*Acta Mathematica Sinica, English Series*
Mar., 2007, Vol. 23, No. 3, pp. 487–490
Published online: Jan. 19, 2006
DOI: 10.1007/s10114-005-0682-6
Http://www.ActaMath.com

**Acta Mathematica Sinica, English Series**


# Singular Measures and Convolution Operators

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**Abstract** We show that in the study of certain convolution operators, functions can be replaced by measures without changing the size of the constants appearing in weak type $(1,1)$ inequalities. As an application, we prove that the best constants for the centered Hardy–Littlewood maximal operator associated with parallelotopes do not decrease with the dimension.

**Keywords** Hardy–Littlewood maximal function, Weak type inequalities, Singular measures, Convolution operators
**MR(2000) Subject Classification** 42A99

## 1 Introduction

The method of discretization for convolution operators, due to de Guzmán (cf. [1], Theorem 4.1.1), and further developed by Menárguez and Soria (cf. Theorem 1 of [2]) consists in replacing functions by finite sums of Dirac deltas in the study of the operator. So far, the main applications of these theorems have been related to the Hardy–Littlewood maximal function, and more precisely, to the determination of bounds for the best constants $c_d$ appearing in the weak type $(1,1)$ inequalities (cf. [2], [3], [4], and [5] for the one-dimensional case, and for higher dimensions, [2] and [6]). In this paper we complement de Guzmán's theorem by proving that one can consider arbitrary measures instead of finite discrete measures, and the same conclusions still hold (Theorem 1). A special case of our theorem (where the space is the real line and the convolution operator is precisely the Hardy–Littlewood maximal function) appears in [5] (see Theorem 2).

Regarding upper bounds for $c_d$, Stein and Strömberg (see [7]) showed that the constants grow at most like $O(d\log d)$ for arbitrary balls, and like $O(d)$ in the case of euclidean balls. With respect to lower bounds for the maximal function associated with cubes, it is shown in [2], Theorem 6, that $c_d \ge \left(\frac{1+2^{1/d}}{2}\right)^d$. These bounds decrease with the dimension to $\sqrt{2}$. Increasing lower bounds are given in Proposition 1.4 of [6], where in particular it is proven that $\liminf_d c_d \ge \frac{47\sqrt{2}}{36}$. But since the best constants are not known, there still is left open the possibility that the $c_d$ would form a decreasing sequence; here (see Theorem 2) we show that, for cubes, the inequality $c_d \le c_{d+1}$ holds in every dimension $d$ (not only for the usual maximal function, but also for lacunary versions of it). In dimensions 1 and 2 the stronger result $c_1 < c_2$ is known, thanks to the recent determination by Antonios Melas of the exact value of $c_1$ as $\frac{11+\sqrt{61}}{12}$ (Corollary 1 of [5]). Since $c_2 \ge \sqrt{\frac{3}{2}} + \frac{3-\sqrt{2}}{4}$, by Proposition 1.4 of [6], Melas's result entails that the first inequality is strict.

Finally, we note that the original question of Stein and Strömberg (see also [8], Problem 7.74 c, proposed by Carbery), as to whether $\lim_d c_d < \infty$ or $\lim_d c_d = \infty$, remains open.

Received April 14, 2004, Accepted August 25, 2004
Research supported by Grant BFM2003-06335-C03-03 of the DGI
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## 2 Convolution Operators and Measures

