CONVERGENCE OF FOURIER-PADÉ APPROXIMANTS FOR STIELTJES FUNCTIONS

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Abstract. We prove convergence of diagonal multipoint Padé approximants of Stieltjes-type functions when certain moment problem is determinate. This is used for the study of the convergence of Fourier-Padé and nonlinear Fourier-Padé approximants for such type of functions.

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1. Introduction

Two of the most important topics in Analysis are Fourier series and continued fractions. It is well known the strong connection between Padé approximants and continued fractions. In this paper, we study the convergence of rational approximants which extend the basic definitions of the classical Padé approximants of power series to the case of series in orthogonal polynomials. In particular, we study the convergence of rational approximants called Fourier-Padé approximants and nonlinear Fourier-Padé approximants for Stieltjes functions. A Stieltjes function is an integral of Stieltjes-Cauchy type of a measure supported on $\mathbb{R}_+$. Gonchar, Rakhmanov, and Suetin studied this kind of problems when the measure involved has bounded support [6] (see also [2], V. 2). The unbounded case is more delicate and requires special treatment. Multipoint Padé approximants play a fundamental role in the study of these classes of rational approximants.

Multipoint Padé approximants are rational approximants which interpolate a function at a given set of points. A systematic study of the convergence properties of multipoint Padé approximants was initiated about 25 years ago by Gonchar and López (see [4], and also [12]). In [8], López considered multipoint Padé approximants for Stieltjes functions. He assumed that the interpolation points and the corresponding measure satisfy a Carleman type condition. If the interpolation points are only 0 and ∞, the corresponding two-point Padé approximants represent continued fractions called positive T-fractions. These fractions are studied in [7], and there, it is proved its convergence when the so called strong moment problem is determinate. We extend the above results by proving that multipoint Padé approximants of a Stieltjes function converge when a certain moment problem is determinate.

Multipoint Padé approximants are a useful tool for solving problems in rational approximation. For example, Gonchar and Rakhmanov proved in [5] a general result concerning the exact rate of best rational approximation for a large class of analytic functions based on the construction of convenient multipoint Padé approximants. This result had a great impact in approximation theory; in particular, in the solution of the so called $1/9$ conjecture.

We begin introducing some notations and the definitions of Fourier-Padé and nonlinear Fourier-Padé approximants.

Let β be a positive Borel measure on $(-\infty, 0)$ with finite moments whose support, $\mathrm{supp}(\beta)$, contains infinitely many points. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the sequence of orthonormal polynomials with respect to β. If f is a function in $L_1(\beta)$, its Fourier coefficients with respect to $\{\varphi_n\}$ are given by

$$c_n(f) = \int_{-\infty}^{0} f(x)\varphi_n(x)d\beta(x), \quad n = 0, 1, \ldots$$

We say that the rational function $F_{l,m} = \dfrac{S_l}{T_m}$ is a *nonlinear Fourier-Padé approximant of type* (l, m) of the formal series

$$\sum_{n=0}^{\infty} c_n(f)\varphi_n \tag{1}$$

if S_l and T_m are polynomials such that deg $S_l \leq l$, deg $T_m \leq m$, $T_m \not\equiv 0$, the rational function $F_{l,m} = \frac{S_l}{T_m}$ is in $L_1(\beta)$, and

$$c_i(F_{l,m}) = c_i(f), \quad i = 0, 1, \ldots, l + m.$$

In order to find $F_{l,m}$ we have to solve a system of nonlinear equations.

This system does not always have a solution. Thus, it is possible that there does not exist a nonlinear Fourier-Padé approximant of type (l, m). However, if a nonlinear Padé approximant exists, it is unique.

The (linear) *Fourier-Padé approximant of type* (l, m) of series (1) is a rational function $\Phi_{l,m} = \dfrac{s_l}{t_m}$ where s_l and t_m are polynomials such that deg $s_l \leq l$, deg $t_m \leq m$, $t_m \not\equiv 0$, and

$$c_i(t_m f - s_l) = 0, \quad i = 0, 1, \ldots, l + m. \tag{2}$$

This is a system of homogeneous linear equations on the coefficients of the polynomials s_l and t_m. The number of equations of the system is equal to $l+m+1$, and the number of unknown parameters equals $l+m+2$. Hence, there always exists a non-trivial solution, but it may not be unique. However, if every polynomial $t_m \not\equiv 0$ determined by the system (2) has degree exactly equal to m, then there exists an unique approximant $\Phi_{l,m}$ of series (1).

