and

$$
\begin{aligned}
nf_r\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) &= n\left(1+\exp\left(\frac{1}{n}\sum_{i=n}^{n}\ln\left(\frac{x_i}{1-x_i}\right)\right)\right)^{-r} \\
&= n\left(1+\prod_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\frac{1}{n}}\right)^{-r} \\
&= n\left(\frac{\left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}{\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}\right)^{r} \\
&= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}}\cdot G_r(x_1,x_2,\ldots,x_n)\,. \qquad (7)
\end{aligned}
$$

Furthermore, note that $t_i \geq -\ln r$ is equivalent to $\dfrac{x_i}{1-x_i} \geq \dfrac{1}{r}$, or $\dfrac{1-x_i}{x_i} \leq r$, which in turn is equivalent to $\dfrac{1}{x_i} \leq r+1$, or $\dfrac{1}{x_i} \geq r+1$. Hence, f is convex if $r \leq 0$ or if $r > 0$ and $\frac{1}{r+1} \leq x_i < 1$ for all i; and concave if $r > 0$ and $0 \leq x_i \leq \dfrac{1}{r+1}$.

Using (6) and (7) together with Jensen's Inequality applied to $f_r(t)$ yields (ii) and (iii) for $x_i > 0$. The validity of these inequalities when $x_i = 0$ for any i follows from the continuity of $F_r - G_r$ at 0 in each of its variables.

Part (a) *also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; and the proposer. There was one incomplete solution.*

KLAMKIN–02. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$
xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \;\geq\; \frac{28}{27}\,.
$$

(b)★ Prove or disprove the following generalization involving n positive real numbers x_1, x_2, ..., x_n which sum to 1:

$$
\left(\prod_{i=1}^{n} x_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right) \;\geq\; \frac{n^3+1}{n^n}\,.
$$

I. *Solution to* (a) *by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.*

Using the condition that $x+y+z=1$, we obtain

$$\begin{aligned} xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) &= xyz+\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z} \\ &= \frac{(xyz)^2+(yz)^2+(zx)^2+(xy)^2}{xyz} \\ &= \frac{(xyz)^2+(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right]}{xyz(x+y+z)^3}. \end{aligned}$$

Thus, the given inequality is equivalent to

$$27(xyz)^2+27(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right] \ \geq\ 28xyz(x+y+z)^3\,,$$

or

$$\begin{aligned} 27s_1-30xyzs_2+54(x^3y^3+y^3z^3+z^3x^3)& \\ -28xyz(x^3+y^3+z^3)-60x^2y^2z^2 &\ \geq\ 0\,, \end{aligned} \qquad (1)$$

where

$$\begin{aligned} s_1 &= x^4y^2+x^2y^4+y^4z^2+y^2z^4+z^4x^2+z^2x^4\,, \\ s_2 &= x^2y+xy^2+y^2z+yz^2+z^2x+zx^2\,. \end{aligned}$$

Let S denote the left side of (1), and define T, U, V, and W as follows:

$$\begin{aligned} T &= s_1-2xyz(x^3+y^3+z^3)\,, \\ U &= 2(x^3y^3+y^3z^3+z^3x^3)-xyzs_2\,, \\ V &= s_1-xyzs_2\,, \\ \text{and}\quad W &= s_1-6x^2y^2z^2\,. \end{aligned}$$

Then, by tedious (but straightforward) computations, we determine that $S=14T+27U+3V+10W$; whence, (1) is equivalent to

$$14T+27U+3V+10W \ \geq\ 0\,. \qquad (2)$$

To establish (2), it suffices to show that T, U, V, $W\geq 0$.

Consider a majorization relation among the vectors $(4,1,1)$, $(4,2,0)$, $(3,2,1)$, and $(3,3,0)$ in $\mathbb{R}^3$. Since $(4,1,1)\prec(4,2,0)$, $(3,2,1)\prec(3,3,0)$, and $(3,2,1)\prec(4,2,0)$, we have, by Muirhead's Inequality,

$$\begin{aligned} 2xyz(x^3+y^3+z^3) &\leq s_1\,, \\ xyzs_2 &\leq 2(x^3y^3+y^3z^3+z^3x^3)\,, \\ \text{and}\quad xyzs_2 &\leq s_1\,; \end{aligned}$$

that is, T, U, $V\ \geq\ 0$. Also, by the AM–GM Inequality, we see that $s_1\geq 6x^2y^2z^2$; that is, $W\geq 0$.

Hence, (2) is proven, and our proof is complete.

II. *Solution to* (b) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.*

The claim is false for $n \geq 4$.

Let $f(x_1, x_2, \ldots, x_n)$ denote the left side of the given inequality. If we set $x_1 = x_2 = \cdots = x_{n-1} = k$ and $x_n = 1 - (n-1)k$, where $0 < k < \frac{1}{n}$, then $\sum_{i=1}^{n} x_i = 1$. Since

$$\begin{aligned}
&f\big(k, k, \ldots, k, 1-(n-1)k\big) \\
&\qquad = k^{n-1}(1-(n-1)k)\left(1 + \frac{n-1}{k^2} + \frac{1}{\big(1-(n-1)k\big)^2}\right) \\
&\qquad = k^{n-3}(1-(n-1)k)\left(k^2 + n - 1 + \frac{k^2}{\big(1-(n-1)k\big)^2}\right),
\end{aligned}$$

we have $\lim_{k \to 0+} f\big(k, k, \ldots, k, 1-(n-1)k\big) = 0$.

Also solved by ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC. Part (a) *was solved by ARKADY ALT, San Jose, CA, USA; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incomplete solution.*

KLAMKIN–03. [2005 : 327, 330] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

If a, b, c are positive real numbers, prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc} - \frac{a^2+b^2+c^2}{ab+bc+ca}\right) \geq 4\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$, the given inequality is equivalent to

$$\frac{2(ab+bc+ca)}{a^2+b^2+c^2} + \frac{ab^4+ac^4+bc^4+ba^4+ca^4+cb^4}{2abc(ab+bc+ca)} \geq 3\,. \qquad (1)$$

By the Cauchy–Schwarz Inequality, we have

$$\begin{aligned}
\left(\frac{1}{a}+\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}\right)&(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4) \\
&\geq \big(b^2+c^2+c^2+a^2+a^2+b^2\big)^2 \\
&= 4\big(a^2+b^2+c^2\big)^2,
\end{aligned}$$