ATOMIC DECOMPOSITION OF WEIGHTED BESOV SPACES

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ABSTRACT

We find the atomic decomposition of functions in the weighted Besov spaces under certain factorization conditions on the weight.

Introduction

Following the achievement establishing the atomic decomposition of Hardy spaces (see [**8**, **22**, **33**]), many other function spaces have been shown to admit similar decompositions. We mention the decomposition of BMO (see [**32**, **25**]), Bergman spaces (see [**9**, **24**]), the predual of Bloch space (see [**12**]), Besov spaces (see [**15**, **5**, **10**]), Lipschitz spaces (see [**18**]), Triebel–Lizorkin spaces (see [**16**, **31**]).

They are obtained by quite different methods, but there is a unified and beautiful approach to get the decomposition for most of the spaces. This is the use of a formula due to A. P. Calderón (see [**6**, **7**]). The reader is referred to the book by M. Frazier, B. Jawerth and G. Weiss [**18**] for a compilation of spaces where Calderón's formula produces the atomic decomposition, as well as applications of it.

Atomic decompositions of weighted versions of different spaces have been also considered in the literature (see [**27**] for weighted Hardy spaces, [**4**] for Lipschitz spaces, and so on).

In this paper we shall be concerned with weighted Besov spaces $B^{p,q}_{\phi,w}$. We shall find some conditions on the weights which are necessary for atomic decomposition on the spaces.

We refer the reader to [**19**, **29**, **18**] for general notions and applications of atomic decomposition and to [**1**, **23**, **30**] for different formulations and properties of Besov classes.

The classes of weights where the results are achieved consist of those which factorize through powers of Dini and b_1 weights. Our arguments for the cases $1 < p, q < \infty$ will be based upon two main points: Calderón's formula and a quite simple Schur Lemma. To obtain the extreme cases $p, q \in \{1, \infty\}$ we need some new results on the classes of weights which enable us to apply the same procedure as in the previous cases. The reader should be aware that the case $1 < q < \infty$ could have been shown by interpolation with the extreme cases, but a direct proof is presented in the paper.

Throughout the paper a weight $w\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable function $w > 0$ a.e., with $1 \leqslant p, q \leqslant \infty$ and p', q' stand for the conjugate exponents; $\mathcal{S}$ denotes the

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Schwartz class of test functions on $\mathbb{R}^n$, $\mathscr{S}'$ the space of tempered distributions, $\mathscr{S}_0$ the set of functions in $\mathscr{S}$ with mean zero and $\mathscr{S}'_0$ its topological dual. We denote by $\mathscr{D}$ the collection of dyadic cubes Q in $\mathbb{R}^n$, that is,

$$Q = Q_{j,z} = \{x \in \mathbb{R}^n : 2^j z_i \leqslant x_i \leqslant 2^j(z_i + 1)\} \quad \text{for } z \in \mathbb{Z}^n, j \in \mathbb{Z}.$$

As usual $|Q|$, $l(Q)$ stand for the volume and the side length of the cube Q, respectively. We shall write $T(Q)$ for $Q \times (\frac{1}{2}l(Q), l(Q)] \in \mathbb{R}^{n+1}_+$ and cQ for a cube with the same centre as Q but with side length equal $cl(Q)$.

Given a weight $w, \phi \in \mathscr{S}_0$ and $1 \leqslant p, q \leqslant \infty$ we shall denote by $B^{p,q}_{w,\phi}$ the space functions $f \colon \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$ such that

$$\|f\|_{B^{p,q}_{w,\phi}} = \left(\int_{\mathbb{R}^+} \frac{\|\phi_t * f\|_p^q}{w(t)^q} \frac{dt}{t} \right)^{1/q} < \infty \quad \text{for } 1 \leqslant q < \infty$$

or

$$\|f\|_{B^{p,\infty}_{w,\phi}} = \inf\{C > 0 \colon \|\phi_t * f\|_p \leqslant C w(t) \text{ a.e. } t > 0\} < \infty \quad \text{for } q = \infty,$$

where $\phi_t(x) = (1/t^n)\phi(x/t)$.

To state the results of the paper, let us first recall the following notions.

A weight w is said to *satisfy Dini condition* if there exists $C > 0$ such that

$$\int_0^s \frac{w(t)}{t} dt \leqslant C w(s) \quad \text{a.e. } s > 0.$$

A weight w is said to be a b_1*-weight* if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^2} dt \leqslant C \frac{w(s)}{s} \quad \text{a.e. } s > 0.$$

We shall denote by $\mathscr{W}_{0,1}$ the space of b_1-weights which satisfy Dini condition. We use the notation $\mathscr{A}_1$ for the class of functions $\phi \in \mathscr{S}_0$ such that

(a) $\int_0^\infty (\hat{\phi}(t\chi))^2 (dt/t) = 1$ for $\chi \neq 0$,
(b) ϕ radial and real,
(c) $\operatorname{supp} \phi \subseteq \{|x| \leqslant 1\}$,
(d) $\int_{\mathbb{R}^n} x_i \phi(x)\, dx = 0$ for $i = 1, \dots, n$.

We refer the reader to Section 1 for the notion of the (A,p)-atom and the unexplained notations. The aim of the paper is to prove the following theorem.