In the present paper we deal with Padé approximants of orthogonal expansions on $(-\infty, 0)$ of *Stieltjes functions*; that is, functions of the form

$$\hat{\rho}(z) = \int_0^\infty \frac{d\rho(x)}{z-x}, \tag{3}$$

where ρ is a finite positive Borel measure in $(0,\infty)$ with an infinite number of points on its support and $z^i\hat{\rho}(z) \in L_1(\beta), i \in \mathbb{Z}_+ = \{0, 1, \dots\}$. Such functions are analytic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\rho)$.

Sequences of the form

$$\{F_{n+j,n}\}, \quad \{\Phi_{n+j,n}\}, \quad n \in \mathbb{N}, \tag{4}$$

where j is a fixed integer, are called *diagonal* approximants. The main result of the present paper is valid for arbitrary sequences of type (4) whenever $j \geq -1$. For simplicity, we restrict our attention to the case $j = -1$ and write

$$F_n = F_{n-1,n}, \quad \Phi_n = \Phi_{n-1,n}, \quad n \in \mathbb{N}.$$

It will be shown below that given $\hat{\rho}(z)$, F_n and Φ_n exist, and they are uniquely determined for all $n \in \mathbb{N}$.

The main result of this paper is the following

Theorem 1. *If the Stieltjes moment problems for ρ and $xd\rho(x^{-1})$ are determinate then*

$$\lim_{n\in\mathbb{N}} F_n(z) = \hat{\rho}(z) \tag{5}$$

and

$$\lim_{n\in\mathbb{N}} \Phi_n(z) = \hat{\rho}(z) \tag{6}$$

uniformly on each compact subset of $\mathbb{C} \setminus [0,\infty)$.

This theorem will be proved in Sections 4 and 5. The key ingredient in the proof is a result on the convergence of multipoint Padé approximants which is obtained in Section 3. Section 2 contains some auxiliary results.

2. Auxiliary Results

Let $\mathcal{M}$ be the class of all finite positive Borel measures on $\mathbb{R}$ with an infinite number of points on its support. There is a one to one correspondence between analytic functions $f : \{z \in \mathbb{C} : \Im(z) > 0\} \longrightarrow \{w \in \mathbb{C} : \Im(w) > 0\}$ such that $\sup_{y\geq 1} |yf(i\,y)| < \infty$ and functions which may be represented as in (3) (see [1], Chap. 3). Using the Stieltjes-Perron inversion formula (see [9], p. 74), we can recover the measure $\mu \in \mathcal{M}$ from $\hat{\mu}$.

If $x^j \in L_1(\mu)$ it is said that μ ($\mu \in \mathcal{M}$) has moment $c_j(\mu)$ of order $j \in \mathbb{Z}$ where

$$c_j(\mu) = \int x^j \, d\mu(x).$$

Let $\mathcal{M}_+$ (respectively, $\mathcal{M}_-$ and $\mathcal{M}_\pm$) denote the class of all measures μ in $\mathcal{M}$ with $\operatorname{supp}(\mu) \subset [0,\infty)$ such that $\{c_j(\mu) < \infty : j \geq 0\}$ (respectively, $\{c_j(\mu) < \infty : j < 0\}$ and $\{c_j(\mu) < \infty : j \in \mathbb{Z}\}$).

The Stieltjes moment problem is said to be determinate for $\mu \in \mathcal{M}_+$ if there does not exist any other measure in $\mathcal{M}_+$ which has the same moments as those of μ. Analogously, one defines determinate Stieltjes moment problems for $\mu \in \mathcal{M}_-$ and $\mu \in \mathcal{M}_\pm$. The bilateral case is usually called the strong Stieltjes moment problem. A sufficient condition for the Stieltjes moment problem to be determinate for $\mu \in \mathcal{M}_+$ is Carleman's condition

$$\sum_{j=0}^{\infty} \frac{1}{\sqrt[2j]{c_j}} = \infty.$$

Of course, the strong moment problem for $\mu \in \mathcal{M}_{\pm}$ is determinate if either the Stieltjes moment problem for μ or the Stieltjes moment problem for $t\,d\mu(t^{-1}))$ are determinate in $\mathcal{M}_{+}$.

Let S be a topological space. We denote $C(S)$ the space of all continuous functions on S and $C_0(S)$ the space of functions in $C(S)$ which vanish at infinity. Let $C_0(S)^*$ be the dual space of $C_0(S)$.

Let $\{\beta_n\}_{n\in\mathbb{N}}$ and β be real Borel measures on S. It is said that β_n converges to β in the weak star topology of $C_0(S)^*$ if for all $f \in C_0(S)$

$$\lim_{n\to\infty} \int f\, d(\beta_n - \beta) = 0$$

holds.

