where r_1, r_2, and r_3 are the inradii of triangles APB, BPC, and CPA, respectively. Hence,

$$\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} \;=\; 3+\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right) \;\geq\; 3+3\cdot\sqrt[3]{\frac{1}{\sin A\sin B\sin C}}\,,$$

by the AM–GM Inequality. From [2], we have $\sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8}$. It follows that

$$\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} \;\geq\; 3+3\left(\sqrt[3]{\frac{8}{3\sqrt{3}}}\right) \;=\; 3+2\sqrt{3}\,.$$

Since $R \geq 2r$ (see [2]), we have

$$\frac{R}{r_1}+\frac{R}{r_2}+\frac{R}{r_3} \;\geq\; 2\left(\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3}\right) \;\geq\; 6+4\sqrt{3}\,.$$

The desired result follows.

Equality occurs when $\triangle ABC$ is equilateral.

References

[1] Matematika Skole, problem 378, pp. 75–76, No. 1, 1968.

[2] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

Also solved by MIHÁLY BENCZE, Brasov, Romania; and the proposer.

KLAMKIN–12. [2005 : 329, 332] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, c be the sides of a spherical triangle. Show that

$$3\cos a\cos b\cos c \;\leq\; \cos^2 a+\cos^2 b+\cos^2 c \;\leq\; 1+2\cos a\cos b\cos c\,.$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the AM–GM Inequality,

$$\cos^2 a+\cos^2 b+\cos^2 c \;\geq\; 3\sqrt[3]{\cos^2 a\cos^2 b\cos^2 c} \;\geq\; 3\sqrt[3]{\cos^3 a\cos^3 b\cos^3 c} \;=\; 3\cos a\cos b\cos c\,,$$

which establishes the left inequality.

[*Ed.:* Next, we give the argument of Benito, Ciaurri, and Fernández for the right inequality.] Let A, B, and C be the vertices of the spherical triangle. We use the well-known fundamental formula of spherical trigonometry,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A .$$

We have

$$\begin{aligned} 0 &\le \sin^2 b \sin^2 c \sin^2 A = \sin^2 b \sin^2 c - \sin^2 b \sin^2 c \cos^2 A \\ &= (1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2 \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\,, \end{aligned}$$

which completes the proof.

[*Ed.:* We now proceed with Zhou's argument for the same inequality.] We may assume that the spherical triangle is spanned by the unit vectors $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{C}$, starting at the centre of the sphere. Let $\overrightarrow{A} = \langle A_1, A_2, A_3\rangle$, $\overrightarrow{B} = \langle B_1, B_2, B_3\rangle$, $\overrightarrow{C} = \langle C_1, C_2, C_3\rangle$, and $M = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$. Then

$$\begin{aligned} 0 &\le [(\overrightarrow{A}\times\overrightarrow{B})\cdot\overrightarrow{C}]^2 = (\det M)^2 = \det(MM^T) \\ &= \begin{vmatrix} \overrightarrow{A}\cdot\overrightarrow{A} & \overrightarrow{A}\cdot\overrightarrow{B} & \overrightarrow{A}\cdot\overrightarrow{C} \\ \overrightarrow{B}\cdot\overrightarrow{A} & \overrightarrow{B}\cdot\overrightarrow{B} & \overrightarrow{B}\cdot\overrightarrow{C} \\ \overrightarrow{C}\cdot\overrightarrow{A} & \overrightarrow{C}\cdot\overrightarrow{B} & \overrightarrow{C}\cdot\overrightarrow{C} \end{vmatrix} = \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix} \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\,, \end{aligned}$$

completing the proof.

Also solved by the proposer.

KLAMKIN–13. [2005 : 329, 332] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a smooth closed convex curve in the plane. Theorems in analysis assure us that there is at least one circumscribing triangle $A_0B_0C_0$ to $\mathcal{C}$ having minimum perimeter. Prove that the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

Similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Suppose, to the contrary, that $A_0B_0C_0$ is a circumscribed triangle with the minimum perimeter, but the excircle opposite A_0 does not touch $\mathcal{C}$. Since $\mathcal{C}$ is convex, the common tangent B_0C_0 separates it from the excircle. By means of a dilatation with centre A, we can therefore shrink that circle to a