Main Theorem. *Let $1 \leqslant p, q \leqslant \infty, \phi \in \mathscr{A}_1$ and w be a weight that can be factored as $w(t) = \lambda^{1/q'}(t)\mu^{-1/q}(t^{-1})$, where $\lambda, \mu \in \mathscr{W}_{0,1}$. Then if*

$$w_Q = \left(\int_{l(Q)/2}^{l(Q)} w(t)^{q'} \frac{dt}{t} \right)^{1/q'}$$

we have $f \in B^{p,q}_{w,\phi}$ if and only if there exist $A > 0$, $\{s_Q\}_{Q \in \mathscr{D}}$ and (A,p)-atoms $\{a_Q\}_{Q \in \mathscr{D}}$ such that $f = \sum_{Q \in \mathscr{D}} s_Q a_Q$ (convergence in $\mathscr{S}'_0$) and

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} \left| \frac{s_Q}{w_Q} \right|^p \right)^{q/p} \right)^{1/q} < \infty.$$

Moreover

$$\|f\|_{B^{p,q}_{w,\phi}} \approx \inf\left\{\left(\sum_{j=-\infty}^{\infty}\left(\sum_{l(Q)=2^j}\left|\frac{s_Q}{w_Q}\right|^p\right)^{q/p}\right)^{1/q} : f = \sum_{Q\in\mathscr{D}} s_Q a_Q\right\}$$

(*with the obvious modifications for the case when p and q are infinite*).

This can be understood as a generalization of the cases proved in **[15, 5, 10]** corresponding to $w(t) = t^\alpha$.

The paper is divided into three sections. Section 1 has a preliminary character and it is devoted to introducing the notation and the main lemmas to be used later on. In Section 2 we get the atomic decomposition for the spaces in the case $1 \leqslant p < \infty$ and $1 < q < \infty$ and we postpone the remaining cases to the last section.

1. *Preliminaries and basic lemmas*

Let us recall some notions on weights that we shall need later.

DEFINITION 1.1. Let $\varepsilon, \delta \in \mathbb{R}$ and w be a weight, then w is said to be a *d_ε-weight* if there exists $C > 0$ such that

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leqslant Cs^\varepsilon w(s) \quad \text{a.e.}\, s > 0, \tag{1.1}$$

and w is said to be a *b_δ-weight* if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^\delta}\frac{dt}{t} \leqslant C\frac{w(s)}{s^\delta} \quad \text{a.e.}\, s > 0. \tag{1.2}$$

If (d_ε) (respectively (b_δ)) denotes the class of d_ε-weights (respectively b_δ-weights) we write

$$\mathscr{W}_{\varepsilon,\delta} = (d_\varepsilon) \cap (b_\delta).$$

REMARK 1.1. The main examples of such weights are given by

$$w_{\alpha,\beta}(t) = t^\alpha(1+|\log t|)^\beta.$$

It is left to the reader to show that $w_{\alpha,\beta} \in \mathscr{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Let us collect some elementary properties to be used in the sequel:

(1.3) $w \in (d_\varepsilon) \Rightarrow w \in (d_{\varepsilon'})$ for any $\varepsilon' > \varepsilon$;

(1.4) let $\bar{w}(t) = w(t^{-1})$, then $w \in (b_\varepsilon) \Leftrightarrow \bar{w} \in (d_\varepsilon)$;

(1.5) $w \in \mathscr{W}_{\varepsilon,\delta} \Rightarrow w(t) \geqslant C\min(t^{-\varepsilon}, t^\delta)$.

The following properties on weights belonging to $\mathscr{W}_{0,1}$ are needed for some results later on.

LEMMA 1.1. *Let $\varepsilon \geqslant 0$, $\delta \geqslant 0$ and $w \in \mathscr{W}_{\varepsilon,\delta}$. Then*

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leqslant C \inf_{s/2 \leqslant u < \infty} u^\varepsilon w(u), \tag{1.6}$$

$$\int_s^\infty \frac{w(t)}{t^\delta}\frac{dt}{t} \leqslant C \inf_{0 < u \leqslant 2s} \frac{w(u)}{u^\delta}. \tag{1.6'}$$

Proof. From (1.4) it is enough to prove (1.6). Let $u \geqslant s$. From (1.1)

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leqslant C\int_0^u t^\varepsilon w(t)\frac{dt}{t} \leqslant Cu^\varepsilon w(u).$$

If we integrate this inequality in $[s, 2s]$ against the weight $1/u^{1+\varepsilon+\delta}$ we have

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t}\int_s^{2s}\frac{du}{u^{1+\varepsilon+\delta}} \leqslant C\int_s^{2s} w(u)\frac{du}{u^{1+\delta}} \leqslant C\int_s^\infty w(u)\frac{du}{u^{1+\delta}}.$$

Hence, if $s/2 \leqslant v \leqslant s$, we have

$$C'\frac{1}{s^{\varepsilon+\delta}}\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leqslant C\int_v^\infty w(u)\frac{du}{u^{1+\delta}} \leqslant C\frac{w(v)}{v^\delta}$$

and, finally,

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leqslant \frac{C}{C'}w(v)\frac{s^{\varepsilon+\delta}}{v^\delta} \leqslant \frac{2^{\varepsilon+\delta}C}{C'}w(v)\,v^\varepsilon.$$

COROLLARY 1.1. *Let $w \in \mathscr{W}_{0,1}$. Then for any $s > 0$ we have*

$$\int_0^\infty \min\left(\frac{s}{t}, 1\right) w(t)\frac{dt}{t} \leqslant C \inf_{s/2 \leqslant u < s} w(u). \tag{1.7}$$

The next result was pointed out to the authors by F. Ruiz and J. Bastero, who showed us the proof we present here.