Hereafter $|\beta|$ denotes the total variation of the real Borel measures β. The next lemma seems to be known but we could not find a reference to its proof, so for completeness we present it here.

Lemma 1. *Let S be a locally compact space in which every open set is $\sigma-$compact. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence of real Borel measures on S which converges to a real Borel measure β in the weak star topology of $C_0(S)^*$. Let $g \in C(S)$ be such that $g \in L_1(|\beta|) \cap L_1(|\beta_n|)$, for all $n \in \mathbb{N}$, and for each $\epsilon > 0$ there exist a compact set $K \subset S$ and $N_0 \in \mathbb{N}$ such that*

$$\forall n \geq N_0 \quad \int_{K^c} |g|\, d|\beta_n| < \epsilon, \tag{7}$$

then

$$\lim_{n\to\infty} \int g\, d(\beta_n - \beta) = 0.$$

Proof. Define the set function $\sigma(E) = \int_E g d\beta$ where E denotes any measurable set contained in S. Since $g \in L_1(|\beta|)$, σ defines a regular measure which is absolutely continuous with respect to β (see, for example, [3], p. 84 and p. 210). Let $\epsilon > 0$ be fixed. Then, there exists a compact set J such that

$$\int_{J^c} |g|\, d|\beta| < \epsilon. \tag{8}$$

On the other hand, according to our assumptions there exists a compact set K such that (7) takes place. Choose an open set U such that $(K \cup J) \subset U$. By Urysohn's lemma (see, for example, [10], p. 39) there exists a continuous function $f : S \to [0, 1]$ such that $f \equiv 1$ on $K \cup J$ and $f \equiv 0$ on U^c. Since $gf \in C_0(S)$ and β_n converges weakly to β there exists $N_1 \geq N_0$ such that

$$\left| \int g\, f\, d(\beta_n - \beta) \right| < \epsilon \tag{9}$$

for all $n \geq N_1$. Combining (7), (8), and (9) we get

$$\begin{aligned} \left| \int g\, d(\beta_n - \beta) \right| &\leq \left| \int g\,(1 - f)\, d(\beta_n - \beta) \right| + \left| \int g\, f\, d(\beta_n - \beta) \right| \\ &\leq \int_{(K \cup J)^c} |g|\, d|\beta_n| + \int_{(K \cup J)^c} |g|\, d|\beta| + \epsilon < 3\epsilon \end{aligned}$$

for all $n \geq N_1$. $\square$

Remark 1. We only use this lemma for $S = (0, \infty)$.

3. Multipoint Padé Approximants

Consider an arbitrary table $A_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,2n}\}$ where $n \in \mathbb{N}$, $z_{n,k} \in \overline{\mathbb{C}} \setminus (0, \infty)$ and they are symmetric with respect to the real axis. We assume that $|z_{n,1}| \geq |z_{n,2}| \geq \ldots \geq |z_{n,2n}|$ and we denote

$$w_n(z) = \prod_{|z_{n,i}|<1} (z - z_{n,i}) \prod_{|z_{n,i}| \geq 1} (1 - \frac{z}{z_{n,i}}),$$

where the first product is over all $z_{n,i}$ such that $|z_{n,i}| < 1$ and the second one is over the rest of $z_{n,i}$. By convention, if $z_{n,i} = \infty$, the corresponding factor equals 1.

Let ρ be a measure on $M_{\pm}$, and let $\{A_n\}_{n \in \mathbb{N}}$ be as above. The multipoint Padé approximant of order n that interpolates $\hat{\rho}$ in A_n is a rational function $\pi_n = \dfrac{p_n}{q_n}$ where p_n and q_n are polynomials such that:

- $\deg(p_n) \leq n - 1$, $\deg q_n \leq n$, and $q_n \not\equiv 0$;
- $\dfrac{q_n(z)\hat{\rho}(z) - p_n(z)}{w_n(z)}$ is an analytic function in $\mathbb{C} \setminus [0, \infty)$;
- If λ_n points $z_{n,i}$ equal zero, then $\lim_{z \to 0-} \frac{q_n(z)\hat{\rho}(z) - p_n(z)}{z^{\lambda_n}} = 0$;

- If γ_n points $z_{n,i}$ equal ∞, then $\lim_{z\to\infty,\, z<0} z^{\gamma_n}(q_n(z)\hat{\rho}(z) - p_n(z)) = 0$.

The last three conditions are equivalent to saying that π_n interpolates $\hat{\rho}$ at $\{z_{n,1}, z_{n,2}, \ldots, z_{n,2n}\}$. Moreover, these conditions determine p_n and q_n up to a constant factor. So, the multipoint Padé approximant of order n is unique.

