, v) \in A_p(-1, 1)$, $1 < p < \infty$, iff there exists a positive constant $C$ such that $\left(\int_I u(x)\,dx\right)\left(\int_I v(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C|I|^p$ for all intervals $I \subset [-1, 1]$, where $|I|$ is the Lebesgue measure of $I$.

(ii) $(u, v) \in A_p^\delta(-1, 1)$ $(\delta > 1)$ iff $(u^\delta, v^\delta) \in A_p(-1, 1)$.

(iii) Given a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}}$, we say that $(u_n, v_n) \in A_p(-1, 1)$ uniformly if there exists a constant $C$, independent of $n$, such that $\left(\int_I u_n(x)\,dx\right)\left(\int_I v_n(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C|I|^p$ for all intervals $I \subset [-1, 1]$.

It is well known that $(u, v) \in A_p$ is a necessary condition for the boundedness of the Hilbert transform $H$ from $L^p(v)$ into $L^p(u)$ and that $(u, v) \in A_p^\delta$ (for some $\delta > 1$) is a sufficient condition [7], [12]. Analogous conditions work for the uniform boundedness, modifying slightly the arguments in [12].

This is connected with the Fourier expansion, and the idea comes from Pollard: let $\{p_n(x)\}_{n=0}^{\infty}$ denote the orthonormal polynomials with respect to $d\mu = \mu'(x)\,dx$ and $\{q_n(x)\}_{n=0}^{\infty}$ the orthonormal polynomials with respect to $(1-x^2)\,d\mu$. Then

$$S_n(f, x) = \int_{-1}^{1} f(t) K_n(x, t) \mu'(t)\,dt$$

and the kernel $K_n(x, t)$ can be decomposed in the form

$$K_n(x, t) = r_n T_1(n, x, t) + s_n T_2(n, x, t) + s_n T_3(n, x, t)$$

where:

$$T_1(n,x,t) = p_{n+1}(x)p_{n+1}(t),$$
$$T_2(n,x,t) = (1-t^2)\frac{p_{n+1}(x)q_n(t)}{x-t},$$
$$T_3(n,x,t) = T_2(n,t,x) = (1-x^2)\frac{p_{n+1}(t)q_n(x)}{t-x}.$$

If $\mu' > 0$ a.e., then $\{r_n\}$ and $\{s_n\}$ are bounded [17]. Let $U$ and $V$ be weights, $1 < p < \infty$, and

$$W_i(f,x) = W_{i,n}(f,x) = \int_{-1}^{1} f(t)T_i(n,x,t)\mu'(t)\,dt \quad (i = 1,2,3).$$

We try to estimate the three terms:

$$\|(W_i f)U\|_{p,\mu'} \le C\|fV\|_{p,\mu'}.$$

Denote:

$$u_n(x) = |p_{n+1}(x)|^p U(x)^p \mu'(x), \quad v_n(x) = |q_n(x)|^{-p}(1-x^2)^{-p}V(x)^p\mu'(x)^{1-p},$$
$$\overline{u}_n(x) = |q_n(x)|^p(1-x^2)^p U(x)^p\mu'(x), \quad \overline{v}_n(x) = |p_{n+1}(x)|^{-p}V(x)^p\mu'(x)^{1-p}.$$

By using Hölder's inequality and $A_p$ results we obtain the following sufficient conditions for the boundedness of $W_i$ $(i = 1,2,3)$:

$$(u_n, v_n) \in A_p^{\delta}(-1,1) \quad \text{uniformly for some } \delta > 1,$$
$$(\overline{u}_n, \overline{v}_n) \in A_p^{\delta}(-1,1) \quad \text{uniformly for some } \delta > 1.$$

On the other hand, the conditions

$$((1-x^2)^{-p/4}U(x)^p\mu'(x)^{1-p/2}, (1-x^2)^{-p/4}V(x)^p\mu'(x)^{1-p/2}) \in A_p(-1,1) \quad (1)$$

and

$$((1-x^2)^{p/4}U(x)^p\mu'(x)^{1-p/2}, (1-x^2)^{p/4}V(x)^p\mu'(x)^{1-p/2}) \in A_p(-1,1) \quad (2)$$

turn out to be necessary for the boundedness of $W_i$ $(i = 1,2,3)$. From (1), (2) and Th. 2 in [8], Máté-Nevai-Totik's conditions for the mean convergence of $S_n f$ can be obtained.