LEMMA 1.2. *Let $w \in (d_\varepsilon)$ (respectively, $w \in (b_\delta)$). Then there exists $\rho > 0$ such that $w \in (d_{\varepsilon-\rho})$ (respectively, $w \in (b_{\delta-\rho})$).*

Proof. From (1.4) it is enough to consider $w \in (d_\varepsilon)$. Write $\lambda(t) = t^\varepsilon w(t)$. Clearly $\lambda \in (d_0)$. Let us define the operator

$$H(\lambda)(t) = \int_0^t \frac{\lambda(s)}{s}ds \quad \text{for } t > 0.$$

Since $H(\lambda) \leqslant C\lambda$ then $H^n(\lambda) \leqslant C^n\lambda$. Applying Fubini's theorem and an easy induction one gets

$$H^n(\lambda)(t) = \frac{1}{(n-1)!}\int_0^t \frac{\lambda(s)}{s}\log^{n-1}\left(\frac{t}{s}\right)ds.$$

Take $\rho > 0$ such that $\rho C < 1$, then $\sum_{n=1}^\infty \rho^{n-1}H^n(\lambda) \leqslant (C/(1-\rho C))\,\lambda$. Hence

$$\int_0^t \frac{\lambda(s)}{s}\left(\frac{t}{s}\right)^\rho ds \leqslant \frac{C}{1-\rho C}\lambda(t).$$

Finally this gives

$$\int_0^t s^{\varepsilon-\rho}w(s)\frac{ds}{s} \leqslant \frac{C}{1-\rho C}t^{\varepsilon-\rho}w(t).$$

DEFINITION 1.2. Let $1 \leqslant p \leqslant \infty$ and $A > 0$. A differentiable function a_Q is called an *(A,p)-smooth atom* if

$$\operatorname{supp} a_Q \subseteq 3Q \quad \text{for some } Q \in \mathscr{D}, \tag{1.8}$$

$$\int a_Q(x)\,dx = \int x_i\, a_Q(x)\,dx = 0 \quad \text{for } i = 1,2,\ldots,n, \tag{1.9}$$

$$|a_Q(x)| \leqslant \frac{A}{l(Q)^{n/p}}, \quad \left|\frac{\partial}{\partial x_i} a_Q(x)\right| \leqslant \frac{A}{l(Q)^{n/p+1}} \quad \text{for } i = 1,2,\ldots,n. \tag{1.10}$$

Let us now establish one of the main lemmas to be used later on. This result is closely related with Calderón's reproducing formula, and gives a procedure to decompose functions in $L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$.

LEMMA A (*see* **[6, 18]**). *Let $f \in L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$ and $\phi \in \mathscr{A}_1$. Given $1 \leqslant p \leqslant \infty$, $Q \in \mathscr{D}$, define*

$$s_Q(f) = |Q|^{-1/p'} \iint_{T(Q)} |\phi_t * f(y)|\,dy\frac{dt}{t},$$

$$a_Q(f)(x) = \frac{1}{s_Q(f)} \iint_{T(Q)} \phi_t(x-y)\,\phi_t * f(y)\,dy\frac{dt}{t}.$$

Then the $a_Q(f)$ are (A,p)-smooth atoms for

$$A = 2^{n+1}\max\left\{|\phi(x)|, \left|\frac{\partial}{\partial x_i}\phi(x)\right| \textit{ for } i = 1,\ldots,n\right\}$$

and

$$f = \sum_{Q\in\mathscr{D}} s_Q(f)\,a_Q(f) = \lim_{M\to\infty,\,N\to\infty} \sum_{k=-M}^{N} \sum_{l(Q)=2^k} s_Q(f)\,a_Q(f) \quad \textit{in } \mathscr{S}'_0.$$

LEMMA 1.3. *Let $1 \leqslant p \leqslant \infty, \phi \in \mathscr{A}_1, f \in L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$. If we write*

$$s_Q(f) = |Q|^{-1/p'} \iint_{T(Q)} |\phi_t * f(y)|\,dy\frac{dt}{t},$$

then for any $j \in \mathbb{Z}$ we have

$$\left(\sum_{l(Q)=2^j} |s_Q(f)|^p\right)^{1/p} \leqslant \int_{2^{j-1}}^{2^j} \|\phi_t * f\|_p \frac{dt}{t}. \tag{1.11}$$

Proof. Assume that $p < \infty$ (the case $p = \infty$ is similar), then

$$\begin{aligned}
\left(\sum_{l(Q)=2^j} |s_Q(f)|^p\right)^{1/p} &= \sup_{\sum \beta_Q^{p'}=1} \left|\sum_{l(Q)=2^j} \beta_Q\, s_Q(f)\right| \\
&= \sup_{\sum \beta_Q^{p'}=1} \left|\int_{2^{j-1}}^{2^j}\int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^j} \beta_Q |Q|^{-1/p'}\chi_Q\right) |\phi_t * f(y)|\,dy\frac{dt}{t}\right| \\
&\leqslant \sup_{\sum \beta_Q^{p'}=1} \int_{2^{j-1}}^{2^j} \left\|\sum_{l(Q)=2^j} \beta_Q |Q|^{-1/p'}\chi_Q\right\|_{p'} \|\phi_t * f\|_p \frac{dt}{t} \\
&\leqslant \int_{2^{j-1}}^{2^j} \|\phi_t * f\|_p \frac{dt}{t}.
\end{aligned}$$

LEMMA 1.4. *Let* $1 \leqslant p \leqslant \infty, A > 0, \{\alpha_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$ *and* $\{a_Q\}_{Q\in\mathcal{D}}$ *satisfies* (1.8) *and* (1.10) *for* A *and* p. *Then for* $t > 0$ *and* $j \in \mathbb{Z}$ *we have*

$$\Big\|\sum_{l(Q)=2^j} \alpha_Q a_Q\Big\|_p \leqslant C\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}, \tag{1.12}$$

$$\Big\|\sum_{l(Q)=2^j} \alpha_Q(\phi_t * a_Q)\Big\|_p \leqslant C\min(t/2^j, 1)\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}. \tag{1.13}$$

Proof. First observe that $\sum_{l(Q)=2^j} s_Q a_Q$ has only a finite number of non-zero terms since there are a finite number of overlapping cubes of the form $\{3Q\}$. Hence

$$\Big\|\sum_{l(Q)=2^j} \alpha_Q a_Q\Big\|_p \leqslant A\Big\|\sum_{l(Q)=2^j} |\alpha_Q|\,|Q|^{-1/p}\chi_{3Q}\Big\|_p \leqslant C\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}.$$