Lemma 2. *With the above notations, the polynomial q_n is the $n-th$ orthogonal polynomial with respect to the varying measure $\dfrac{d\rho(x)}{w_n(x)}$; that is,*

$$\int_0^\infty x^k\, q_n(x)\, \frac{d\rho(x)}{w_n(x)} = 0, \quad k = 0, 1, \ldots n-1.$$

Consequently, as w_n is positive on $(0,\infty)$, all the zeros of q_n are simple and lie on $(0,\infty)$.

This result is well known and its proof can be found, for example, in [8]. Using Lemma 2, the numerator can be expressed by

$$p_n(z) = \int \frac{w_n(x)q_n(z) - w_n(z)q_n(x)}{z-x} \frac{d\rho(x)}{w_n(x)},$$

and the remainder equals

$$\hat{\rho}(z) - \pi_n(z) = \frac{w_n(z)}{q_n^2(z)} \int \frac{q_n^2(x)}{z-x} \frac{d\rho(x)}{w_n(x)}. \tag{10}$$

Furthermore, as the zeros $x_{n,1}, x_{n,2}, \ldots, x_{n,n}$ of the orthogonal polynomial q_n, are simple and lie on $(0,\infty)$, the multipoint Padé approximant is given by

$$\pi_n(z) = \sum_{j=1}^{n} \frac{\lambda_{n,j}}{z - x_{n,j}},$$

where $\lambda_{n,j}$ are the Christophel-Darboux coefficients given by

$$\begin{aligned} \lambda_{n,j} &= \frac{w_n(x_{n,j})}{q_n'(x_{n,j})} \int \frac{q_n(x)}{x - x_{n,j}} \frac{d\rho(x)}{w_n(x)} \\ &= \frac{w_n(x_{n,j})}{q_n'(x_{n,j})^2} \int \left(\frac{q_n(x)}{x - x_{n,j}}\right)^2 \frac{d\rho(x)}{w_n(x)}. \end{aligned}$$

See [4] and [12], Chap. 6 for the proof of the above formula. The last identity is a consequence of the following result that is a transformation of the Gauss-Jacobi lemma (see [13], Theorem 3.4.1)

Lemma 3. *Set* $d\rho_n = \sum_{j=1}^{n} \lambda_{n,j} d\delta_{x_{n,j}}$, *where* $\delta_{x_{n,j}}$ *is the Dirac measure at* $x_{n,j}$. *We have*

$$\sum_{j=1}^{n} \lambda_{n,j} \frac{h(x_{n,j})}{w_n(x_{n,j})} = \int \frac{h(x)}{w_n(x)} d\rho_n(x) = \int \frac{h(x)}{w_n(x)} d\rho(x) \tag{11}$$

for each polynomial h *of degree less than or equal to* $2n-1$.

With this notation, π_n can be rewritten as

$$\pi_n(z) = \int \frac{1}{z-x} d\rho_n(x). \tag{12}$$

Given $z_0 \in \mathbb{C} \setminus (0,\infty)$, $n \in \mathbb{N}$, and $\epsilon > 0$, let $I_{n,\epsilon}(z_0)$ denote the number of points $z_{n,1}, z_{n,2}, \ldots, z_{n,2n}$ in $\{z : |z - z_0| < \epsilon\}$. Analogously $I_{n,\epsilon}(\infty)$ is the number of points $z_{n,1}, z_{n,2}, \ldots, z_{n,2n}$ in $\{z : |z - z_0| > \epsilon\}$.
The next result presents a nice refinement of results in [8]. The results in [8] are based on Carleman's condition which is sufficient for the determination of the moment problem. We manage to prove those results in terms of the determination of the moment problem itself. Moreover, the interpolation points can approach the support of the measure through the complex plane.

Theorem 2. *Assume that the set of interpolation points* $\{A_n\}_{n\in\mathbb{N}}$ *has no limit points on* $(0,\infty)$.

