

SINGULAR MEASURES AND CONVOLUTION OPERATORS

J. M. ALDAZ AND JUAN L. VARONA

ABSTRACT. We show that in the study of certain convolution operators, functions can be replaced by measures without changing the size of the constants appearing in weak type $(1, 1)$ inequalities. As an application, we prove that the best constants for the centered Hardy-Littlewood maximal operator associated to parallelotopes do not decrease with the dimension.

1. INTRODUCTION

The method of discretization for convolution operators, due to M. de Guzmán (cf. [1], Theorem 4.1.1), and further developed by M. T. Menárguez and F. Soria (cf. Theorem 1 of [2]) consists in replacing functions by finite sums of Dirac deltas in the study of the operator. So far, the main applications of these theorems have been related to the Hardy-Littlewood maximal function, and more precisely, to the determination of bounds for the best constants c_d appearing in the weak type $(1, 1)$ inequalities (cf. [2], [3], [4], and [5] for the one dimensional case, and for higher dimensions, [2] and [6]). In this paper we complement de Guzmán's Theorem by proving that one can consider arbitrary measures instead of finite discrete measures, and the same conclusions still hold (Theorem 1). A special case of our theorem (where the space is the real line and the convolution operator is precisely the Hardy-Littlewood maximal function) appears in [5] (see Theorem 2).

Regarding upper bounds for c_d , E. M. Stein and J. Strömberg (see [7]) showed that the constants grow at most like $O(d \log d)$ for arbitrary balls, and like $O(d)$ in the case of euclidean balls. With respect to lower bounds for the maximal function associated to cubes, it is shown in [2], Theorem 6, that $c_d \geq \left(\frac{1+2^{1/d}}{2}\right)^d$. These bounds decrease with the dimension to $\sqrt{2}$. Increasing lower bounds are given in Proposition 1.4 of [6], where in particular it is proven that $\liminf_d c_d \geq \frac{47\sqrt{2}}{36}$. But since the best constants are not known, this still left open the possibility that

2000 *Mathematics Subject Classification.* 42A99.

Key words and phrases. Hardy-Littlewood maximal function, weak type inequalities, singular measures, convolution operators.

Research supported by grant BFM2003-06335-C03-03 of the DGI.

THIS PAPER HAS BEEN PUBLISHED IN: *Acta Math. Sin. (Engl. Ser.)* **23** (2007), no. 3, 487–490.

Copyright © Springer-Verlag.

the c_d would form a decreasing sequence; here (see Theorem 2) we show that, for cubes, the inequality $c_d \leq c_{d+1}$ holds in every dimension d (not only for the usual maximal function, but also for lacunary versions of it). In dimensions 1 and 2 the stronger result $c_1 < c_2$ is known, thanks to the recent determination by Antonios D. Melas of the exact value of c_1 as $\frac{11+\sqrt{61}}{12}$ (Corollary 1 of [5]). Since $c_2 \geq \sqrt{\frac{3}{2}} + \frac{3-\sqrt{2}}{4}$, by Proposition 1.4 of [6], Melas's result entails that the first inequality is strict.

Finally, we note that the original question of Stein and Strömberg (see also [8], Problem 7.74 c, proposed by A. Carbery) as to whether $\lim_d c_d < \infty$ or $\lim_d c_d = \infty$, remains open.

2. CONVOLUTION OPERATORS AND MEASURES

We shall state the main theorem of this note in terms of a locally compact group X . Denote by $C(X)$ the family of all continuous functions $g: X \rightarrow \mathbb{R}$, by $C_c(X)$ the continuous functions with compact support, and by λ the left Haar measure on X . If $X = \mathbb{R}^d$, λ^d will stand for the d -dimensional Lebesgue measure. As usual, we shall write dx instead of $d\lambda(x)$. A finite real valued Borel measure μ on X is *Radon* if $|\mu|$ is inner regular with respect to the compact sets. It is well known that if X is a locally compact separable metric space, then every finite Borel measure is automatically Radon. Let \mathcal{N} be a neighborhood base at 0 such that each element of \mathcal{N} has compact closure, and let $\{h_U : U \in \mathcal{N}\}$ be an approximate identity, i.e., a family of nonnegative Borel functions such that for every $U \in \mathcal{N}$, $\text{supp } h_U \subset U$ and $\|h_U\|_1 = 1$. Furthermore, since for every neighborhood U of 0 there is a continuous function g_U with values in $[0, 1]$, $g_U(0) = 1$, and $\text{supp } g_U \subset U$, we may assume that each function in the approximate identity is continuous (obtain h_U by normalizing g_U). Let μ be a finite, nonnegative Radon measure on X . Recall that

$$h * f(x) = \int f(y^{-1}x)h(y) dy \quad \text{and} \quad \mu * f(x) = \int f(y^{-1}x) d\mu(y).$$

