KLAMKIN SOLUTIONS

These are the solutions to the special section of problems appearing in the September 2005 issue and dedicated to the memory of Murray S. Klamkin.

KLAMKIN–01. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x and y be positive real numbers from the interval $\left[0,\frac{1}{2}\right]$. Prove that

$$2 \;\le\; \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \;\le\; \frac{2}{\left(\sqrt{x}\sqrt{y}+\sqrt{1-x}\sqrt{1-y}\right)^{\frac{1}{2}}}\,.$$

(b)★ Is there a generalization of the above inequality to three or more numbers?

I. *Composite of solutions to* (a) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

The left inequality follows from the well-known fact that $t+\frac{1}{t}\ge 2$ for $t>0$, by taking $t=\left(\frac{1-x}{1-y}\right)^{1/4}$.

To prove the right inequality, we make the substitutions $x=\sin^2\alpha$ and $y=\sin^2\beta$, where $\alpha,\beta\in\left[0,\frac{\pi}{4}\right]$. Then the inequality becomes

$$\sqrt{\frac{\cos\alpha}{\cos\beta}}+\sqrt{\frac{\cos\beta}{\cos\alpha}} \;\le\; \frac{2}{\sqrt{\sin\alpha\sin\beta+\cos\alpha\cos\beta}}\,,$$

which is equivalent to $\dfrac{\cos\alpha+\cos\beta}{\sqrt{\cos\alpha\cos\beta}}\le\dfrac{2}{\sqrt{\cos(\alpha-\beta)}}$, or

$$(\cos\alpha+\cos\beta)^2\cos(\alpha-\beta) \;\le\; 4\cos\alpha\cos\beta \tag{1}$$

For notational convenience, we set $u=\frac{1}{2}(\alpha+\beta)$ and $v=\frac{1}{2}(\alpha-\beta)$. Then

$$\begin{aligned}(\cos\alpha+\cos\beta)^2 &= \big(\cos(u+v)+\cos(u-v)\big)^2\\ &= (2\cos u\cos v)^2 \;=\; 4(1-\sin^2 u)(1-\sin^2 v) \qquad (2)\\ \text{and}\quad \cos(\alpha-\beta) &= \cos(2v) \;=\; 1-2\sin^2 v\,. \qquad (3)\end{aligned}$$

Also,

$$\begin{aligned}\cos\alpha\cos\beta &= \tfrac{1}{2}\big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\big]\\ &= \tfrac{1}{2}\big(\cos(2u)+\cos(2v)\big) \;=\; 1-\sin^2 u-\sin^2 v\,. \qquad (4)\end{aligned}$$

Substituting (2), (3), and (4) into (1), gives

$$(1-\sin^2 u)(1-\sin^2 v)(1-2\sin^2 v) \;\le\; 1-\sin^2 u-\sin^2 v\,,$$

which simplifies to

$$\sin^2 u \sin^2 v \;\leq\; 2(1-\sin^2 u)(1-\sin^2 v)\sin^2 v\,. \tag{5}$$

Now we will prove (5). Since α, $\beta \in \left[0,\frac{\pi}{4}\right]$, we have $u \in \left[0,\frac{\pi}{4}\right]$ and $v \in \left[-\frac{\pi}{8},\frac{\pi}{8}\right]$. Then $\sin^2 u < \frac{1}{2}$ and $\sin^2 v \leq \sin^2\frac{\pi}{8} < \frac{1}{2}$; hence,

$$2(1-\sin^2 u)(1-\sin^2 v) \;>\; 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \;=\; \tfrac{1}{2} \;\geq\; \sin^2 u\,,$$

from which (5) follows.