i) *If there exist* $z_0 \in \mathbb{C} \setminus [0,\infty)$ *and a sequence of indices* Λ *such that* $\lim_{n\in\Lambda} I_{n,\epsilon}(z_0) = \infty$ *for all* $\epsilon > 0$, *then*

$$\lim_{n\to\infty,\, n\in\Lambda} \pi_n(z) = \hat{\rho}(z), \tag{13}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

ii) *If the Stieltjes moment problem for* ρ *is determinate; there exists* $K > 0$ *such that*

$$\left| 1 - \frac{x}{z_{n,j}} \right| \geq K, \tag{14}$$

for each $j \in \mathbb{N}$, *for all* $x \in (0,\infty)$ *and* $n \in \Lambda$; *and* $\lim_{n\in\Lambda} I_{n,\epsilon}(\infty) = \infty$ *for all* $\epsilon > 0$, *then* (13) *holds.*

iii) *If the Stieltjes moment problem for* $x\,d\rho(x^{-1})$ *is determinate; there exists* $L>0$ *such that*

$$|1 - x\,z_{n,n-j}| \geq L \tag{15}$$

for each $j \geq 0$*, for all* $x \in (0,\infty)$ *and* $n \in \Lambda$*; and* $\lim_{n\in\Lambda} I_{n,\epsilon}(0) = \infty$ *for all* $\epsilon > 0$*, then* (13) *holds.*

iv) *If the strong Stieltjes moment problem for* ρ *is determinate;* $\lim_{n\in\Lambda} I_{n,\epsilon}(0) = \infty$ *and* $\lim_{n\in\Lambda} I_{n,\epsilon}(\infty) = \infty$ *for all* $\epsilon > 0$*; and both conditions* (14) *and* (15) *are satisfied, then* (13) *holds.*

Remark 2. Conditions (14) and (15) mean that the interpolation points converge to ∞ and 0, respectively, nontangentially to $(0,\infty)$.

Proof. First, we are going to see that i) and ii) hold. We begin proving that the sequence of multipoint Padé approximants $\{\pi_n\}_{n\in\Lambda}$ is uniformly bounded on compact subsets of $\mathbb{C}\setminus[0,\infty)$.
Let E be a compact subset of $(\mathbb{C}\setminus[0,\infty)) \cap (\{|z|<r\})$, where r is a fixed number, $0 < r < +\infty$. In case that there exists $z_{n,k} = \infty$, we can put $h(x) = w_n(x)$ $(deg\, h \leq 2n-1)$ in formula (11) and we obtain

$$\sum_{j=1}^{n} \lambda_{n,j} = \int_0^{+\infty} d\rho_n(x) = \int_0^{+\infty} d\rho(x) = |\rho| < +\infty.$$

Therefore,

$$|\pi_n(z)| = \left| \sum_{j=1}^{n} \frac{\lambda_{n,j}}{z - x_{n,j}} \right| \leq \sum_{j=1}^{n} \frac{|\lambda_{n,j}|}{|z - x_{n,j}|} \leq \frac{1}{d}|\rho|, \quad z \in E,$$

where $d = \inf\{|z-x| : z \in E,\, x \in [0,\infty)\}$. So, $\{\pi_n\}$ is uniformly bounded on compact subsets of $\mathbb{C}\setminus[0,\infty)$.
Now, we study the case $z_{n,k} \neq \infty$, for all $k = 1,\ldots,2n$. Under the both assumptions in i) and ii), we can choose $l > 0$ and $z_{n,i}$ such that $|z_{n,i}| \geq l$, for large enough $n \in \Lambda$. From (12)

$$|\pi_n(z)| = \left| \int_0^{+\infty} \frac{1 - \frac{x}{z_{n,i}}}{z - x} \frac{d\rho_n(x)}{1 - \frac{x}{z_{n,i}}} \right| \leq \int_0^{+\infty} \left| \frac{1 - \frac{x}{z_{n,i}}}{z - x} \right| \frac{d\rho_n(x)}{\left|1 - \frac{x}{z_{n,i}}\right|}.$$

If z_0 is a limit point of the set $\{z_{n,1},\ldots,z_{n,2n}\}$, $n \in \Lambda$, and $\epsilon > 0$ is a fixed number, there exists $z_{n,i}$ with $n \in \Lambda$ such that

$$\left|\frac{1}{1-\frac{x}{z_{n,i}}}\right| = \frac{|z_{n,i}|}{|z_{n,i}-x|} \le \frac{|z_0|+\epsilon}{m},$$

where $m = \inf\{|z-x| : x \in (0,+\infty),\ z \in \{|z-z_0| \le \epsilon\}\}$.

When we are in the case ii), ∞ is a limit point, and from (14) we have

$$\left|\frac{1}{1-\frac{x}{z_{n,i}}}\right| \le \frac{1}{K}.$$

Hence, in both cases, the sequence of positive Borel measures $\left\{\dfrac{d\rho_n}{1-\frac{x}{z_{n,i}}}\right\}_{n\in\Lambda}$ is uniformly bounded.