Let $g \in C_c(X)$; we shall utilize the following well known results: $\mu * (h_U * g) = (\mu * h_U) * g$, and $h_U * g \rightarrow g$ uniformly as $U \downarrow 0$. The idea of the proof below consists simply in replacing the measure μ with the continuous function $\mu * h_U$, using the fact that $\|\mu * h_U\|_1 = \mu(X)$.

The L_1 norm refers always in this paper to Haar measure.

Lemma 1. *Let $\{k_\beta\}$ be a family of nonnegative lower semicontinuous real valued functions, defined on X . Set $k^*v := \sup_\beta |v * k_\beta|$, where v is either a function or a measure. Then, for every finite real valued Radon measure μ on X , and every $\alpha > 0$,*

$$\lambda^d\{k^*\mu > \alpha\} \leq \sup \{\lambda^d\{k^*f > \alpha\} : \|f\|_1 = |\mu|(X)\}.$$

The same result holds if $\{k_n\}$ is a sequence of nonnegative real valued Borel functions.

Proof. Consider first the case where $\{k_\beta\}$ is a family of lower semicontinuous functions. We shall assume that functions and measures are nonnegative. There is no loss of generality in doing so since $k^*f \leq k^*|f|$ and $k^*\mu \leq k^*|\mu|$ always. Also, by lower semicontinuity, $\int k_\beta d\mu = \sup\{\int g_{\gamma,\beta} d\mu : 0 \leq g_{\gamma,\beta} \leq k_\beta, g_{\gamma,\beta} \in C_c(X)\}$ (Corollary 7.13 of [9]). It follows that for every x , $\sup_\beta \mu * k_\beta(x) = \sup_{\gamma,\beta} \{\mu * g_{\gamma,\beta}(x) : 0 \leq g_{\gamma,\beta} \leq k_\beta, g_{\gamma,\beta} \in C_c(X)\}$. Therefore we may assume that the family $\{k_\beta\}$ consists of nonnegative continuous functions with compact support.

Next, let $\{h_U : U \in \mathcal{N}\}$ be an approximate identity as above, with each h_U continuous, and let $C \subset \{k^*\mu > \alpha\}$ be a compact set. It suffices to show that there exists a function f with $\|f\|_1 = \mu(X)$ and $C \subset \{k^*f > \alpha\}$. We shall take f to be $\mu * h_{U_0}$, for a suitably chosen neighborhood U_0 . Since $\{k^*\mu > \alpha\} = \cup_\beta \{\mu * k_\beta > \alpha\}$ and each $\mu * k_\beta$ is continuous, there exists a finite subcollection of indices $\{\beta_1, \dots, \beta_\ell\}$ with $C \subset \cup_{1 \leq i \leq \ell} \{\mu * k_{\beta_i} > \alpha\}$, so the continuous function $\max_{1 \leq i \leq \ell} \mu * k_{\beta_i}$ attains a minimum value $\alpha + a$ on C , with a strictly positive. Because μ is a finite measure and $h_U * k_{\beta_i}$ converges uniformly to k_{β_i} as $U \rightarrow 0$, $\mu * h_U * k_{\beta_i}$ also converges uniformly to $\mu * k_{\beta_i}$. Hence, there exists an $U_0 \in \mathcal{N}$ such that for every $V \subset U_0$, $V \in \mathcal{N}$, and every $i \in \{1, \dots, \ell\}$, $\|\mu * k_{\beta_i} - \mu * h_V * k_{\beta_i}\|_\infty < a/2$. In particular, it follows that

$$C \subset \left\{ \max_{1 \leq i \leq \ell} \mu * h_{U_0} * k_{\beta_i} > \alpha \right\} \subset \left\{ k^*(\mu * h_{U_0}) > \alpha \right\}.$$