Next, we introduce a particular kind of measures.

**Definition.** We say that $d\mu = \mu'(x)\,dx \in H$ if $\mu'(x) = (1-x)^{\alpha}(1+x)^{\beta}w(x)$, where:

(i) $w > 0$ a.e. and $C_1 < w(x) < C_2$ for $x \in (1-\varepsilon, 1)$ and $x \in (-1, -1+\varepsilon)$.

(ii) $|p_n(x)| \le C(1-x+a_n)^{-(\alpha/2+1/4)}(1+x+b_n)^{-(\beta/2+1/4)}w(x)^{-1/2}$.

(iii) $|q_n(x)| \le C(1-x+a_n)^{-(\alpha/2+3/4)}(1+x+b_n)^{-(\beta/2+3/4)}w(x)^{-1/2}$ and $\{a_n\}$, $\{b_n\}$ are positive sequences such that $\lim a_n = \lim b_n = 0$.

There exist particular weights belonging to the class $H$: the generalized Jacobi weights $(GJ)$ $d\mu(x) = \mu'(x)\,dx$, being

$$\mu'(x) = \varphi(x)(1-x)^{\Gamma_1} \prod_{k=2}^{N-1} |x-x_k|^{\Gamma_k}(1+x)^{\Gamma_N}$$

where $\Gamma_k \ge 0$ $(k = 1, 2, \ldots, N)$, $1 > x_2 > \cdots > x_{N-1} > -1$, $\varphi > 0$ and continuous on $[-1, 1]$ and $\omega(\delta)/\delta \in L^1(0, 1)$, $\omega$ being the modulus of continuity of $\varphi$.

**Theorem 1.** *Let $d\mu \in H$, $U(x) = (1-x)^a(1+x)^b u(x)$, $V(x) = (1-x)^A(1+x)^B v(x)$, with $u > 0$ a.e., $v > 0$ a.e. and such that $C_1 < u(x), v(x) < C_2$ for $x \in (1-\varepsilon, 1)$ and $x \in (-1, -1+\varepsilon)$. If*

$$|(\alpha+1)(1/p-1/2) + (a+A)/2| < (a-A)/2 + \min\{1/4, (\alpha+1)/2\}, \quad A \le a,$$
$$|(\beta+1)(1/p-1/2) + (b+B)/2| < (b-B)/2 + \min\{1/4, (\beta+1)/2\}, \quad B \le b,$$

*and*

$$(w^{1-p/2}u^p, w^{1-p/2}v^p) \in A_p^\delta(-1, 1) \quad \textit{for some} \quad \delta > 1,$$

*then:*

$$\int_{-1}^{1} |S_n(f, x)U(x)|^p \mu'(x)\,dx \le C \int_{-1}^{1} |f(x)V(x)|^p \mu'(x)\,dx.$$

This theorem is a consequence of the following lemmas:

**Lema 1.** *Let $\{u_n(x)\}$, $\{v_n(x)\}$, $\{U_n(x)\}$, $\{V_n(x)\}$ be sequences of weights defined on a finite interval $(a, b)$. Let $c \in (a, b)$ and $\varepsilon > 0$ be fixed and independent of $n$. Assume that there exist some positive constants $\lambda_i$ $(i = 1, 2, 3, 4)$ such that $\lambda_1 \le U_n(x), V_n(x) \le \lambda_2$ on $(a, c+\varepsilon)$ and $\lambda_3 \le u_n(x), v_n(x) \le \lambda_4$ on $(c-\varepsilon, b)$. If $(u_n, v_n) \in A_p(a, c)$ and $(U_n, V_n) \in A_p(c, b)$ uniformly, then $(u_n U_n, v_n V_n) \in A_p(a, b)$ uniformly.*