Thus, it is sufficient to see that $\left|\dfrac{1-\frac{x}{z_{n,i}}}{z-x}\right|$ is uniformly bounded on each compact subset E of $\mathbb{C}\setminus[0,\infty)$.

If $x \ge 2r$, $z \in E$

$$\left|\frac{1-\frac{x}{z_{n,i}}}{z-x}\right| = \left|\frac{\frac{1}{x}-\frac{1}{z_{n,i}}}{1-\frac{z}{x}}\right| \le \frac{\frac{1}{x}+\frac{1}{|z_{n,i}|}}{1-\left|\frac{z}{x}\right|} \le 2\left(\frac{1}{2r}+\frac{1}{l}\right) = A_1.$$

If $x < 2r$, $z \in E$

$$\left|\frac{1-\frac{x}{z_{n,i}}}{z-x}\right| \le \frac{1}{d}(1+\frac{2r}{l}) = A_2.$$

Then

$$\left|\frac{1-\frac{x}{z_{n,i}}}{z-x}\right| \le \max\{A_1,A_2\} = A.$$

Therefore, the sequence of multipoint Padé approximants $\{\pi_n\}_{n\in\Lambda}$ is uniformly bounded on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

In order to prove i), according to Montel's, theorem it is sufficient to prove pointwise convergence of π_n to $\hat{\rho}$. This follows since the assumptions of i) and (10) hold, and so any partial limit of $\{\hat{\rho}-\pi_n\}_{n\in\Lambda}$ has zeros with a limit point in $\mathbb{C}\setminus[0,\infty)$.

Now, we prove ii). Here we use the notation

$$w_{n,j}(x) = \left(1-\frac{x}{z_{n,1}}\right)\left(1-\frac{x}{z_{n,2}}\right)\cdots\left(1-\frac{x}{z_{n,2j}}\right).$$

Observe that A_n is symmetric with respect to the real axis, so we can assume that $w_{n,j}(x) > 0$, $x \in (0, \infty)$. Moreover, for all $j \geq 1$ and $0 \leq i \leq 2j$, we have

$$\lim_{n\to\infty,\, n\in\Lambda} z_{n,i} = \infty. \tag{16}$$

Then, using Lemma 3 with $h(x) = \dfrac{x^i w_n(x)}{w_{n,j}(x)}$, (14), and the dominated convergence theorem, we obtain

$$\lim_{n\in\Lambda} \int_0^{+\infty} \frac{x^i d\rho_n(x)}{w_{n,j}(x)} = \lim_{n\in\Lambda} \int_0^{+\infty} \frac{x^i d\rho(x)}{w_{n,j}(x)} = \int_0^{+\infty} x^i d\rho(x). \tag{17}$$

In particular, for $i = 0$ the above relation means that the sequence $\left\{ \dfrac{d\rho_n(x)}{w_{n,j}(x)} \right\}_{n\in\Lambda}$ is bounded. Then, from Alaoglu's theorem there exists a subsequence $\Lambda_1 \subset \Lambda$, and a measure $\alpha_j \in \mathcal{M}$ such that

$$\lim_{n\in\Lambda_1} \frac{d\rho_n(x)}{w_{n,j}(x)} = \alpha_j(x) \tag{18}$$

in the weak star topology of $C_0[0, \infty)^*$.

Furthermore, for each i such that $0 \leq i \leq 2j - 1$ and $N > 1$ we have

$$\lim_{N\to\infty} \lim_{n\in\Lambda_1} \int_{(N,\infty)} \frac{x^i d\rho_n(x)}{w_{n,j}(x)} \leq \lim_{N\to\infty} \lim_{n\to\infty} \frac{1}{N} \int_0^{+\infty} \frac{x^{i+1} d\rho_n(x)}{w_{n,j}(x)} = 0.$$

Thus, from Lemma 1 and (18), we obtain

$$\lim_{n\in\Lambda_1} \int_0^{+\infty} \frac{x^i d\rho_n(x)}{w_{n,j}(x)} = \int_0^{+\infty} x^i d\alpha_j(x)$$

and according to (17)

$$\int_0^{+\infty} x^i d\rho(x) = \int_0^{+\infty} x^i d\alpha_j(x). \tag{19}$$

Using (16), (17), and (18), we have

$$\begin{aligned}
\lim_{n\in\Lambda_1} \pi_n(z) &= \lim_{n\in\Lambda_1} \int_0^{+\infty} \frac{w_{n,j}(x)}{z-x} \frac{d\rho_n(x)}{w_{n,j}(x)} \\
&= \lim_{n\in\Lambda_1} \int_0^{+\infty} \left(\frac{1}{z-x} - \sum_{i=1}^{j} \prod_{1\leq k_1 < \ldots < k_i \leq j} \frac{(-1)^{i-1}}{z_{n,k_1} z_{n,k_2} \ldots z_{n,k_i}} \frac{x^i}{z-x} \right) \frac{d\rho_n(x)}{w_{n,j}(x)} \\
&= \int_0^{+\infty} \frac{d\alpha_j(x)}{z-x}.
\end{aligned}$$