The case where $\{k_n\}$ is a sequence of nonnegative bounded Borel functions, can be proven by reduction to the previous one. Choose a finite Radon measure μ and fix $\alpha > 0$. Given $\varepsilon \in (0, 1)$, for every n let $g_n \geq k_n$ be a bounded, lower semicontinuous function with

$$\|g_n - k_n\|_1 < \frac{\varepsilon^2}{2^{n+1}\mu(X)}$$

(cf. Proposition 7.14 of [9]). Then, for any $f \in L_1(\lambda)$, using the Fubini-Tonelli Theorem and left invariance we have

$$\begin{aligned} & \|g^*f - k^*f\|_1 \\ &= \left\| \sup_n \int g_n(y^{-1}x)f(y) dy - \sup_n \int k_n(y^{-1}x)f(y) dy \right\|_1 \\ &\leq \sum_n \iint (g_n(y^{-1}x) - k_n(y^{-1}x))|f(y)| dy dx \\ &= \sum_n \int |f(y)| \int (g_n(y^{-1}x) - k_n(y^{-1}x)) dx dy \\ &= \sum_n \|f\|_1 \|g_n - k_n\|_1 < \|f\|_1 \varepsilon^2 (\mu(X))^{-1}. \end{aligned}$$

In particular, if $\|f\|_1 = \mu(X)$, we have that

$$\|g^*f - k^*f\|_1 < \varepsilon^2,$$

from which

$$\lambda\{g^*f - k^*f \geq \varepsilon\} \leq \frac{\|g^*f - k^*f\|_1}{\varepsilon} < \varepsilon$$

follows. Now $\{g^*f > \alpha + \varepsilon\} \subset \{k^*f > \alpha\} \cup \{g^*f - k^*f > \varepsilon\}$, so

$$\begin{aligned} (\alpha + \varepsilon)\lambda\{k^*\mu > \alpha + \varepsilon\} &\leq (\alpha + \varepsilon)\lambda\{g^*\mu > \alpha + \varepsilon\} \\ &\leq (\alpha + \varepsilon)\sup\{\lambda\{g^*f > \alpha + \varepsilon\} : \|f\|_1 = \mu(X)\} \\ &\leq (\alpha + \varepsilon)(\sup\{\lambda\{k^*f > \alpha\} : \|f\|_1 = \mu(X)\} + \varepsilon), \end{aligned}$$

and the result is obtained by letting $\varepsilon \downarrow 0$. \square

Theorem 1. *Let $\{k_\beta\}$ be a family of nonnegative lower semicontinuous real valued functions, defined on X , and let $c > 0$ be a fixed constant. Then the following are equivalent:*

(i) *For every function $f \in L_1(\lambda)$, and every $\alpha > 0$,*

$$\alpha\lambda\{k^*f > \alpha\} \leq c\|f\|_1.$$

(ii) *For every finite real valued Radon measure μ on X , and every $\alpha > 0$,*

$$\alpha\lambda\{k^*\mu > \alpha\} \leq c|\mu|(X).$$

The same result holds if $\{k_n\}$ is a sequence of nonnegative real valued Borel functions.

Proof. (i) is the special case of (ii) where $d\mu(y) = f(y)dy$. For the other direction, by Lemma 1 and part (i) we have

$$\alpha\lambda\{k^*\mu > \alpha\} \leq \alpha\sup\{\lambda\{k^*f > \alpha\} : \|f\|_1 = |\mu|(X)\} \leq c|\mu|(X).$$

\square

Remark 1. By the discretization theorem of M. de Guzmán (see [1], Theorem 4.1.1), further refined by M. T. Menárguez and F. Soria (Theorem 1 of [2]), in \mathbb{R}^d conditions (i) and (ii) of Theorem 1 are both equivalent to

(iii) *For every finite collection $\{\delta_{x_1}, \dots, \delta_{x_N}\}$ of Dirac deltas on X , and every $\alpha > 0$,*

$$\alpha\lambda\left\{k^*\sum_{i=1}^N \delta_{x_i} > \alpha\right\} \leq cN.$$

From the viewpoint of obtaining lower bounds, the usefulness of (ii) is due to the fact that it allows to choose among a wider class of potential examples than just finite sums of Dirac deltas. Both (ii) and (iii) will be utilized in the next section.