**Lema 2.** *Let $\{x_n\}$ be a sequence of positive numbers which converges to zero. Then $(x^r(x+x_n)^s, x^R(x+x_n)^S) \in A_p^\delta(0, 1)$ uniformly if and only if:*

$$r > -1,\ R < p-1,\ R \le r,\ R+S \le r+s,\ r+s > -1,\ R+S < p-1.$$

From the above theorem, we have the following result, which was established by Badkov [3] (using other methods and without the restriction $\Gamma_k \ge 0$, $2 \le k \le N-1$):

**Corollary 1.** *Let $w \in (GJ)$ and $U(x) = (1-x)^a(1+x)^b \prod_{k=2}^{N-1} |x - x_k|^{c_k}$. If*

$$|(\Gamma_1 + 1)(1/2 - 1/p) - a| < \min\{1/4, (\Gamma_1 + 1)/2\},$$
$$|(\Gamma_N + 1)(1/2 - 1/p) - b| < \min\{1/4, (\Gamma_N + 1)/2\}$$

*and*

$$|(\Gamma_k + 1)(1/2 - 1/p) - c_k| < \min\{1/2, (\Gamma_k + 1)/2\} \quad (k = 2, \ldots, N-1),$$

*then*

$$\int_{-1}^{1} |S_n(f,x)U(x)|^p \mu'(x)\,dx \le C \int_{-1}^{1} |f(x)U(x)|^p \mu'(x)\,dx.$$

## Weak convergence

Another aim in this paper is to examine the weak behaviour of the orthogonal Fourier expansion, that is to study if there exists a constant $C$, independent of $n$, $y$ and $f$, such that:

$$\int_{|S_n(f,x)|>y} d\mu(x) \le C y^{-p} \int_{-1}^{1} |f(x)|^p \, d\mu(x), \quad y > 0,$$

i.e., if $S_n$ is uniformly bounded from $L^p(d\mu)$ into $L^p_*(d\mu)$, $1 < p < \infty$.

The previous inequality only can be true, besides the mean convergence interval, in its endpoints. For the Fourier-Legendre expansion ($d\mu = dx$), Chanillo [5] proved that the partial sum operator is not weak type $(4,4)$.

The following result gives necessary conditions for the weak boundedness [6].

**Theorem 2.** *Let $d\mu$ be such that* $\operatorname{supp}(d\mu) = [-1,1]$, *$\mu' > 0$ a.e., $U$ and $V$ be weights, $1 < p < \infty$. If there exists a constant $C$ such that*

$$\|S_n f\|_{L^p_*(U^p\,d\mu)} \le C \|f\|_{L^p(V^p\,d\mu)}$$

*holds for all integers $n \ge 0$ and every $f \in L^p(V^p\,d\mu)$, then:*

- *(i)* $U^p, V^{-q} \in L^1(d\mu)$,
- *(ii)* $\mu'(x)^{-1/2}(1-x^2)^{-1/4} \in L^p_*(U^p \mu'\,dx)$,
- *(iii)* $\mu'(x)^{-1/2}(1-x^2)^{-1/4} \in L^q(V^{-q} \mu'\,dx)$.

This result is a consequence of the following lemmas:

**Lema 3.** *Let $U$ and $V$ be weights and $1 < p < \infty$. If there exists a constant $C$ such that for every $f \in L^p(V^p\,d\mu)$ the inequality*

$$\|S_n f\|_{L^p_*(U^p\,d\mu)} \le C \|f\|_{L^p(V^p\,d\mu)}$$

*holds for all integers $n \ge 0$, then*

$$\|p_n\|_{L^q(V^{-q}\,d\mu)} \|p_n\|_{L^p_*(U^p\,d\mu)} \le C.$$

**Lema 4 ([8, Th. 2]).** *Let* $\text{supp}(d\mu) = [-1,1]$, $\mu' > 0$ *a.e. in* $[-1,1]$ *and* $0 < p \le \infty$. *There exists a constant* $C$ *such that if* $g$ *is a Lebesgue-measurable function in* $[-1,1]$, *then*

$$\|\mu'(x)^{-1/2}(1-x^2)^{-1/4}\|_{L^p(|g|^p\,dx)} \le \liminf_{n\to\infty} \|p_n\|_{L^p(|g|^p\,dx)}.$$