Use the previous estimate (1.12) and Young's inequality to get

$$\Big\|\sum_{l(Q)=2^j} \alpha_Q(\phi_t * a_Q)\Big\|_p \leqslant \|\phi_t\|_1\Big\|\sum_{l(Q)=2^j} \alpha_Q a_Q\Big\|_p \leqslant \|\phi\|_1\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}.$$

On the other hand, assume that $l(Q) = 2^j$ and $t < 2^j$. Note that $x \notin 5Q$ and $y \in 3Q$ implies that $|(x-y)/t| > 1$ and so $\phi((x-y)/t)\,a_Q(y) = 0$. Hence $\operatorname{supp}\phi_t * a_Q \subseteq 5Q$. Moreover

$$|a_Q(y) - a_Q(x)| \leqslant C \sup_{\xi\in[x,y]} |\nabla a_Q(\xi)|\,|x-y| \leqslant \frac{CA|x-y|}{l(Q)^{n/p+1}}.$$

Therefore, using the fact that $\int \phi_t = 0$, we have

$$\begin{aligned}|\phi_t * a_Q(x)| &= \Big|\int \phi_t(x-y)(a_Q(y)-a_Q(x))\,dy\Big| \\ &\leqslant \frac{CA}{l(Q)^{n/p+1}}\int |\phi_t(x-y)|\,|x-y|\,dy = CA\frac{t}{l(Q)^{n/p+1}}\int |\phi(z)|\,|z|\,dz.\end{aligned}$$

We have proved that $|\phi_t * a_Q(x)| \leqslant C\frac{t}{l(Q)^{n/p+1}}\chi_{5Q}$. Hence

$$\|\sum s_Q \phi_t * a_Q\|_p \leqslant C\frac{t}{2^j}\Big\|\sum_{l(Q)=2^j} |\alpha_Q|\,|Q|^{-1/p}\chi_{5Q}\Big\|_p \leqslant C\frac{t}{2^j}\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}.$$

LEMMA 1.5. *Let* $1 \leqslant p \leqslant \infty, \{\alpha_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$ *and let* $\{a_Q\}_{Q\in\mathcal{D}}$ *be* (A,p)*-smooth atoms. Then there exist* $\{c_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$ *with* $c_Q \neq 0$ *for finitely many* $Q \in \mathcal{D}$ *such that for* $p < \infty$ *and* $p = \infty$ *respectively, we have*

$$\int_{\mathbb{R}^n} \frac{|\sum_{l(Q)=2^j} \alpha_Q(a_Q - c_Q)|}{(1+|x|)^{n+1}}\,dx \leqslant C\min(1, 2^{-j})\Big(\sum_{l(Q)=2^j} |\alpha_Q|^p\Big)^{1/p}, \tag{1.14}$$

$$\int_{\mathbb{R}^n} \frac{|\sum_{l(Q)=2^j} \alpha_Q(a_Q - c_Q)|}{(1+|x|)^{n+1}}\,dx \leqslant C\min\Big(1, \frac{|j|}{2^j}\Big)\sup_{l(Q)=2^j} |\alpha_Q|. \tag{1.14'}$$

Proof. Let

$$c_Q = \begin{cases} a_Q(0) & \text{if } l(Q) \geqslant 1 \text{ and } 3Q \cap B(0, l(Q)) \neq \varnothing, \\ 0 & \text{otherwise.}\end{cases}$$

For $j<0$ we have $c_Q=0$ and then we simply use the estimate

$$\left|\sum_{l(Q)=2^j}\alpha_Q(a_Q(x)-c_Q)\right|\leqslant A\sum_{l(Q)=2^j}|\alpha_Q|\,|Q|^{-1/p}\chi_{3Q}(x).$$

Hölder's inequality gives

$$\int_{\mathbb{R}^n}\frac{|\sum_{l(Q)=2^j}\alpha_Q(a_Q-c_Q)|}{(1+|x|)^{n+1}}dx\leqslant C\Big(\sum_{l(Q)=2^j}|\alpha_Q|^p\Big)^{1/p}.$$

For $j\geqslant 0$ we argue as follows. Note that, for fixed j, there exist a finite number, independent of j, of dyadic cubes of length 2^j such that $3Q\cap B(0,l(Q))\neq\varnothing$. Call such a family $\mathscr{F}_j$ and denote by $E_j=\bigcup_{Q\in\mathscr{F}_j}3Q$.

In the same way as in the case $j<0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\sum_{Q\notin\mathscr{F}_j}\alpha_Q(a_Q-c_Q)|}{(1+|x|)^{n+1}}dx&\leqslant\left(\int\Big|\sum_{Q\notin\mathscr{F}_j}\alpha_Q\,a_Q\Big|^p\right)^{1/p}\left(\int_{|x|\geqslant 2^j}(1+|x|)^{-(n+1)p'}dx\right)^{1/p'}\\&\leqslant C2^{-j(1+n/p)}\Big(\sum_{Q\notin\mathscr{F}_j}|\alpha_Q|^p\Big)^{1/p}.\end{aligned}$$

For $Q\in\mathscr{F}_j$ and $x\notin E_j$ we use the simple estimate $|a_Q(x)-C_Q|\leqslant 2A2^{-jn/p}$. Hence

$$\begin{aligned}\int_{\mathbb{R}^n\setminus E_j}\frac{|\sum_{Q\in\mathscr{F}_j}\alpha_Q(a_Q(x)-c_Q)|}{(1+|x|)^{n+1}}&\leqslant C2^{-jn/p}\int_{|x|\geqslant 2^j}\frac{dx}{(1+|x|)^{(n+1)}}\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&\leqslant C2^{-jn/p}\int_{2^j}^{\infty}t^{-2}dt\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&=C2^{-j(1+n/p)}\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&\leqslant C2^{-j(1+n/p)}\Big(\sum_{Q\in\mathscr{F}_j}|\alpha_Q|^p\Big)^{1/p}.\end{aligned}$$