II. *Solution to* (a) *and* (b) *by Li Zhou, Polk Community College, Winter Haven, FL, USA, expanded slightly by the editor.*

For any integer $n \geq 2$, any real number r, and any real numbers $x_1, x_2, \ldots, x_n \in [0,1)$, let

$$F_r(x_1,x_2,\ldots,x_n) \;=\; \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{-\frac{1}{n}} \cdot \sum_{i=1}^{n}(1-x_i)^r\,,$$

and $$G_r(x_1,x_2,\ldots,x_n) \;=\; n\left(\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}\right)^{-r}.$$

We prove that

(i) $F_r(x_1,x_2,\ldots,x_n) \geq n$;

(ii) $F_r(x_1,x_2,\ldots,x_n) \leq G_r(x_1,x_2,\ldots,x_n)$ if $r > 0$ and $0 \leq x_i \leq \dfrac{1}{1+r}$ for all i;

(iii) $F_r(x_1,x_2,\ldots,x_n) \;\geq\; G_r(x_1,x_2,\ldots,x_n)$ if $r \leq 0$ or if $r > 0$ and $\dfrac{1}{1+r} \leq x_i \leq 1$ for all i.

Note that (a) is the special case of (i) and (ii) when $n=2$ and $r=\frac{1}{2}$.

Proof: Part (i) follows immediately from the AM–GM Inequality. For the other two parts, let $f_r(t) = (1+e^t)^{-r}$. Then

$$\begin{aligned} f_r'(t) &= -r(1+e^t)^{-r-1}e^t \\ \text{and}\qquad f_r''(t) &= r(r+1)(1+e^t)^{-r-2}e^{2t} - r(1+e^t)^{-r-1}e^t \\ &= r(1+e^t)^{-r-2}e^t(re^t-1)\,. \end{aligned}$$

Hence, f is convex if $r \leq 0$ or if $r > 0$ and $t \geq -\ln r$; and concave if $r > 0$ and $t \leq -\ln r$. Now let $t_i = \ln\left(\dfrac{x_i}{1-x_i}\right)$ for $0 < x_i < 1$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} \sum_{i=1}^{n} f_r(t_i) &= \sum_{i=1}^{n}\left(1+\frac{x_i}{1-x_i}\right)^{-r} = \sum_{i=1}^{n}(1-x_i)^r \\ &= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}} \cdot F_r(x_1,x_2,\ldots,x_n) \end{aligned} \tag{6}$$

and

$$\begin{aligned} nf_r\left(\frac{1}{n}\sum_{i=1}^n t_i\right) &= n\left(1+\exp\left(\frac{1}{n}\sum_{i=n}^n \ln\left(\frac{x_i}{1-x_i}\right)\right)\right)^{-r} \\ &= n\left(1+\prod_{i=1}^n \left(\frac{x_i}{1-x_i}\right)^{\frac{1}{n}}\right)^{-r} \\ &= n\left(\frac{\left(\prod_{i=1}^n (1-x_i)\right)^{\frac{1}{n}}}{\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n (1-x_i)\right)^{\frac{1}{n}}}\right)^r \\ &= \left(\prod_{i=1}^n (1-x_i)^r\right)^{\frac{1}{n}} \cdot G_r(x_1,x_2,\dots,x_n)\,. \qquad (7) \end{aligned}$$

Furthermore, note that $t_i \geq -\ln r$ is equivalent to $\frac{x_i}{1-x_i} \geq \frac{1}{r}$, or $\frac{1-x_i}{x_i} \leq r$, which in turn is equivalent to $\frac{1}{x_i} \leq r+1$, or $\frac{1}{x_i} \geq r+1$. Hence, f is convex if $r \leq 0$ or if $r > 0$ and $\frac{1}{r+1} \leq x_i < 1$ for all i; and concave if $r > 0$ and $0 \leq x_i \leq \frac{1}{r+1}$.

Using (6) and (7) together with Jensen's Inequality applied to $f_r(t)$ yields (ii) and (iii) for $x_i > 0$. The validity of these inequalities when $x_i = 0$ for any i follows from the continuity of $F_r - G_r$ at 0 in each of its variables.

Part (a) *also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; and the proposer. There was one incomplete solution.*

KLAMKIN–02. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \;\geq\; \frac{28}{27}\,.$$

(b)★ Prove or disprove the following generalization involving n positive real numbers x_1, x_2, …, x_n which sum to 1:

$$\left(\prod_{i=1}^n x_i\right)\left(1+\sum_{i=1}^n \frac{1}{x_i^2}\right) \;\geq\; \frac{n^3+1}{n^n}\,.$$