for each i then we also have $0 \le y_1 \le y_2 \le \cdots \le y_n$. The Rearrangement Inequality states that $\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for any permutation σ of $\{1, 2, \dots, n\}$. Since $\sum_{k=1}^{n} a_k^3 = \sum_{i=1}^{n} x_i y_i$ and $\sum_{k=1}^{n} a_k a_{k+1}^2 = \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for an appropriate permutation σ, the result follows.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, University of Toledo, Toledo, OH, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Ricardo comments that this problem appears as Problem 11.7 on p. 148 of Elementary Inequalities *by D.S. Mitrinović (P. Nordhoff, 1964), but that no solution is provided there.*

3305. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Prove that

$$\begin{aligned}\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \\ &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} = \sqrt{13+2\sqrt{13}}\,.\end{aligned}$$

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, in memory of Jim Totten.

We will prove that

$$\begin{aligned}\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} \\ &= \tan\frac{6\pi}{13} - 4\sin\frac{5\pi}{13} = \sqrt{13+2\sqrt{13}} \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} &= -\tan\frac{\pi}{13} + 4\sin\frac{3\pi}{13} \\ &= -\tan\frac{3\pi}{13} + 4\sin\frac{4\pi}{13} = \sqrt{13-2\sqrt{13}}\,. \qquad (2)\end{aligned}$$

We will make use of two elegant results due to K.F. Gauss and included in the *Sectio VII* of the *Disquisitiones Arithmeticæ* (DA).

Lemma (DA, art. 362, II). Let $n > 1$ be an odd number and $\omega = \dfrac{2k\pi}{n}$, where k is any of the numbers $1, 2, \ldots, n-1$. Then,

$$\tan\omega \;=\; 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) + \cdots \mp \sin\big((n-1)\omega\big)\big]\,.$$

Theorem (DA, art. 356). Let $n > 1$ be an odd prime number, $\mathfrak{R}$ be the set of the (positive and less than n) quadratic residues modulo n, and $\mathfrak{N}$ be the set of the (positive and less than n) quadratic non-residues modulo n. Then,

$$\sum_{r\in\mathfrak{R}} \cos\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \cos\frac{2\pi m}{n} \;=\; \begin{cases} \sqrt{n} & if \quad n \equiv 1 \pmod 4\ , \\ 0 & if \quad n \equiv 3 \pmod 4\ , \end{cases}$$

and

$$\sum_{r\in\mathfrak{R}} \sin\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \sin\frac{2\pi m}{n} \;=\; \begin{cases} 0 & if \quad n \equiv 1 \pmod 4\ , \\ \sqrt{n} & if \quad n \equiv 3 \pmod 4\ . \end{cases}$$

For $n = 13$ with $\omega = \dfrac{2k\pi}{n}$ and $1 \le k \le 12$ the Lemma yields

$$\tan\omega = 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) - \sin(8\omega) + \sin(10\omega) - \sin(12\omega)\big]\,.$$

We compute with different values of k in this identity as follows.
If $k = 1$ and $\omega = \dfrac{2\pi}{13}$, then the Lemma yields

$$\tan\frac{2\pi}{13} = 2\left(\sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{6\pi}{13} + \sin\frac{2\pi}{13}\right)\,. \quad (3)$$

If $k = 3$ and $\omega = \dfrac{6\pi}{13}$, then the Lemma yields

$$\tan\frac{6\pi}{13} = 2\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right)\,. \quad (4)$$

If $k = 4$ and $\omega = \dfrac{8\pi}{13}$, then $\tan\dfrac{5\pi}{13} = -\tan\dfrac{8\pi}{13}$ and the Lemma yields

$$\tan\frac{5\pi}{13} = 2\left(\sin\frac{3\pi}{13} + \sin\frac{6\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{5\pi}{13}\right)\,. \quad (5)$$

By comparing the equations (3), (4), and (5) we see that the first three expressions in equation (1) are equal.
If $k = 2$ and $\omega = \dfrac{4\pi}{13}$, then the Lemma yields

$$\tan\frac{4\pi}{13} = 2\left(\sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{6\pi}{13} - \sin\frac{\pi}{13} + \sin\frac{4\pi}{13}\right)\,. \quad (6)$$

If $k = 5$ and $\omega = \dfrac{10\pi}{13}$, then $\tan\dfrac{3\pi}{13} = -\tan\dfrac{10\pi}{13}$ and the Lemma yields

$$\tan\frac{3\pi}{13} = 2\left(\sin\frac{6\pi}{13} - \sin\frac{\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{3\pi}{13}\right)\,. \quad (7)$$

If $k = 6$ and $\omega = \dfrac{12\pi}{13}$, then $\tan\dfrac{\pi}{13} = -\tan\dfrac{12\pi}{13}$ and the Lemma yields

$$\tan\frac{\pi}{13} = 2\left(\sin\frac{2\pi}{13} - \sin\frac{4\pi}{13} + \sin\frac{6\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{\pi}{13}\right) . \quad (8)$$

By comparing the equations (6), (7), and (8) we see that the first three expressions in equation (2) are equal.