3. BEHAVIOR OF CONSTANTS FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

Let $B \subset \mathbb{R}^d$ be an open, bounded, convex set, symmetric about zero. We shall call B a ball, since each norm on \mathbb{R}^d yields sets of this type, and each bounded B , convex and symmetric about zero, defines a norm. The (centered) Hardy-Littlewood maximal operator associated to B is defined for locally integrable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$M_{d,B}f(x) := \sup_{r>0} \frac{\chi_{rB}}{r^d \lambda^d(B)} * |f|(x).$$

We denote by $c_{d,B}$ the best constant in the weak type $(1,1)$ inequality $\alpha \lambda^d\{M_{d,B}f > \alpha\} \leq c \|f\|_1$, where c is independent of $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Let $s := \{r_n\}_{-\infty}^{\infty}$ be a lacunary (bi)sequence (i.e., a sequence that satisfies $r_{n+1}/r_n \geq c$ for some fixed constant $c > 1$ and every $n \in \mathbb{Z}$). Then the associated maximal operator is defined via

$$M_{s,d,B}f(x) := \sup_{n \in \mathbb{Z}} \frac{\chi_{r_n B}}{r_n^d \lambda^d(B)} * |f|(x).$$

The arguments given below are applicable to both the maximal function and to lacunary versions of it, so we shall not introduce a different notation for the best constants in the lacunary case. In particular, Lemma 2 and Theorem 2 refer to all of these maximal operators, but only the usual maximal operator shall be mentioned in the proofs.

Given a finite sum $\mu = \sum_1^k \delta_{x_i}$ of Dirac deltas, where the x_i 's need not be all different, let $\sharp(x+B)$ be the number of point masses from μ contained in $x+B$.

Lemma 2. *Let B be a ball in \mathbb{R}^d . Then for every linear transformation $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\det T \neq 0$, $c_{d,B} = c_{d,T(B)}$.*

Proof. Given $\mu := \sum_1^k \delta_{x_i}$ and $T\mu := \sum_1^k \delta_{T(x_i)}$, we have that

$$M_{d,B}\mu(x) := \sup_{r>0} \frac{\sharp(x+rB)}{r^d \lambda^d(B)}$$

and

$$M_{d,T(B)}T\mu(x) := \sup_{r>0} \frac{\sharp(x+rT(B))}{r^d \lambda^d(T(B))}.$$

Then $x \in \{M_{d,B}\mu > \alpha\}$ iff $T(x) \in \{M_{d,T(B)}T\mu > (\alpha/|\det T|)\}$. Since

$$|\det T| \lambda^d\{M_{d,B}\mu > \alpha\} = \lambda^d\{M_{d,T(B)}T\mu > (\alpha/|\det T|)\},$$

we have

$$\alpha \lambda^d\{M_{d,B}\mu > \alpha\} = (\alpha/|\det T|) \lambda^d\{M_{d,T(B)}T\mu > (\alpha/|\det T|)\},$$

and the result follows. \square

Theorem 2. *For each $d \in \mathbb{N} \setminus \{0\}$ let B_d be a d -dimensional parallelootope centered at zero. Then $c_{d,B_d} \leq c_{d+1,B_{d+1}}$ for both the maximal operator and for lacunary operators.*

Proof. Since every such B_d is the image under a nonsingular linear transformation of the d -dimensional cube Q_d centered at zero with sides parallel to the axes and volume 1, we may assume that in fact $B_d = Q_d$. With the convex bodies fixed, we will write c_d and M_d rather than c_{d,B_d} and M_{d,B_d} . Given $\alpha > 0$, $\mu_d = \sum_1^k \delta_{x_i}$ on \mathbb{R}^d and a constant $c > 0$ such that $\alpha \lambda^d \{M_d \mu_d > \alpha\} > c \mu_d(\mathbb{R}^d)$, we want to find a measure μ_{d+1} on \mathbb{R}^{d+1} such that $\alpha \lambda^{d+1} \{M_{d+1} \mu_{d+1} > \alpha\} > c \mu_{d+1}(\mathbb{R}^{d+1})$. This will imply that $c_d \leq c_{d+1}$. Let $L := (k/\alpha)^{1/d}$. Note that if $r \geq L$, then for every $x \in \mathbb{R}^d$, $\frac{\#(x+rQ_d)}{r^d} \leq \alpha$. Choose $N \gg L$ such that $\alpha \frac{N-L}{N} \lambda^d \{M_d \mu_d > \alpha\} > ck$, and let $\mu_{d+1} := \mu_d \times \lambda_{[-N,N]}$, where $\lambda_{[-N,N]}$ stands for the restriction of linear Lebesgue measure to the interval $[-N, N]$. We claim that $\{M_d \mu_d > \alpha\} \times [-N+L, N-L] \subset \{M_{d+1} \mu_{d+1} > \alpha\}$. In order to establish the claim, the following notation shall be used: If $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, by (x, x_{d+1}) we denote the point $(x_1, \dots, x_d, x_{d+1}) \in \mathbb{R}^{d+1}$. Now if $x \in \{M_d \mu_d > \alpha\}$, then there exists an $r(x) \in (0, L)$ such that $r(x)^{-d} \mu_d(x + r(x)Q_d) > \alpha$, so for every $y \in [-N+L, N-L]$,