We shall state the main theorem of this note in terms of a locally compact group $X$. Denote by $C(X)$ the family of all continuous functions $g\colon X \to \mathbb{R}$, by $C_c(X)$ the continuous functions with compact support, and by $\lambda$ the left Haar measure on $X$. If $X = \mathbb{R}^d$, $\lambda^d$ will stand for the $d$-dimensional Lebesgue measure. As usual, we shall write $dx$ instead of $d\lambda(x)$. A finite real-valued Borel measure $\mu$ on $X$ is *Radon* if $|\mu|$ is inner regular with respect to the compact sets. It is well known that if $X$ is a locally compact separable metric space, then every finite Borel measure is automatically Radon. Let $\mathscr{N}$ be a neighborhood base at 0 such that each element of $\mathscr{N}$ has compact closure, and let $\{h_U : U \in \mathscr{N}\}$ be an approximate identity, i.e., a family of nonnegative Borel functions such that for every $U \in \mathscr{N}$, $\operatorname{supp} h_U \subset U$ and $\|h_U\|_1 = 1$. Furthermore, since for every neighborhood $U$ of 0 there is a continuous function $g_U$ with values in $[0,1]$, $g_U(0) = 1$, and $\operatorname{supp} g_U \subset U$, we may assume that each function in the approximate identity is continuous (obtain $h_U$ by normalizing $g_U$). Let $\mu$ be a finite, nonnegative Radon measure on $X$. Recall that $h * f(x) = \int f(y^{-1}x)h(y)\,dy$ and $\mu * f(x) = \int f(y^{-1}x)\,d\mu(y)$. Let $g \in C_c(X)$; we shall utilize the following well-known results: $\mu * (h_U * g) = (\mu * h_U) * g$, and $h_U * g \to g$ uniformly as $U \downarrow 0$. The idea of the proof below consists simply in replacing the measure $\mu$ with the continuous function $\mu * h_U$, using the fact that $\|\mu * h_U\|_1 = \mu(X)$.

The $L_1$ norm refers always in this paper to the Haar measure.

**Lemma 1** *Let $\{k_\beta\}$ be a family of nonnegative lower semicontinuous real-valued functions, defined on $X$. Set $k^*v := \sup_\beta |v * k_\beta|$, where $v$ is either a function or a measure. Then, for every finite real-valued Radon measure $\mu$ on $X$, and every $\alpha > 0$,*

$$\lambda^d\{k^*\mu > \alpha\} \le \sup\big\{\lambda^d\{k^*f > \alpha\} : \|f\|_1 = |\mu|(X)\big\}.$$

*The same result holds if $\{k_n\}$ is a sequence of nonnegative real-valued Borel functions.*

*Proof* Consider first the case where $\{k_\beta\}$ is a family of lower semicontinuous functions. We shall assume that functions and measures are nonnegative. There is no loss of generality in doing so since $k^*f \le k^*|f|$ and $k^*\mu \le k^*|\mu|$ always. Also, by lower semicontinuity, $\int k_\beta\,d\mu = \sup\{\int g_{\gamma,\beta}\,d\mu : 0 \le g_{\gamma,\beta} \le k_\beta,\ g_{\gamma,\beta} \in C_c(X)\}$ (Corollary 7.13 of [9]). It follows that for every $x$, $\sup_\beta \mu * k_\beta(x) = \sup_{\gamma,\beta}\{\mu * g_{\gamma,\beta}(x) : 0 \le g_{\gamma,\beta} \le k_\beta,\ g_{\gamma,\beta} \in C_c(X)\}$. Therefore we may assume that the family $\{k_\beta\}$ consists of nonnegative continuous functions with compact support.

Next, let $\{h_U : U \in \mathscr{N}\}$ be an approximate identity as above, with each $h_U$ continuous, and let $C \subset \{k^*\mu > \alpha\}$ be a compact set. It suffices to show that there exists a function $f$ with $\|f\|_1 = \mu(X)$ and $C \subset \{k^*f > \alpha\}$. We shall take $f$ to be $\mu * h_{U_0}$, for a suitably chosen neighborhood $U_0$. Since $\{k^*\mu > \alpha\} = \cup_\beta\{\mu * k_\beta > \alpha\}$ and each $\mu * k_\beta$ is continuous, there exists a finite subcollection of indices $\{\beta_1, \dots, \beta_\ell\}$ with $C \subset \cup_1^\ell\{\mu * k_{\beta_i} > \alpha\}$, so the continuous function $\max_{1\le i\le\ell} \mu * k_{\beta_i}$ attains a minimum value $\alpha + a$ on $C$, with $a$ strictly positive. Because $\mu$ is a finite measure and $h_U * k_{\beta_i}$ converges uniformly to $k_{\beta_i}$ as $U \to 0$, $\mu * h_U * k_{\beta_i}$ also converges uniformly to $\mu * k_{\beta_i}$. Hence, there exists a $U_0 \in \mathscr{N}$ such that for every $V \subset U_0$, $V \in \mathscr{N}$, and every $i \in \{1, \dots, \ell\}$, $\|\mu * k_{\beta_i} - \mu * h_V * k_{\beta_i}\|_\infty < a/2$. In particular, it follows that

