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Is Symbolic Integration Better Than Numerical Integration in Satellite Dynamics?

A. ABAD AND A. ELIPE
Grupo de Mecánica Espacial, Univ. de Zaragoza
50009 Zaragoza, Spain
J. F. SAN-JUAN
Grupo de Mecánica Espacial, Univ. de Zaragoza
50009 Zaragoza, Spain
and
Depto. Matemáticas y Computación, Univ. de la Rioja
26004 Logroño, Spain
S. SERRANO
Grupo de Mecánica Espacial, Univ. de Zaragoza
50009 Zaragoza, Spain
and
Dpto. Matemática Aplicada, Univ. de Zaragoza
50009 Zaragoza, Spain

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Abstract—Computer algebra systems are developing very fast and it is now possible to use new computational power very efficiently to analytically integrate dynamical systems. However, the task of producing an appropriate program is time consuming and requires a considerable amount of skill and practice. Here the merits of numerical versus computer algebraic approaches are compared in the context of a specific problem. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

From time to time, there appear debates about the convenience of using either numerical computations or analytical theories to describe dynamical systems: usually the positions taken are quite radical. On the one hand, partisans of numerical integrations claim there are many efficient numerical methods, even with dense output, and since computers are becoming faster and cheaper, they do not see a particular reason for using analytical theories that usually give a less accurate solution due to truncation of expansions at finite order. On the other hand, people in favor of analytical methods advance almost the same arguments to defend their use; there are many commercial software packages that allow the construction of approximate analytical

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theories which give results of order high enough, are much less time computing, and so make analytical theories competitive with purely numerical methods, as it was shown in [1].

While the example provided in [1] is indeed competitive with respect to numerical integration (in the sense that the formulas there obtained are simple and give similar precision to well-tested numerical methods and even better behavior for long time integration), one may, nevertheless, extract erroneous conclusions; in fact, the example used to illustrate their conclusions is quite simple—an equatorial orbit under the J_2 perturbation (the second term in the Legendre polynomial expansion of the Earth's gravity field [2])—i.e., a problem that is integrable in terms of elliptic functions. In general, in artificial satellite theory (AST) one has to cope with significantly more involved situations, handling huge amount of terms and particular objects like the so-called Poisson series. All of this makes general purpose "off the shelf" algebraic manipulators ineffective and it becomes necessary to create a specific computer algebra system (CAS) to handle these types of objects in an efficient way.

In this letter, we illustrate the difficulties one can expect in obtaining a theory of high order in the main problem of the AST, which may require months of effort by trained people to develop the necessary purpose built symbolic algebraic routines. However, once one has obtained the desired formulas, the evaluation is very fast, and the results are reliable and comparable in precision with efficient numerical methods. Besides, and this is usually not emphasized sufficiently, they can be used to obtain a deeper insight into the qualitative problem [3], by determining the phase flow evolution, equilibria, bifurcations, etc.

2. THE MAIN PROBLEM IN AST

Celestial mechanics provides ample problems for use as work benches for both numerical methods and algebraic manipulators. The orbital motion of a particle under the attraction of the Earth is one of the most deeply studied due to its practical importance.

One of the most common ways to represent how Earth's gravity field acts on the motion of a satellite is (see [2]) by a finite sum of spherical harmonics. Assuming a symmetrical Earth, the potential function is

$$V = -\frac{\mu}{r} - \frac{\mu}{r} \sum_{n\geq 2} J_n \left(\frac{\alpha}{r}\right)^n P_n\left(\frac{z}{r}\right), \tag{1}$$

where α stands for the mean radius of the equator of the Earth, μ is the Gaussian constant, P_n is the Legendre polynomial of degree n, r is the radial distance, and z is the third component of the position vector of the satellite. The coefficients J_n are constants representing the shape of the Earth and are called *zonal harmonics*. When only the coefficient J_2 is taken into account, we have the *main problem*, that is routinely used in the first steps in mission analysis. Thus, this potential is a perturbation on the pure Keplerian motion, and since $J_2 \approx 0.001$, it is taken as a small parameter ε.