$$\begin{aligned} & r(x)^{-d-1} \mu_{d+1}((x, y) + r(x)Q_{d+1}) \\ &= r(x)^{-d-1} (\mu_d(x + r(x)Q_d) \times \lambda_{[-N,N]}([y - \frac{r(x)}{2}, y + \frac{r(x)}{2}])) \\ &= r(x)^{-d} \mu_d(x + r(x)Q_d) > \alpha, \end{aligned}$$

as desired. But now

$$\begin{aligned} & \alpha \lambda^{d+1} \{M_{d+1} \mu_{d+1} > \alpha\} \geq 2\alpha(N-L) \lambda^d \{M_d \mu_d > \alpha\} \\ &= 2\alpha N \frac{N-L}{N} \lambda^d \{M_d \mu_d > \alpha\} > 2Nck = c \mu_{d+1}(\mathbb{R}^{d+1}). \end{aligned}$$

□

Remark 2. Recall from the Introduction that for the ℓ_∞ balls (i.e., cubes with sides parallel to the axes) $c_1 < c_2$. Since the ℓ_1 unit ball in dimension 2 is a square, it follows from Lemma 2 that the best constant in dimension 2 is equal for the ℓ_1 and the ℓ_∞ norms. It follows that $c_1 < c_2$ in the ℓ_1 case also. It would be interesting to know whether or not the best constants associated to the ℓ_p balls are all the same. Note that establishing bounds of the type $a^{-1}c_{d,2} \leq c_{d,p} \leq ac_{d,2}$ (where the constant $a \geq 1$ is independent of the dimension d and $c_{d,p}$ denotes the best constant associated to the ℓ_p ball), would show that the bounds $O(d)$ (which hold for euclidean balls by [7]) extend to ℓ_p balls.

REFERENCES

- [1] Guzmán, M. de, “Real variable methods in Fourier analysis,” North-Holland Mathematics Studies, 46. Notas de Matemática [Mathematical Notes], 75. North-Holland Publishing Co., Amsterdam-New York, 1981.
- [2] Menárguez, M.T., and Soria, F., Weak type (1, 1) inequalities for maximal convolution operators, *Rend. Circ. Mat. Palermo (2)* **41** (1992), 342–352.

- [3] Aldaz, J.M., Remarks on the Hardy-Littlewood maximal function, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 1–9.
- [4] Melas, A.D., On the centered Hardy-Littlewood maximal operator, *Trans. Amer. Math. Soc.* **354** (2002), 3263–3273.
- [5] Melas, A.D., The best constant for the centered Hardy-Littlewood maximal inequality, *Ann. of Math. (2)*, **157** (2003), 647–688.
- [6] Aldaz, J.M., A remark on the centered n -dimensional Hardy-Littlewood maximal function, *Czech. Math. J.* **50(125)** (2000), 103–112.
- [7] Stein, E.M., and Strömberg, J.O., Behaviour of maximal functions in \mathbb{R}^n for large n , *Ark. Mat.* **21** (1983), 259–269.
- [8] Brannan, D.A., and Hayman, W.K., Research problems in complex analysis, *Bull. London Math. Soc.* **21** (1989), 1–35.
- [9] Folland, G.B., “Real Analysis. Modern Techniques and Their Applications,” Pure and Applied Mathematics, John Wiley and Sons, New York, 1984.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, CALLE LUIS DE ULLOA S/N, 26004 LOGROÑO, SPAIN
E-mail address: aldaz@dmc.unirioja.es, jvarona@dmc.unirioja.es