Finally, we observe that it follows from the mean value theorem and (1.10) that there exists $C'>0$ such that

$$|a_Q(x)-c_Q|\leqslant C'\frac{|x|}{l(Q)^{n/p+1}}\quad\text{for } x\in E_j, Q\in\mathscr{F}_j.$$

It is clear that if $x\in E_j$ then there exist K, independent of j, such that $|x|\leqslant K2^j$. Therefore

$$\begin{aligned}\int_{E_j}\frac{|\sum_{Q\in\mathscr{F}_j}\alpha_Q(a_Q-c_Q)|}{(1+|x|)^{n+1}}dx&\leqslant A2^{-jn/p-1}\int_{|x|\leqslant K2^j}\frac{|x|}{(1+|x|)^{n+1}}dx\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&\leqslant A2^{-jn/p-1}\left(\int_0^{K2^j}\frac{t^n}{(1+t)^{n+1}}dt\right)\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&\leqslant C2^{-jn/p-1}(1+\log(2^j))\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\\&\leqslant C2^{-jn/p-1}|j|\sum_{Q\in\mathscr{F}_j}|\alpha_Q|\leqslant C2^{-jn/p-1}|j|\Big(\sum_{Q\in\mathscr{F}_j}|\alpha_Q|^p\Big)^{1/p}.\end{aligned}$$

Combining the previous estimates we have (1.14) and (1.14′).

Observe that a net $\{\phi_i\}_{i\in\Lambda}$ converges to ϕ in $\mathscr{S}'_0$ if there exist $\{c_i\}_{i\in\Lambda}\subset\mathbb{C}$ such that $\phi_i - c_i$ converges to ϕ in S'. Using this it is elementary to prove the following lemma.

LEMMA 1.6. *Let $\{f_j\}_{j\in\mathbb{Z}}$ be measurable functions defined in $\mathbb{R}^n$ such that there exist real numbers $\{c_j\}_{j\in\mathbb{Z}}$ with*

$$\int_{\mathbb{R}^n}\frac{\sum_{j\in\mathbb{Z}}|f_j-c_j|}{1+|x|^{n+1}}\,dx<\infty.$$

Then $\sum_{j\in\mathbb{Z}}f_j$ converges in $\mathscr{S}'_0$ to some function $f\in L^1(\mathbb{R}^n, dx/(1+|x|^{n+1}))$.

2. *Atomic decompositions for $B^{p,q}_{\phi,w}$ for $1\leqslant p<\infty$ and $1<q<\infty$.*

Let us first state a version of the Schur Lemma that will be useful for our purposes and whose proof follows easily from Hölder's inequality.

LEMMA B. *Let $1<q<\infty$ and $1/q+1/q'=1$. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be two σ-finite measure spaces, let $K\colon\Omega_1\times\Omega_2\to\mathbb{R}^+$ be a measurable function and write $T_K(f)$ for*

$$T_K(f)(w_2)=\int_{\Omega_1}K(w_1,w_2)f(w_1)\,d\mu_1(w_1).$$

If there exist $C>0$ and measurable functions $h_i\colon\Omega_i\to\mathbb{R}^+$ (for $i=1,2$) such that

$$\int_{\Omega_1}K(w_1,w_2)\,h_1^{q'}(w_1)\,d\mu_1(w_1)\leqslant Ch_2^{q'}(w_2)\quad \mu_2\text{-a.e.},\tag{2.1}$$

$$\int_{\Omega_2}K(w_1,w_2)\,h_2^{q}(w_2)\,d\mu_2(w_2)\leqslant Ch_1^{q}(w_1)\quad \mu_1\text{-a.e.}\tag{2.2}$$

Then T_K defines a bounded operator from $L^q(\Omega_1,\mu_1)$ into $L^q(\Omega_2,\mu_2)$.

THEOREM 2.1. *Let $1\leqslant p<\infty, 1<q<\infty, \phi\in\mathscr{A}_1, w(t)=\lambda^{1/q'}(t)\,\mu^{-1/q}(t^{-1})$, where $\lambda,\mu\in\mathscr{W}_{0,1}$. Setting*

$$w_Q=\left(\int_{\frac{1}{2}l(Q)}^{l(Q)}w^{q'}(t)\frac{dt}{t}\right)^{1/q'}$$

we have $f\in B^{p,q}_{w,\phi}$ if and only if there exist $A>0, \{s_Q\}_{Q\in\mathscr{D}}\subseteq\mathbb{C}$ and (A,p)-smooth atoms $\{a_Q\}_{Q\in\mathscr{D}}$ such that $f=\sum_{Q\in\mathscr{D}}s_Q a_Q$ in $\mathscr{S}'_0$ and

$$\sum_{j=-\infty}^{j=\infty}\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{q/p}<\infty.\tag{2.3}$$

Moreover

$$\|f\|_{B^{p,q}_{w,\phi}}\approx\inf\left\{\left(\sum_{j=-\infty}^{j=\infty}\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{q/p}\right)^{1/q}: f=\sum_{Q\in\mathscr{D}}s_Q a_Q\right\}.$$

Proof. Assume that $f\in B^{p,q}_{w,\phi}$ and write

$$w_j=\left(\int_{2^{j-1}}^{2^j}w^{q'}(t)\frac{dt}{t}\right)^{1/q'}.$$

The atomic decomposition is obtained from Lemma A. The only thing to prove is the estimate (2.3). Using (1.11) we have

$$\left(\sum_{l(Q)=2^j}\left(\frac{s_Q}{w_Q}\right)^p\right)^{1/p} \leqslant \int_0^\infty \frac{w(t)}{w_j}\chi_{(2^{j-1},2^j)}(t)\frac{\|\phi_t * f\|_p}{w(t)}\frac{dt}{t}.$$