$$C \subset \Big\{\max_{1\le i\le\ell} \mu * h_{U_0} * k_{\beta_i} > \alpha\Big\} \subset \big\{k^*(\mu * h_{U_0}) > \alpha\big\}.$$

The case where $\{k_n\}$ is a sequence of nonnegative bounded Borel functions can be proved by reduction to the previous one. Choose a finite Radon measure $\mu$ and fix $\alpha > 0$. Given $\varepsilon \in (0,1)$, for every $n$ let $g_n \ge k_n$ be a bounded, lower semicontinuous function with $\|g_n - k_n\|_1 < \frac{\varepsilon^2}{2^{n+1}\mu(X)}$ (cf. Proposition 7.14 of [9]). Then, for any $f \in L_1(\lambda)$, using the Fubini–Tonelli theorem and left invariance we have

$$\begin{aligned}\|g^*f - k^*f\|_1 &= \left\|\sup_n \int g_n(y^{-1}x)f(y)\,dy - \sup_n \int k_n(y^{-1}x)f(y)\,dy\right\|_1\\ &\le \sum_n \iint (g_n(y^{-1}x) - k_n(y^{-1}x))|f(y)|\,dy\,dx\end{aligned}$$

$$= \sum_n \int |f(y)| \int (g_n(y^{-1}x) - k_n(y^{-1}x))\, dx\, dy$$
$$= \sum_n \|f\|_1 \|g_n - k_n\|_1 < \|f\|_1 \varepsilon^2 (\mu(X))^{-1}.$$

In particular, if $\|f\|_1 = \mu(X)$, we have that $\|g^* f - k^* f\|_1 < \varepsilon^2$, from which $\lambda\{g^* f - k^* f \geq \varepsilon\} \leq \frac{\|g^* f - k^* f\|_1}{\varepsilon} < \varepsilon$ follows. Now $\{g^* f > \alpha + \varepsilon\} \subset \{k^* f > \alpha\} \cup \{g^* f - k^* f > \varepsilon\}$, so

$$\begin{aligned}(\alpha + \varepsilon)\lambda\{k^* \mu > \alpha + \varepsilon\} &\leq (\alpha + \varepsilon)\lambda\{g^* \mu > \alpha + \varepsilon\}\\ &\leq (\alpha + \varepsilon) \sup\{\lambda\{g^* f > \alpha + \varepsilon\} : \|f\|_1 = \mu(X)\}\\ &\leq (\alpha + \varepsilon)(\sup\{\lambda\{k^* f > \alpha\} : \|f\|_1 = \mu(X)\} + \varepsilon),\end{aligned}$$

and the result is obtained by letting $\varepsilon \downarrow 0$.

**Theorem 1** *Let $\{k_\beta\}$ be a family of nonnegative lower semicontinuous real-valued functions, defined on $X$, and let $c > 0$ be a fixed constant. Then the following are equivalent*:

(i) *For every function $f \in L_1(\lambda)$, and every $\alpha > 0$, $\alpha\lambda\{k^* f > \alpha\} \leq c\|f\|_1$.*

(ii) *For every finite real-valued Radon measure $\mu$ on $X$, and every $\alpha > 0$, $\alpha\lambda\{k^* \mu > \alpha\} \leq c|\mu|(X)$.*

*The same result holds if $\{k_n\}$ is a sequence of nonnegative real-valued Borel functions.*

*Proof* (i) is the special case of (ii) where $d\mu(y) = f(y)\, dy$. For the other direction, by Lemma 1 and part (i) we have $\alpha\lambda\{k^* \mu > \alpha\} \leq \alpha \sup\{\lambda\{k^* f > \alpha\} : \|f\|_1 = |\mu|(X)\} \leq c|\mu|(X)$.

