2770. [2002 : 398] *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, MD, USA.*

In $\triangle ABC$, suppose that $a \le b \le c$ and $\angle ABC \ne \frac{\pi}{2}$. Prove that

$$2 + \sec B \;\le\; \left(1 + \tfrac{b}{a}\right)\left(1 + \tfrac{b}{c}\right) .$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

We are given that $a \le b \le c$, which implies that $\angle ABC < \frac{\pi}{2}$. The proof is accomplished by using the following successive transformations of the desired inequality:

$$\begin{aligned}
\sec B &\le \left(1 + \frac{b}{a}\right)\left(1 + \frac{b}{c}\right) - 2\,,\\
\frac{1}{\cos B} &\le \frac{ab + ac + b^2 + bc}{ac} - 2\,,\\
\cos B &\ge \frac{ac}{ab + b^2 + bc - ac}\,,\\
\frac{a^2 + c^2 - b^2}{2ac} &\ge \frac{ac}{ab + b^2 + bc - ac}\,,\\
\frac{(a+b+c)(a-b+c)}{2ac} - 1 &\ge \frac{ac}{b(a+b+c) - ac}\,,\\
\frac{(a+b+c)(a-b+c)}{2ac} &\ge \frac{b(a+b+c)}{b(a+b+c) - ac}\,,\\
(a-b+c)[b(a+b+c) - ac] &\ge 2abc\,,\\
(a-b+c)b(a+b+c) - a^2c - abc - ac^2 &\ge 0\,,\\
(a+b+c)(ab - b^2 + bc - ac) &\ge 0\,,\\
(a+b+c)(b-a)(c-b) &\ge 0\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; WINFER TABARES and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2771★. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that

$$(a+b)^b \;=\; a^b + b^a .$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain (modified slightly by the editor).

Clearly, $(a,1)$ is a solution for all positive integers a. We show that these are the only solution pairs.

Assuming that $b > 1$, we have

$$a^b + b^a \;=\; (a+b)^b \;=\; \sum_{k=0}^{b} \binom{b}{k} a^{b-k} b^k > a^b + b^b\,,$$

and thus, $a > b > 1$. Let $d = \gcd(a,b)$. Set $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$. Then we have $a_1 > b_1$ and $\gcd(a_1,b_1) = 1$, and the given equation becomes

$$\begin{aligned} d^b(a_1+b_1)^b &= d^b a_1^b + d^a b_1^a\,, \\ \text{or} \qquad (a_1+b_1)^b &= a_1^b + d^{a-b} b_1^a\,. \end{aligned} \tag{1}$$

If $d > 2$, then (1) has no solutions in positive integers by the Fermat-Wiles Theorem.

If $d = 2$, then (1) becomes

$$(a_1+b_1)^{2b_1} \;=\; a_1^{2b_1} + (2^{a_1-b_1} b_1^{a_1})^2\,, \tag{2}$$

which implies that $a_1 + b_1$ and a_1 have the same parity. Thus, b_1 must be even. Let $b_1 = 2b_2$. Then (2) becomes

$$\left((a_1+b_1)^{b_2}\right)^4 \;=\; (a_1^{b_2})^4 + (2^{a_1-b_1} b_1^{a_1})^2\,.$$

But it is well known that the equation $x^4 - y^4 = z^2$ has no non-trivial integer solutions. [*Ed:* See, for example, *Number Theory with Computer Applications* by Ramanujachary Kumanduri and Cristina Romero, p. 352.]

If $d = 1$, then $(a,b) = 1$ and the given equation can be written as

$$\begin{aligned} a^b + \binom{b}{1} a^{b-1} b + \cdots + \binom{b}{b-1} a b^{b-1} + b^b &= a^b + b^a\,, \\ \text{or} \quad a^{b-1}b^2 + \frac{b(b-1)}{2} a^{b-2} b^2 + \cdots + a b^b + b^b &= b^a\,. \end{aligned} \tag{3}$$

Suppose first that $b > 2$. Then $a > 3$, and (3) becomes

$$a^{b-1} + \frac{b(b-1)}{2} a^{b-2} + \cdots + a b^{b-2} + b^{b-2} \;=\; b^{a-2}\,. \tag{4}$$

If b has an odd prime divisor p, then $p \,\Big|\, \frac{b(b-1)}{2}$. Hence, (4) implies that $p | a^{b-1}$ and thus, $p|a$. However, this contradicts $(a,b) = 1$. Therefore, $b = 2^k$ where $k \in \mathbb{N}$. Since we are assuming that $b > 2$, we have $k > 1$

and $\frac{b(b-1)}{2} = 2^{k-1}(2^k - 1)$, which is even. Then (4) implies that a is even which again is a contradiction.

Hence, $b = 2$, and the given equation becomes $(a+2)^2 = a^2 + 2^a$, or $a + 1 = 2^{a-2}$. By simple induction on n, it is easily seen that $n \le 2^{n-3}$ for all integers $n \ge 6$, and that $n \ne 2^{n-3}$ for $1 \le n \le 5$. Hence, $a + 1 = 2^{a-2}$ has no solutions in integers. Our proof is now complete.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina. A partial solution was submitted by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Guersenzvaig considered a similar problem and showed that for positive integers a, b, and c, the Diophantine equation $(a+b)^b = a^b + b^{ac}$ holds if and only if either $b = 1$ or $(a,b,c) = (1,2,3)$. His proof also used the Fermat-Wiles Theorem.

2772. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f\big(yf(x) - 1\big) \;=\; x^2 f(y) - f(x) \text{ for all real } x \text{ and } y.$$

Solution by D. Kipp Johnson, Beaverton, OR, USA.

First we note that the constant function $f(x) = 0$ is a solution of the given equation.

Let $f(x)$ be a solution such that $f(x) \ne 0$ for some x. We show that this implies $f(x) = x$, thereby establishing that the given equation has only two solutions: $f(x) = 0$ and $f(x) = x$.

Letting $x = 0$ in the given equation gives

$$f(0)\left[f\big(yf(0) - 1\big) + 1\right] \;=\; 0\,.$$

Suppose $f(0) \ne 0$. Since $yf(0) - 1$ can take any value x (just replace y by $(x+1)/f(0)\big)$, we obtain a possible candidate for a solution, namely, the constant function $f(x) = -1$. A quick check shows that this function is not a solution of the given equation. Therefore, $f(0) = 0$.

Now, suppose that $f(x) = 0$ for some $x \ne 0$. Then the original equation gives $0 = x^2 f(y)$, which can only happen if $f(y) = 0$ for all y, a contradiction, because we have already assumed that f is not zero everywhere. Thus, we can conclude that $f(x) = 0$ if and only if $x = 0$.

Letting $x = y = 1$ in the original equation gives $f(1)\, f\big(f(1) - 1\big) = 0$, and since $f(1) \ne 0$, we must have $f\big(f(1) - 1\big) = 0$, implying $f(1) - 1 = 0$. Hence, $f(1) = 1$.

When $x = 1$, the original equation then becomes

$$f(y-1) \;=\; f(y) - 1\,. \tag{1}$$