2850. [2003 : 242] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find all integral solutions of

$$x^2 - 4xy + 6y^2 - 2x - 20y = 29\,.$$

Composite of essentially the same solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; Con Amore Problem Group, The Danish University of Education, Copenhagen, Denmark; Douglass L. Grant, University College of Cape Breton, Sydney, NS; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; D. Kipp Johnson, Beaverton, OR, USA; David Loeffler, student, Trinity College, Cambridge, UK; Digby Smith, Mount Royal College, Calgary, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

The given equation can be written as

$$(x - 2y - 1)^2 + 2(y - 6)^2 \;=\; 102\,.$$

Thus, $x - 2y - 1$ is even. Setting $x - 2y - 1 = 2u$ and $y - 6 = v$, we then have $2u^2 + v^2 = 51$ for some integers u and v. Clearly, $|u| \leq 5$. By setting $u = 0$, ± 1, ± 2, ± 3, ± 4, and ± 5, we find that v is an integer only when $u = \pm 1$ or ± 5. These values yield eight pairs: $(u,v) = (\pm 1, \pm 7)$ or $(\pm 5, \pm 1)$. Simple substitutions then give eight solution pairs (x,y) for the given equation:

$$(29,13)\,,\;\; (25,13)\,,\;\; (1,-1)\,,\;\; (-3,-1)\,,\;\; (25,7)\,,\;\; (5,7)\,,\;\; (21,5)\,,\;\; (1,5)\,.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); DIONNE T. BAILEY, ELSIE M. CAMPBEII, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; PAOLO CUSTODI, Fara Novarese, Italy; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; NEVEN JURIČ, Zagreb, Croatia; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB SERKEY, Leonia, NJ, USA; MIKE SPIVEY, Samford University, Birmingham, AL, USA; MIHAÏ STOËNESCU, Bischwiller, France; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; ROGER ZARNOWSKI, Angelo State University, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There were four incorrect or incomplete solutions.

As pointed out by Con Amore Problem Group, Curtis, and Konečný, the given equation represents an ellipse E, on which there can be only a finite number of lattice points. The solutions to the problem simply identify all eight lattice points on E.