This means that

$$\lim_{n\in\Lambda_1} \pi_n(z) = \hat{\alpha}_j(z), \quad \text{for all } j \geq 1,$$

and $\hat{\alpha_1} = \hat{\alpha_j}$ in $\mathbb{C} \setminus [0,\infty)$ for all $j \geq 1$. Futhermore, the Stieltjes-Perron inversion formula implies that $\alpha_1 = \alpha_j$ for all $j \geq 1$. From (19) it follows that

$$\int_0^{+\infty} x^i d\alpha_1(x) = \int_0^{+\infty} x^i d\rho(x), \quad \text{for all } i \geq 0.$$

Finally, since the Stieltjes moment problem for ρ is determinate, $\alpha_1 = \rho$ and (13) holds.

We see now that iii) follows from ii). Indeed, changing variables z by $1/\zeta$ and x by $1/t$ we obtain

$$\hat{\rho}(z) - \pi_n(z) = \hat{\rho}(1/\zeta) - \pi_n(1/\zeta)$$

$$= -\zeta\hat{\beta}(\zeta) - \frac{\zeta^n p_n(1/\zeta)}{\zeta^n q_n(1/\zeta)} = -\zeta\left(\hat{\beta}(\zeta) - \frac{P_n(\zeta)}{Q_n(\zeta)}\right),$$

where $d\beta(t) = t\,d\rho(t^{-1})$, $P_n(\zeta) = \zeta^{n-1}p_n(\zeta^{-1})$, and $Q_n(\zeta) = \zeta^n q_n(\zeta^{-1})$. Observe that $\tilde{\pi}_n = \frac{P_n}{Q_n}$ is the multipoint Padé approximant of type $[n-1,n]$ that interpolates $\hat{\beta}$ in the set of points $\tilde{A}_n = \{z_{n,1}^{-1}, z_{n,2}^{-1}, \ldots, z_{n,2n}^{-1}\}$, and the hypothesis in iii) transform into the corresponding hypothesis ii) for this new rational interpolant $\tilde{\pi}_n$ of $\hat{\beta}$.

In order to prove iv) it is sufficient to observe that if the strong Stieltjes moment problem for ρ is determinate, the result can be obtained combining the ideas used in proving ii) and iii). $\square$

Remark 3. An analogous result can be obtained for positive Borel measures λ whose support is contained in the interval $[-1,1]$, when the interpolation points lie in $\mathbb{C} \setminus (-1,1)$ and they are symmmetric with respect to the real axis, doing the changes of variables:

$$z = \frac{1+\zeta}{1-\zeta}, \quad z \in \mathbb{C} \setminus [0,+\infty), \quad \zeta \in \mathbb{C} \setminus [-1,1],$$

$$x = \frac{1+t}{1-t}, \quad x \in (0,+\infty), \quad t \in (-1,1).$$

4. Nonlinear Fourier-Padé Approximants

First we need to prove that there exists an unique nonlinear Fourier-Padé approximant for every $n \in \mathbb{N}$. In this case we will have to find a rational function F_n such that

$$c_i(\hat{\rho} - F_n) = 0, \quad k = 0, 1, \ldots, 2n-1.$$

This is equivalent to

$$\int_{-\infty}^{0} (\hat{\rho}(t) - F_n(t)) t^j d\beta(t) = 0, \quad j = 0, 1, \ldots, 2n-1. \tag{20}$$

Consider an arbitrary set of $2n$ points of $[-\infty, 0]$

$$Z_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,2n}\}$$

with $z_{n,k} \leq z_{n,k+1}$. This set of points forms a simplex in the space $[-\infty, 0]^{2n}$. It is denoted by K^{2n}. Set π_n to be the multipoint Padé approximant that interpolates the function $\hat{\rho}$ at Z_n. In accordance with Section 3, we have $\pi_n = \dfrac{p_n}{q_n}$, where from Lemma 2, the polynomial $q_n(x) = x^n + \cdots$ is uniquely determined by the orthogonal relations

$$\int_0^{\infty} q_n(t) t^j \frac{d\rho(t)}{W_n(t)} = 0, \quad j = 0, 1, \ldots, n-1,$$

with $W_n(x) = \prod_{i=1}^{2n}(x - z_{n,i})$, where $x - z_{n,i} = 1$ if $z_{n,i} = \infty$. Now, we define a monic polynomial Ω_n, $deg(\Omega_n) = 2n$ by the relations

$$\int_{-\infty}^{0} \Omega_n(t) t^j \left(\frac{1}{q_n^2(t)} \int_0^{\infty} \frac{q_n^2(x)}{x-t} \frac{d\rho(x)}{W_n(x)} \right) d\beta(t) = 0, \quad j = 0, 1, \ldots, 2n-1. \tag{21}$$