**Remark 1** By the discretization theorem of de Guzmán (see [1], Theorem 4.1.1), further refined by Menárguez and Soria (Theorem 1 of [2]), in $\mathbb{R}^d$ conditions (i) and (ii) of Theorem 1 are both equivalent to

(iii) *For every finite collection $\{\delta_{x_1}, \dots, \delta_{x_N}\}$ of Dirac deltas on $X$, and every $\alpha > 0$, $\alpha\lambda\{k^* \sum_1^N \delta_{x_i} > \alpha\} \leq cN$.*

From the viewpoint of obtaining lower bounds, the usefulness of (ii) is due to the fact that it allows us to choose among a wider class of potential examples than just finite sums of Dirac deltas. Both (ii) and (iii) will be utilized in the sext section.

## 3 Behavior of Constants for the Hardy–Littlewood Maximal Operator

Let $B \subset \mathbb{R}^d$ be an open, bounded, convex set, symmetric about zero. We shall call $B$ a ball, since each norm on $\mathbb{R}^d$ yields sets of this type, and each bounded $B$, convex and symmetric about zero, defines a norm. The (centered) Hardy–Littlewood maximal operator associated with $B$ is defined for locally integrable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ as $M_{d,B} f(x) := \sup_{r>0} \frac{\chi_{rB}}{r^d \lambda^d(B)} * |f|(x)$. We denote by $c_{d,B}$ the best constant in the weak type $(1,1)$ inequality $\alpha\lambda^d\{M_{d,B} f > \alpha\} \leq c\|f\|_1$, where $c$ is independent of $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Let $s := \{r_n\}_{-\infty}^{\infty}$ be a lacunary (bi)sequence (i.e., a sequence that satisfies $r_{n+1}/r_n \geq c$ for some fixed constant $c > 1$ and every $n \in \mathbb{Z}$). Then the associated maximal operator is defined via $M_{s,d,B} f(x) := \sup_{n\in\mathbb{Z}} \frac{\chi_{r_n B}}{r_n^d \lambda^d(B)} * |f|(x)$. The arguments given below are applicable to both the maximal function and to lacunary versions of it, so we shall not introduce a different notation for the best constants in the lacunary case. In particular, Lemma 2 and Theorem 2 refer to all of these maximal operators, but only the usual maximal operator shall be mentioned in the proofs.

Given a finite sum $\mu = \sum_1^k \delta_{x_i}$ of Dirac deltas, where the $x_i$'s need not be all different, let $\sharp(x + B)$ be the number of point masses from $\mu$ contained in $x + B$.

**Lemma 2** *Let $B$ be a ball in $\mathbb{R}^d$. Then for every linear transformation $T\colon \mathbb{R}^d \to \mathbb{R}^d$ with* $\det T \neq 0$, $c_{d,B} = c_{d,T(B)}$.

*Proof* Given $\mu := \sum_1^k \delta_{x_i}$ and $T\mu := \sum_1^k \delta_{T(x_i)}$, we have that

$$M_{d,B}\mu(x) := \sup_{r>0} \frac{\sharp(x + rB)}{r^d \lambda^d(B)} \quad \text{and} \quad M_{d,T(B)} T\mu(x) := \sup_{r>0} \frac{\sharp(x + rT(B))}{r^d \lambda^d(T(B))}.$$

Then $x \in \{M_{d,B}\mu > \alpha\}$ iff $T(x) \in \{M_{d,T(B)}T\mu > (\alpha/|\det T|)\}$. Since

$$|\det T|\lambda^d\{M_{d,B}\mu > \alpha\} = \lambda^d\{M_{d,T(B)}T\mu > (\alpha/|\det T|)\},$$

we have $\alpha\lambda^d\{M_{d,B}\mu > \alpha\} = (\alpha/|\det T|)\lambda^d\{M_{d,T(B)}T\mu > (\alpha/|\det T|)\}$, and the result follows.