In AST, the Hamiltonian formalism is preferred to Newton's equations, and thus, Cartesian coordinates and velocities are abandoned in favor of appropriate sets of canonical variables $(\mathbf{x}, \mathbf{X})$ that satisfy the following first-order system of ODE (Hamilton equations):

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{X}}, \qquad \frac{d\mathbf{X}}{dt} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}},$$

where $\mathcal{H}(\mathbf{x}, \mathbf{X})$ is a scalar function known as the Hamiltonian and that is made of the sum of the kinetic and potential energies. Usually, these equations cannot be solved in closed form. Ideally, analytical theories hope to obtain new sets of canonical variables $(\mathbf{y}, \mathbf{Y})$ such that Hamilton's equations in the new variables can be solved in a simple way. Unfortunately, this situation rarely happens, and an alternative approach consists of trying to find new equations that are somehow simpler than the old ones, e.g., the famous planetary or lunar theories in the 19^{th} century, and methods like the ones of Poincaré, von Zeipel, and the more recent based on Lie series [4].

A Lie transformation may be defined as an infinitesimal contact transformation $\varphi : (\mathbf{y}, \mathbf{Y}; \varepsilon) \longmapsto (\mathbf{x}, \mathbf{X})$, such that $\mathbf{x}(\mathbf{y}, \mathbf{Y}; \varepsilon)$ and $\mathbf{X}(\mathbf{y}, \mathbf{Y}; \varepsilon)$ satisfy

$$\frac{d\mathbf{x}}{d\varepsilon} = \frac{\partial W}{\partial \mathbf{X}}, \qquad \frac{d\mathbf{X}}{d\varepsilon} = -\frac{\partial W}{\partial \mathbf{x}},$$

with initial conditions $\mathbf{x}(\mathbf{y}, \mathbf{Y}; 0) = \mathbf{y}$ and $\mathbf{X}(\mathbf{y}, \mathbf{Y}; 0) = \mathbf{Y}$. The function $W(\mathbf{x}, \mathbf{X})$ is the generator of the transformation.

Let us now consider a Hamiltonian expanded in series of a small parameter ε,

$$H(\mathbf{x}, \mathbf{X}; \varepsilon) = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} H_{n,0}(\mathbf{x}, \mathbf{X}); \tag{2}$$

then, find $W(\mathbf{x}, \mathbf{X}; \varepsilon) = \sum_{n \geq 0} (\varepsilon^n/n!) W_{n+1}(\mathbf{x}, \mathbf{X})$, the generating function of a Lie transformation such that the new Hamiltonian K (itself a power series in ε)

$$K(\mathbf{y}, \mathbf{Y}; \varepsilon) = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} K_n(\mathbf{y}, \mathbf{Y}) = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} H_{0,n}(\mathbf{y}, \mathbf{Y}) \tag{3}$$

satisfies some prefixed conditions depending on the type of transformation sought; usually it is required that K belongs to the kernel of the Lie derivative of the unperturbed term $H_{0,0}$.

Deprit [4] gives a method to build the transformation term by term in a recursive way by means of the Lie triangle

$$H_{i,j} = H_{i+1,j-1} + \sum_{0 \leq k \leq i} \binom{i}{k} \{H_{k,j-1}; W_{i+1-k}\}, \qquad \text{with } i \geq 0, \quad j \geq 1, \tag{4}$$

where $\{-;-\}$ stands for the Poisson bracket. Note that an attendant problem with this approach is that there appear many intermediate terms $H_{i,j}$ which must be computed and stored.

For the *zonal problem* in ATS, and in particular for the main problem, there are three outstanding and extant Lie transformations, namely, the *elimination of the parallax* [5], the *elimination of the perigee* [6], and the *Delaunay normalization* [7], that are applied in a cascade. To describe them in detail is a task beyond the scope of this letter; the reader who is interested in them (if any) is referred to the original papers where they were developed. We just mention that the first two transformations are really simplifications, that is to say, they only reduce the Hamiltonian to a simpler form which reduces the complexity of high order computations (see [8] for details). In contrast, the Delaunay normalization eliminates (in a sense averages over) certain coordinates, and the resulting differential system may be solved directly by quadratures.

We have performed the above three transformations for the zonal problem up to order 7 in the perturbation; the number of terms $\mathcal{H}_{0,i}$ of the resulting new Hamiltonian and W_i of the generating function are given in Table 1. The result is a quite large number of terms, but they are still manageable. However, to obtain the final Hamiltonians and the generators, many additional terms are necessary, since we use the Lie triangle (4), and in this case, the number of terms appearing in the intermediate Hamiltonians $\mathcal{H}_{i,j}$ is huge, as can be seen from Figure 1: the elimination of the parallax produces 67215 terms; 1170399 result from the elimination of the perigee; 670499 terms come from the Delaunay normalization, which makes a grand total of 1908113 terms. Clearly, all of this requires a significant amount of computer power, and the skills to handle it. In our case, it is done automatically by ATESAT (automatization of theories and ephemeris in artificial satellite theory), a piece of interactive software developed by us [9,10], that automatically generates the Lie transformation chosen and produces a FORTRAN or C-code for computing the ephemeris thus obtained [11]. This software is currently used by

Table 1. Number of terms of the new Hamiltonians and generators for several orders and transformations.

