Using tables or integration by parts, we find

$$\begin{aligned} I &= \frac{1}{26}\left\{\left(27y+\frac{y^4}{2}\right)\ln y - 27y - \frac{y^4}{8}\right\}\Bigg|_1^3 \\ &= \frac{1}{52}\{243\cdot\ln 3 - 128\}\,. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; ROBERT BILINSKI, Outremont, QC (first part only); PAUL BRACKEN, Concordia University, Montréal, QC; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, La Seu d'Urgell, Spain; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Woo observed that one requires neither calculus nor any elaborate argument to show that $y(x)$ is strictly decreasing on $[0,1]$. The value $x = \frac{27-y^3}{26y}$ clearly decreases as y increases through its positive values because the numerator decreases while the denominator increases.

Since the problem stated that y is the only real root of the given equation for x a fixed non-negative number, there was no need for us to verify it. Nevertheless, many solvers did verify it, perhaps because uniqueness is critical for the integration. Descartes' Rule of Signs does the job simply. Alternatively, many solvers simply restricted the domain and range to the first quadrant as in the featured solution. The situation is different when x is a sufficiently large negative number; in that case the given cubic equation will be satisfied by two negative values of y in addition to its one positive value.

2778. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z \neq 1$ is a complex number such that $z^n = 1$ $(n \geq 1)$. Prove that

$$\big|nz-(n+z)\big| \;\leq\; \frac{(n+1)(2n+1)}{6}\,|z-1|^2\,.$$

Preliminary comment. The original proposal from Bencze was to show that

$$|nz-(n+2)| \;\leq\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

The editor mistakenly turned the first 2 into a z. Happily, the modified problem is still of interest, although it has lost some of its intuitive meaning.

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

Differentiating the familiar identity

$$\sum_{k=0}^{n} x^k \;=\; \frac{x^{n+1}-1}{x-1}$$

with respect to x, we get

$$\sum_{k=1}^{n} kx^{k-1} \;=\; \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}\,.$$

Multiplying both sides by x and differentiating again, we arrive at

$$\sum_{k=1}^{n} k^2x^{k-1} \;=\; g(x)\,,$$

where

$$g(x) \;=\; \frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}\,.$$

Taking $x=z$ and using $|z|=1$ (which we were given), we obtain

$$|g(z)| \;\le\; \sum_{k=1}^{n} k^2|z|^{k-1} \;=\; \frac{n(n+1)(2n+1)}{6}\,. \qquad (1)$$

On the other side, taking into account that $z^n=1$, $z\neq 1$, we get

$$g(z) \;=\; \frac{n(nz^2-2(n+1)z+n+2)}{(z-1)^3} \;=\; \frac{n(nz-(n+2))}{(z-1)^2}\,. \qquad (2)$$

From (1) and (2) we therefore conclude that

$$|nz-(n+2)| \;\le\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

This was the inequality that the proposer had intended for us to verify. For the inequality as it appears in our problem, it remains to show that

$$|nz-(n+z)| \;<\; |nz-(n+2)|\,,$$

where $|z|=1$ and n is a positive integer. This is a routine calculation, comparing the square of both sides and using $z\overline{z}=|z|^2=1\ge \mathrm{Re}(z)$. Alternatively, a sketch of the circles $|nz-(n+z)|$ and $|nz-(n+2)|$ (for $z=e^{it}$, $0\le t<2\pi$) makes the inequality clear. Note that the inequality is strict.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous proved the stronger inequality:

$$|nz-(n+z)| \;\le\; \frac{n^2}{3}|z-1|^2\,.$$

He pointed out that even his inequality is far from best possible, and conjectured that the best factor of $|z-1|^2$ on the right side would be

$$\frac{\sqrt{4n(n-1)\sin^2\left(\frac{\pi}{n}\right)+1}}{4\sin^2\left(\frac{\pi}{n}\right)}\,.$$