Now we take $A = \tan\dfrac{2\pi}{13} + 4\sin\dfrac{6\pi}{13}$ and $B = \tan\dfrac{4\pi}{13} + 4\sin\dfrac{\pi}{13}$. Clearly A and B are positive numbers. From (3) and (6) it follows that

$$A + B = 4\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)$$

and

$$A - B = 4\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) .$$

Then,

$$A^2 - B^2 = 16\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) .$$

Applying the identity $2\sin a\sin b = \cos(a-b) - \cos(a+b)$, we have

$$A^2 - B^2 = 8\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right) .$$

However, for $n = 13 \equiv 1 \pmod 4$, the sets $\mathfrak{R}$ and $\mathfrak{N}$ in the Theorem are $\mathfrak{R} = \{1,4,9,3,12,10\}$ and $\mathfrak{N} = \{2,8,6,11,5,7\}$; thus, by the Theorem

$$2\left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13}\right) = \sqrt{13}\,,$$

and therefore

$$A^2 - B^2 = 4\sqrt{13}\,.$$

Similarly, using the identity $2\sin^2 a = 1 - \cos(2a)$ we deduce that

$$\begin{aligned} AB &= 4\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) \\ &\quad \times\left(\sin\frac{\pi}{13} - \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} - \sin\frac{6\pi}{13}\right) \\ &= 6\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right) \\ &= \frac{6\sqrt{13}}{2} = 3\sqrt{13}\,. \end{aligned}$$

For positive real numbers A and B with $A > B$, the solutions of the equations $A^2 - B^2 = 4\sqrt{13}$ and $AB = 3\sqrt{13}$ are

$$A = \sqrt{13 + 2\sqrt{13}} \qquad \text{and} \qquad B = \sqrt{13 - 2\sqrt{13}}\,.$$

This completes the proof of the identities (1) and (2). The following similar identities can be deduced when $n = 11$:

$$\begin{aligned}\tan\frac{\pi}{11} + 4\sin\frac{3\pi}{11} &= -\tan\frac{2\pi}{11} + 4\sin\frac{5\pi}{11} \\ &= \tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \tan\frac{4\pi}{11} + 4\sin\frac{\pi}{11} \\ &= \tan\frac{5\pi}{11} - 4\sin\frac{4\pi}{11} \\ &= \sqrt{11}\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (2 solutions); JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

All solvers noted that $\tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \neq \sqrt{13 + 2\sqrt{13}}$*, as did George Apostolopoulos, Messolonghi, Greece; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON. Wagon used Mathematica to check that the first and third identities are correct and the second is incorrect. The proposer offered a partially correct solution.*

Woo wondered if similar results hold for $\frac{\pi}{5}$, $\frac{\pi}{7}$, $\frac{\pi}{11}$ *or* $\frac{\pi}{17}$*. For the case of* $\frac{\pi}{11}$ *Benito et al. answered (above) in the affirmative. The interested reader may want to investigate the other cases. Woo also challenges the readers to find geometric proofs for the equalities in (1) and (2).*

3306. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Find a real number t and polynomials $f(x)$, $g(x)$, and $h(x)$ with integer coefficients, such that

$$f(t) = \sqrt{2}\,, \qquad g(t) = \sqrt{3}\,, \qquad \text{and} \qquad h(t) = \sqrt{7}\,.$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set $\theta = \sqrt{2} + \sqrt{3} + \sqrt{7}$ and $\psi = \sqrt{2}\sqrt{3}\sqrt{7}$. Computing θ^3, θ^5, and θ^7, we obtain

$$\begin{aligned}\sqrt{2} + \sqrt{3} + \sqrt{7} &= \theta\,, \\ 16\sqrt{2} + 15\sqrt{3} + 11\sqrt{7} + 3\psi &= \frac{1}{2}\theta^3\,, \\ 281\sqrt{2} + 241\sqrt{3} + 161\sqrt{7} + 60\psi &= \frac{1}{4}\theta^5\,, \\ 4796\sqrt{2} + 3975\sqrt{3} + 2611\sqrt{7} + 1043\psi &= \frac{1}{8}\theta^7\,.\end{aligned}$$

This is a linear system for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, and ψ. Solving for $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{7}$