Order	0	1	2	3	4	5	6
After elimination of the parallax							
$\mathcal{H}_{0,i}$	4	2	9	12	30	36	70
W_i	0	7	32	112	264	643	1340
After elimination of the perigee							
$\mathcal{H}_{0,i}$	4	2	6	20	35	68	106
W_i	0	2	34	289	1038	2984	5242
After Delaunay normalization							
$\mathcal{H}_{0,i}$	2	2	9	36	111	991	2682
W_i	0	2	15	104	474	7092	23687

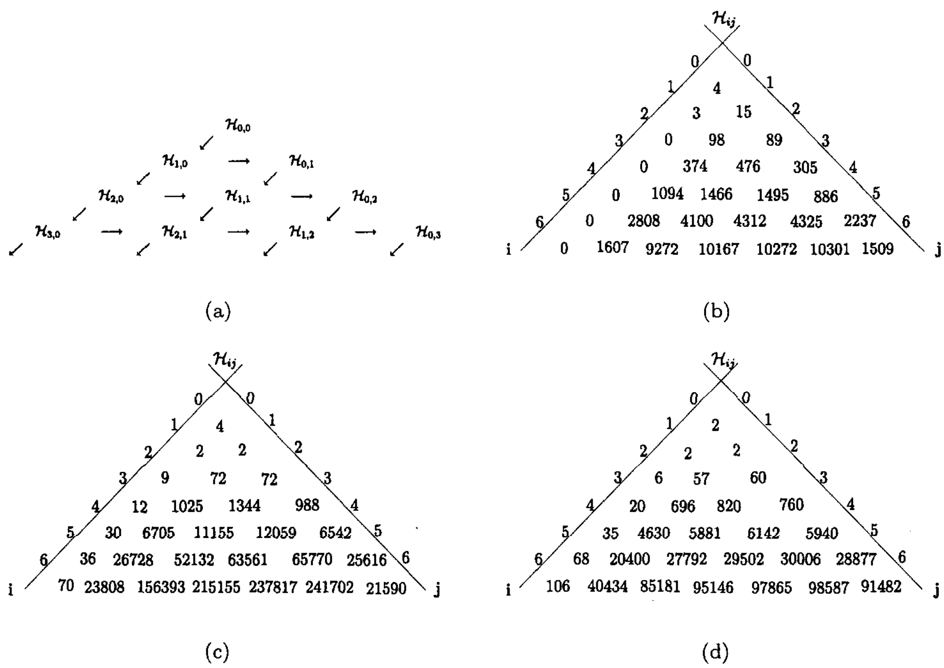

Figure 1. Number of terms of the intermediate $\mathcal{H}_{i,j}$ generated in the Lie triangle (a) in the three transformations, elimination of the parallax (b) and elimination of the perigee (c) and Delaunay normalization (d).

several laboratories, and the code for computing the ephemeris has been recently included in MSLIB, the mathematical routines of the French Centre National d'Études Spatiales.

To give an indication on the performance of ATESAT, let us mention that working on a PC Pentium II at 233 MHz, under Linux, the automatic generation of the C program for computing the ephemeris takes less than one minute for orders one, two, and three; orders four, five, and six take 10, 170, and 930 minutes, respectively. Once the C program ephemeris is given, computing of the position and velocity at any instant is fast, taking less than five seconds in theories up to fourth order, although it increases to 10 and 23 seconds for the fifth and sixth orders. The numerical precision of ATESAT ephemeris depends, logically, on the order of the theory used and the size of the small parameter ε; roughly, we can say that a fourth order gives machine precision. Thus, we agree with García *et al.* that symbolic integration is competitive with respect to numerical integration provided the final formula obtained with analytical methods is already known. Otherwise, one must counterbalance the effort to produce a symbolic integration, which as in the case presented here, may be considerable.

Our conclusion is that numerical and analytical methods both have their uses. For problems which require repetitive computations, as in AST, then the generation of a symbolic formula may well be worth the investment of time and effort needed to produce it.

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