Hence from duality and Hölder's inequality it follows that

$$\left(\sum_{j=-\infty}^{\infty}\left(\sum_{l(Q)=2^j}\left(\frac{s_Q}{w_Q}\right)^p\right)^{q/p}\right)^{1/q} = \sup_{\sum\beta_j^{q'}=1}\int_0^\infty\left(\sum_{j=-\infty}^{\infty}\frac{\beta_j}{w_j}w(t)\chi_{(2^{j-1},2^j)}(t)\right)\frac{\|\phi_t * f\|_p}{w(t)}\frac{dt}{t}$$

$$\leqslant \sup_{\sum\beta_j^{q'}=1}\left\|\sum_{j=-\infty}^{\infty}\beta_j\frac{w(t)}{w_j}\chi_{(2^{j-1},2^j)}(t)\right\|_{q'}\|f\|_{B^{p,q}_{w,\phi}} = \|f\|_{B^{p,q}_{w,\phi}}.$$

Conversely, let us assume that $f=\sum_{Q\in\mathscr{D}}s_Q a_Q$ where $\{s_Q\}_{Q\in\mathscr{D}}$ satisfies (2.3). We now use (1.14) in Lemma 1.5 and Lemma 1.6 to prove that $\sum_{Q\in\mathscr{D}}s_Q a_Q$ converges in $\mathscr{S}'_0$ to a function $f\in L^1(\mathbb{R}^n, dx/((1+|x|)^{n+1}))$. It suffices to prove that $w_j\min(1,2^{-j})\in l_{q'}(\mathbb{Z})$. We have

$$\sum_{j\in\mathbb{Z}} w_j^{q'}\min(1,2^{-j})^{q'} \leqslant C\sum_{j\in\mathbb{Z}}\int_{2^{j-1}}^{2^j} w^{q'}(t)\min\left(1,\frac{1}{t}\right)^{q'}\frac{dt}{t} \leqslant C\int_0^\infty w^{q'}(t)\min\left(1,\frac{1}{t}\right)^{q'}\frac{dt}{t}$$

$$\leqslant C\left(\int_0^1 \lambda(t)\mu^{-q'/q}(t^{-1})\frac{dt}{t}+\int_1^\infty \lambda(t)\mu^{-q'/q}(t^{-1})\left(\frac{1}{t}\right)^{q'}\frac{dt}{t}\right).$$

Using (1.5) we have $\mu(s)\geqslant\min(1,s)$. Hence

$$\sum_{j\in\mathbb{Z}} w_j^{q'}\min(1,2^{-j})^{q'} \leqslant C\left(\int_0^1\lambda(t)\frac{dt}{t}+\int_1^\infty\lambda(t)\,t^{q'-1}t^{-q'}\frac{dt}{t}\right)$$

$$\leqslant C\left(\int_0^1\lambda(t)\frac{dt}{t}+\int_1^\infty\lambda(t)\frac{dt}{t^2}\right)<\infty.$$

Since $\|\phi_t * f\|_p \leqslant \sum_{j\in\mathbb{Z}}\|\sum_{l(Q)=2^j}s_Q(\phi_t * a_Q)\|_p$ we can use (1.13) in Lemma 1.4 to get

$$\frac{\|\phi_t * f\|_p}{w(t)} \leqslant C\sum_{j=-\infty}^{\infty}\frac{w_j}{w(t)}\min\left(\frac{t}{2^j},1\right)\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}.$$

Let us write $(\Omega_1,\Sigma_1,\mu_1)=(\mathbb{N},\mathscr{P}(\mathbb{N}),dv)$, where v denotes the counting measure, and $(\Omega_2,\Sigma_2,\mu_2)=((0,+\infty),\mathscr{B}((0,+\infty)),dt/t)$ and consider the following kernel

$$K(j,t)=\frac{w_j}{w(t)}\min\left(\frac{t}{2^j},1\right).$$

Take

$$\alpha_j=\left(\int_{2^{j-1}}^{2^j}\lambda(t)\frac{dt}{t}\right)^{1/qq'}\left[\inf_{2^{-j}<t\leqslant 2^{-j+1}}\mu(t)\right]^{1/qq'} \quad\text{and}\quad g(t)=\lambda^{1/qq'}(t)\,\mu^{1/qq'}(t^{-1}).$$

Clearly we have

$$g^q(t)=w(t)\,\mu(t^{-1}),\quad g^{q'}(t)\,w(t)=\lambda(t). \tag{2.4}$$

On the other hand $w_j\leqslant(\int_{2^{j-1}}^{2^j}\lambda(t)\,dt/t)^{1/q'}(\inf_{2^{-j}<t\leqslant 2^{-j+1}}\mu(t))^{-1/q}$. This implies that

$$w_j\inf_{2^{-j}<t\leqslant 2^{-j+1}}\mu(t)\leqslant\alpha_j^q,\quad w_j\alpha_j^{q'}\leqslant\int_{2^{j-1}}^{2^j}\lambda(t)\frac{dt}{t}. \tag{2.5}$$

Hence from (2.4), Corollary 1.1 and (2.5)

$$\int_0^\infty K(j,t)\,g^q(t)\frac{dt}{t} = w_j\int_0^\infty \mu(t^{-1})\min\left(\frac{t}{2^j},1\right)\frac{dt}{t}$$
$$= w_j\int_0^\infty \mu(s)\min\left(\frac{2^{-j}}{s},1\right)\frac{ds}{s}$$
$$\leqslant Cw_j \inf_{2^{-j}<t\leqslant 2^{-j+1}} \mu(t) \leqslant C\alpha_j^q.$$