*In particular, if*

$$\liminf_{n\to\infty} \|p_n\|_{L^p(|g|^p\,dx)} = 0$$

*then* $g = 0$ *a.e.*

We are going to study the weak boundedness of the Fourier-Jacobi expansion. Since for $-1 < \alpha, \beta \le -1/2$ the conditions $|1/p - 1/2| < \min\{1/(4\alpha + 4), 1/(4\beta+4)\}$ are trivial for $p \in (1,\infty)$, we suppose, by symmetry, $\alpha \ge \beta$ and $\alpha > -1/2$. Then, the mean convergence interval is $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$.

*Remark 1.* If $U(x) = V(x) = 1$, the inequality $(\alpha+1)(1/p - 1/2) < 1/4$ is not satisfied for $p = 4(\alpha+1)/(2\alpha+3)$. It implies that $S_n$ is not weak type $(p,p)$ for the lower endpoint of the interval of mean convergence. The same happens with generalized Jacobi polynomials.

*Remark 2.* The conditions in Theorem 2 are the same as those of Máté-Nevai-Totik's theorem. Thus, the conditions obtained by Máté, Nevai and Totik are necessary not only for the mean convergence but also for the weak convergence.

*Remark 3.* It can be proved that Máté-Nevai-Totik's conditions are not sufficient for the weak convergence. In order to prove this, consider the Fourier-Legendre expansion ($d\mu = dx$), $p = 4$ and take

$$U(x) = \left|\log\frac{1+x}{4}\right|^{-5/8}\left|\log\frac{1-x}{4}\right|^{-5/8},$$
$$V(x) = \left|\log\frac{1+x}{4}\right|^{-3/8}\left|\log\frac{1-x}{4}\right|^{-3/8}.$$

Let $S_n$ denote the $n$th partial sum of the Fourier-Jacobi expansion with respect to $\mu'(x) = (1-x)^\alpha(1+x)^\beta$, being $\alpha \ge \beta$ and $\alpha > -1/2$. Then, the interval of mean convergence is given by $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$. Theorem 2 works to prove that $S_n$ is not weak type on $L^p(\mu')$ for $p = 4(\alpha+1)/(2\alpha+3)$, but it is not useful to show that $S_n$ is not weak type for $p = 4(\alpha+1)/(2\alpha+1)$. It leads us to make use of other arguments.

**Theorem 3.** *Let* $r = 4(\alpha+1)/(2\alpha+1)$. *Then, there exists no constant* $C$, *independent of* $n$ *and* $f \in L^r(\mu')$, *such that*

$$\|S_n f\|_{L^r_*(\mu')} \le C\|f\|_{L^r(\mu')}.$$

*Proof.* Decompose the kernel $K_n(x,t)$, as before, in the form

$$K_n(x,t) = r_n T_1(n,x,t) + s_n T_2(n,x,t) + s_n T_3(n,x,t).$$

By using the estimates

$$|p_n(x)| \le C(1-x)^{-\alpha/2-1/4}, \quad |q_n(x)| \le C(1-x)^{-\alpha/2-3/4}, \quad x \in (0,1),$$

Hölder's inequality and standard arguments of $A_p$ theory, the boundedness of $T_1$ and $T_3$ can be proved.

Now, it is not difficult to prove that

$$\int_{|p_n(x)H(f(t)q_{n-1}(t)(1-t^2)\mu'(t),x)|>y} \mu'(x)\,dx \le Cy^{-r}\|f\|_{r,\mu'}^r$$

is not satisfied for any fixed constant $C$. The proof is by contradiction, constructing a sequence of functions $\{f_{m,n}\}$ such that the constant $C$ appearing in the previous inequality grows with $m$. □

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