This is possible because $\frac{1}{q_n^2(t)} \int_0^{\infty} \frac{q_n^2(x)}{x-t} \frac{d\rho(x)}{W_n(x)}$ has a constant sign in $(-\infty, 0)$ since all the zeros of q_n lie on $(0, \infty)$. Thus (21) determines an unique polynomial Ω_n that has $2n$ simple zeros $Y_n = \{y_{n,1}, y_{n,2}, \ldots, y_{n,2n}\}$ on the interval $(-\infty, 0)$. The correspondence $Z_n \to Y_n$ defines a mapping of the simplex K^{2n} into itself. This mapping is continuous and, therefore by Brouwer's theorem (see [11], p. 730) it has a fixed point. Keeping the same notation Z_n for the fixed point, we get $\Omega_n = W_n$ and (21) can be rewritten as

$$\int_{-\infty}^{0} W_n(t)t^j \left(\frac{1}{q_n^2(t)} \int_0^\infty \frac{q_n^2(x)}{x-t} \frac{d\rho(x)}{W_n(x)}\right) d\beta(t) = 0, \quad j = 0, 1, \ldots, 2n-1.$$

Taking into account formula (10) with π_n the multipoint Padé approximant of $\hat{\rho}$ associated with Z_n, we get

$$\int_{-\infty}^{0} (\hat{\rho}(t) - \pi_n(t))t^j d\beta(t) = 0, \quad j = 0, 1, \ldots, 2n-1.$$

Setting $F_n = \pi_n$, we obtain that relations in (20) hold for this function. Since all zeros of $T_n = q_n$ lie on $(0, \infty)$, F_n is a nonlinear Fourier-Padé approximant of $\hat{\rho}$ and thus it is unique.

Note that together with the existence of F_n, we have proved that the sequence $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of multipoint Padé approximants of $\hat{\rho}$ corresponding to the table $\{Z_n\}_{n\in\mathbb{N}}$. From this fact it follows that the denominator T_n of the rational function F_n satisfies the orthogonal relations

$$\int_0^\infty T_n(x)x^j \frac{d\rho(x)}{W_n(x)} = 0, \quad j = 0, 1, \ldots, n-1,$$

and all the zeros of T_n lie on $(0, \infty)$.

Taking into account what has been said above, (5) of Theorem 1 follows from Theorem 2.

5. Fourier-Padé approximants

Let us rewrite relation (2) which determines the polynomials s_n and t_n in $\Phi_n = \dfrac{s_n}{t_n}$ in the following equivalent form:

$$\int_{-\infty}^{0} (t_n(t)\hat{\rho}(t) - s_n(t))t^j d\beta(t) = 0, \quad j = 0, 1, \ldots, 2n-1. \tag{22}$$

This system always has a non-trivial solution. Fix an arbitrary pair of polynomials s_n, $t_n \not\equiv 0$, satisfying (22) with t_n monic. Then the function $t_n\hat{\rho} - s_n$ has at least $2n$ zeros on $(-\infty, 0)$. Choose an arbitrary set

$$Z_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,2n}\}$$

of zeros of $t_n\hat{\rho}-s_n$ in $(-\infty,0)$. Set $W_n(x)=(x-z_{n,1})(x-z_{n,2})\cdots(x-z_{n,2n})$, then $\pi_n=\dfrac{s_n}{t_n}$ is a multipoint Padé approximant of $\hat{\rho}$ corresponding to Z_n. Then (see Section 3), t_n satisfies orthogonal relations, has degree n, and all its zeros lie in $(0,\infty)$. From this it follows (see Section 1) that there exists an unique approximant $\Phi_n=\dfrac{s_n}{t_n}$ of $\hat{\rho}$. Also, it is easy to show that the number of zeros of $t_n\hat{\rho}-s_n$ on $(-\infty,0)$ is precisely equal to $2n$. Thus, the polynomial W_n is also uniquely defined. Now, (6) of Theorem 1 follows from Theorem 2.

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