**Theorem 2** *For each $d \in \mathbb{N} \setminus \{0\}$ let $B_d$ be a $d$-dimensional parallelotope centered at zero. Then $c_{d,B_d} \le c_{d+1,B_{d+1}}$ for both the maximal operator and for lacunary operators.*

*Proof* Since every such $B_d$ is the image under a nonsingular linear transformation of the $d$-dimensional cube $Q_d$ centered at zero with sides parallel to the axes and volume 1, we may assume that in fact $B_d = Q_d$. With the convex bodies fixed, we will write $c_d$ and $M_d$ rather than $c_{d,B_d}$ and $M_{d,B_d}$. Given $\alpha > 0$, $\mu_d = \sum_1^k \delta_{x_i}$ on $\mathbb{R}^d$ and a constant $c > 0$ such that $\alpha\lambda^d\{M_d\mu_d > \alpha\} > c\mu_d(\mathbb{R}^d)$, we want to find a measure $\mu_{d+1}$ on $\mathbb{R}^{d+1}$ such that $\alpha\lambda^{d+1}\{M_{d+1}\mu_{d+1} > \alpha\} > c\mu_{d+1}(\mathbb{R}^{d+1})$. This will imply that $c_d \le c_{d+1}$. Let $L := (k/\alpha)^{1/d}$. Note that if $r \ge L$, then for every $x \in \mathbb{R}^d$, $\frac{\sharp(x+rQ_d)}{r^d} \le \alpha$. Choose $N \gg L$ such that $\alpha\frac{N-L}{N}\lambda^d\{M_d\mu_d > \alpha\} > ck$, and let $\mu_{d+1} := \mu_d \times \lambda_{[-N,N]}$, where $\lambda_{[-N,N]}$ stands for the restriction of linear Lebesgue measure to the interval $[-N,N]$. We claim that $\{M_d\mu_d > \alpha\} \times [-N+L, N-L] \subset \{M_{d+1}\mu_{d+1} > \alpha\}$. In order to establish the claim, the following notation shall be used: If $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, by $(x, x_{d+1})$ we denote the point $(x_1,\dots,x_d,x_{d+1}) \in \mathbb{R}^{d+1}$. Now if $x \in \{M_d\mu_d > \alpha\}$, then there exists an $r(x) \in (0,L)$ such that $r(x)^{-d}\mu_d(x + r(x)Q_d) > \alpha$, so for every $y \in [-N+L, N-L]$,

$$\begin{aligned} r(x)^{-d-1}\mu_{d+1}((x,y) + r(x)Q_{d+1}) &= r(x)^{-d-1}(\mu_d(x + r(x)Q_d) \times \lambda_{[-N,N]}([y - \tfrac{r(x)}{2}, y + \tfrac{r(x)}{2}]) \\ &= r(x)^{-d}\mu_d(x + r(x)Q_d) > \alpha, \end{aligned}$$

as desired. But now

$$\begin{aligned} \alpha\lambda^{d+1}\{M_{d+1}\mu_{d+1} > \alpha\} &\ge 2\alpha(N-L)\lambda^d\{M_d\mu_d > \alpha\} = 2\alpha N\frac{N-L}{N}\lambda^d\{M_d\mu_d > \alpha\} \\ &> 2Nck = c\mu_{d+1}(\mathbb{R}^{d+1}). \end{aligned}$$

**Remark 2** Recall from the Introduction that, for the $\ell_\infty$ balls (i.e., cubes with sides parallel to the axes), $c_1 < c_2$. Since the $\ell_1$ unit ball in dimension 2 is a square, it follows from Lemma 2 that the best constant in dimension 2 is equal for the $\ell_1$ and the $\ell_\infty$ norms. It follows that $c_1 < c_2$ in the $\ell_1$ case also. It would be interesting to know whether or not the best constants associated with the $\ell_p$ balls are all the same. Note that establishing bounds of the type $a^{-1}c_{d,2} \le c_{d,p} \le ac_{d,2}$ (where the constant $a \ge 1$ is independent of the dimension $d$ and $c_{d,p}$ denotes the best constant associated with the $\ell_p$ ball) would show that the bounds $O(d)$ (which hold for Euclidean balls by [7]) extend to $\ell_p$ balls.

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