On the other hand, applying (2.4) and (2.5),

$$\sum_{j=-\infty}^{\infty} K(j,t)\,\alpha_j^{q'} \leqslant \frac{C}{w(t)}\sum_{j=-\infty}^{\infty}\min\left(\frac{t}{2^j},1\right)w_j\alpha_j^{q'}$$
$$\leqslant \frac{C}{w(t)}\sum_{j=-\infty}^{\infty}\min\left(\frac{t}{2^j},1\right)\int_{2^{j-1}}^{2^j}\lambda(s)\frac{ds}{s}$$
$$\leqslant \frac{C}{w(t)}\int_0^\infty \min\left(\frac{t}{s},1\right)\lambda(s)\frac{ds}{s} \leqslant C\frac{\lambda(t)}{w(t)} = Cg(t)^{q'}.$$

Hence, by Lemma B,

$$\left\|\frac{\|\phi_t * f\|_p}{w(t)}\right\|_{L^{q(dt/t)}} \leqslant C\left\|T_K\left(\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}\right)\right\|_{L^{q(dt/t)}}$$
$$\leqslant C\left\|\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}\right\|_{l^q} < \infty.$$

3. *Atomic decompositions for $B_{\phi,w}^{\infty,q}$, $B_{\phi,w}^{p,\infty}$ and $B_{\phi,w}^{p,1}$*

THEOREM 3.1. *Let $1\leqslant p<\infty$, $w\in\mathscr{W}_{0,1}$, $\phi\in\mathscr{A}_1$. Then $f\in B_{w,\phi}^{p,\infty}$ if and only if there exist $A>0$, $\{s_Q\}_{Q\in\mathscr{D}}\subseteq\mathbb{C}$ and (A,p)-smooth atoms $\{a_Q\}_{Q\in\mathscr{D}}$ such that $f=\sum_{Q\in\mathscr{D}}s_Q a_Q$ in $\mathscr{S}_0'$ and*

$$\left(\sum_{l(Q)=2^j}|s_Q|^p\right)^{1/p} \leqslant C\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t}. \tag{3.1}$$

Moreover

$$\|f\|_{B_{w,\phi}^{p,\infty}} \approx \inf\left\{\sup_{j\in\mathbb{Z}}\frac{\left(\sum_{l(Q)=2^j}|s_Q|^p\right)^{1/p}}{\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t}} : f=\sum_{Q\in\mathscr{D}}s_Q a_Q\right\}.$$

Proof. Assume that $f\in B_{w,\phi}^{p,\infty}$. Apply Lemma A and Lemma 1.3 to obtain (3.1).

Conversely assume that $\{s_Q\}_{Q\in\mathscr{D}}$ satisfies (3.1). Invoking Lemmas 1.5 and 1.6 we can see that $\sum_{Q\in\mathscr{D}}s_Q(a_Q-c_Q)$ converges in $\mathscr{S}_0'$ to a function $f\in L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$ as soon as we notice that $\min(1,2^{-j})\int_{2^{j-1}}^{2^j}w(t)\frac{dt}{t}\in l_1(\mathbb{Z})$. Then

$$\sum_{j\in\mathbb{Z}}\min(1,2^{-j})\int_{2^{j-1}}^{2^j}w(t)\frac{dt}{t} \leqslant C\int_R w(t)\min(1,t^{-1})\frac{dt}{t}$$
$$\leqslant C\int_0^1 w(t)\frac{dt}{t}+\int_1^\infty w(t)\frac{dt}{t^2}.$$

Note that $\phi_t * f = \sum_{j=-\infty}^{\infty} \sum_{l(Q)=2^j} s_Q(\phi_t * a_Q)$ in $\mathscr{S}'_0$. Hence

$$\|\phi_t * f\|_p \leqslant \sum_{j=-\infty}^{\infty} \left\| \sum_{l(Q)=2^j} s_Q(\phi_t * a_Q) \right\|_p . \tag{3.2}$$

Applying (3.2) and (1.13) in Lemma 1.4 we have

$$\begin{aligned}\|\phi_t * f\|_p &\leqslant C \sum_{j=-\infty}^{\infty} \min\left(\frac{t}{2^j}, 1\right) \int_{2^{j-1}}^{2^j} w(s) \frac{ds}{s} \leqslant C \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} \min\left(\frac{t}{s}, 1\right) w(s) \frac{ds}{s} \\ &\leqslant C\left(\int_0^t w(s) \frac{ds}{s} + t \int_t^\infty w(s) \frac{ds}{s^2} \right) \leqslant C w(t).\end{aligned}$$

THEOREM 3.2. *Let* $1 \leqslant p < \infty, \phi \in \mathscr{A}_1$, *w a weight such that* $w_1(t) = w^{-1}(t)^{-1} \in \mathscr{W}_{0,1}$. *Set* $w_j = \sup\{w(t); 2^{j-1} < t \leqslant 2^j\}$ *and* $w_Q = w_{l(Q)}$. *Then* $f \in B^{p,1}_{w,\phi}$ *if and only if there exist* $A > 0, \{s_Q\}_{Q \in \mathscr{D}} \subseteq \mathbb{C}$ *and* (A,p)-*atoms* $\{a_Q\}_{Q \in \mathscr{D}}$ *such that* $f = \sum_{Q \in \mathscr{D}} s_Q a_Q$ *in* $\mathscr{S}'_0$ *and*

$$\sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} < \infty. \tag{3.3}$$

Moreover

$$\|f\|_{B^{p,1}_{w,\phi}} \approx \inf\left\{ \sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} : f = \sum_{Q \in \mathscr{D}} s_Q a_Q \right\}.$$

Proof. Assume that $f \in B^{p,1}_{w,\phi}$. Once more use Lemma A and (1.11) to obtain

$$\left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} \leqslant \int_{2^{j-1}}^{2^j} \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t}.$$

Adding them up we get

$$\sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} \leqslant \|f\|_{B^{p,1}_{w,\phi}}.$$

Conversely take $\{s_Q\}_{Q \in \mathscr{D}}$ satisfying (3.3). As in Theorem 2.1 we shall prove that $\sum_{Q \in \mathscr{D}} s_Q a_Q$ converges in S'_0 to a function $f \in L^1(\mathbb{R}^n, dx/(1+|x|)^{n+1})$. It then suffices to prove that $w_j \min(1, 2^{-j}) \in l_\infty(\mathbb{Z})$, which easily follows from (1.5).

As in Theorem 2.1 we apply (1.13) to get

$$\|\phi_t * f\|_p \leqslant \sum_{j=-\infty}^{\infty} \left\| \sum_{l(Q)=2^j} s_Q(\phi_t * a_Q) \right\|_p \leqslant C \sum_{j=-\infty}^{\infty} \min\left(\frac{t}{2^{j-1}}, 1 \right) \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^+} \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t} &\leqslant C \sum_{j=-\infty}^{\infty} \left(\int_{\mathbb{R}^+} \frac{1}{w(t)} \min\left(\frac{t}{2^{j-1}}, 1 \right) \frac{dt}{t} \right) \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p} \\ &= C \sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p} \left(\int_0^\infty \min\left(\frac{2^{-j+1}}{s}, 1 \right) w_1(s) \frac{ds}{s} \right).\end{aligned}$$

From Corollary 1.1 we can estimate that

$$\left(\int_0^\infty \min\left(\frac{2^{-j+1}}{s}, 1 \right) w_1(s) \frac{ds}{s} \right) \leqslant C \inf_{2^{-j} < t < 2^{-j+1}} w_1(t) \leqslant C \frac{1}{w_j}.$$

This shows that

$$\int_{\mathbb{R}^+} \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t} \leqslant C \sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p}.$$

Analysing the previous proofs one realise that the only difficulty in extending to the case $p = \infty$ comes from the failure of (1.14), which in this case can be replaced by (1.14′). This problem can be overcome by using Lemma 1.2.

THEOREM 3.3. *Let $1 \leqslant q \leqslant \infty, \phi \in \mathscr{A}_1, w(t) = \lambda^{1/q'}(t)\mu^{-1/q}(t^{-1})$, where $\lambda, \mu \in \mathscr{W}_{0,1}$. Setting*

$$w_Q = \left(\int_{l(Q)/2}^{l(Q)} w^{q'}(t) \frac{dt}{t} \right)^{1/q'} \quad \textit{for } q > 1,$$

or

$$w_Q = \sup\{w(t); \tfrac{1}{2}l(Q) < t \leqslant l(Q)\} \quad \textit{for } q = 1,$$

we have $f \in B^{\infty;q}_{w,\phi}$ if and only if there exist $A > 0, \{s_Q\}_{Q\in\mathscr{D}} \subseteq \mathbb{C}$ and (A,p)-smooth atoms $\{a_Q\}_{Q\in\mathscr{D}}$ such that $f = \sum_{Q\in\mathscr{D}} s_Q a_Q$ in $\mathscr{S}'_0$ and

$$\sum_{j=-\infty}^{+\infty} \sup_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^q < \infty. \tag{2.8}$$

Moreover,

$$\|f\|_{B^{\infty;q}_{w,\phi}} \approx \inf\left\{ \left(\sum_{j=-\infty}^{+\infty} \sup_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^q \right)^{1/q} : f = \sum_{Q\in\mathscr{D}} s_Q a_Q \right\}.$$

(the obvious modifications for $q = \infty$).

Proof. Set

$$w_j = \left(\int_{2^{j-1}}^{2^j} w^{q'}(t) \frac{dt}{t} \right)^{1/q'}$$

or

$$w_j = \sup\{w(t) \colon 2^{j-1} < t \leqslant 2^j\}.$$

Using Lemma 1.2, it follows that $\mu, \lambda \in (b_\varepsilon)$ for some $0 < \varepsilon < 1$ and this is enough to show that $w_j \min(1, |j|/2^j) \in l^{q'}(\mathbb{Z})$.

Indeed, for $q = \infty$ we have $w(t) = \lambda(t)$ and $w_j \min(1, |j|/2^j) \leqslant C w_j \min(1, 1/2^{\varepsilon j})$. From this

$$\sum_{j\in\mathbb{Z}} w_j \min\left(1, \frac{|j|}{2^j}\right) \leqslant C\left(\int_0^1 \lambda(t) \frac{dt}{t} + \int_1^\infty \lambda(t) \frac{dt}{t^{\varepsilon+1}} \right) < \infty.$$

For $q = 1$ we have $w(t) = \mu^{-1}(t^{-1})$ and using (1.5) we have

$$w_j \min\left(1, \frac{|j|}{2^j}\right) \leqslant C \frac{1}{\inf_{2^{-j} < t < 2^{-j+1}} \mu(t)} \min\left(1, \frac{1}{2^{\varepsilon j}}\right) \leqslant C < \infty,$$

where C is independent of j.

For $1 < q < \infty$ take $0 < \alpha < 1$ such that $q'(1-\alpha) = 1 - \varepsilon$ and use

$$w_j \min\left(1, \frac{|j|}{2^j}\right) \leqslant C w_j \min\left(1, \frac{1}{2^{\alpha j}}\right).$$

Therefore

$$\sum_{j\in\mathbb{Z}} w_j^{q'}\left(\min\left(1,\frac{|j|}{2^j}\right)\right)^{q'} \leqslant C\left(\int_0^1 \lambda(t)\,\mu^{-q'/q}(t^{-1})\frac{dt}{t}+\int_1^\infty \lambda(t)\,t^{q'-1}\frac{dt}{t^{\alpha q'+1}}\right)<\infty.$$

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