ASYMPTOTICS FOR STIELTJES POLYNOMIALS, PADÉ-TYPE APPROXIMANTS, AND GAUSS–KRONROD QUADRATURE

By

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Abstract. We study the asymptotic properties of Stieltjes polynomials outside the support of the measure as well as the asymptotic behaviour of their zeros. These properties are used to estimate the rate of convergence of sequences of rational functions, whose poles are partially fixed, which approximate Markov-type functions. An estimate for the speed of convergence of the Gauss–Kronrod quadrature formula in the case of analytic functions is also given.

1 Introduction

1.1 General Remarks Let ω be a nonnegative function on the interval $[-1,1]$ with $\omega \in L^1[-1,1]$. By dx we denote Lebesgue measure on $[-1,1]$. Let $\{p_n\}_{n\in \mathbb{N}}$ be the sequence of orthonormal polynomials with respect to the weight function ω; that is, $p_n(z) = \kappa_n z^n + \cdots$, $\kappa_n > 0$, and

$$\int_{-1}^{1} p_m(x)\, p_k(x)\, \omega(x)\, dx = \delta_{km}. \tag{1}$$

It is well-known and easy to verify that there exists a unique monic polynomial S_n of degree n which satisfies the orthogonality relations

$$\int_{-1}^{1} x^k\, S_n(x)\, p_{n-1}(x)\, \omega(x)\, dx = 0, \quad k = 0, 1, \ldots, n-1. \tag{2}$$

The polynomial S_n is called the nth Stieltjes polynomial with respect to the weight function ω. This class of polynomials $\{S_n\}$ was introduced by Stieltjes [22] for

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the Legendre weight $w \equiv 1$ in terms of the associated function of the second kind. For more information, see the remark after Lemma 2 below.

In the last two decades, Stieltjes polynomials have attracted considerable attention. This interest has been motivated by their connection with Gauss–Kronrod quadrature formulas

$$\int_{-1}^{1} f(x)w(x)\,dx = \sum_{k=1}^{n} \sigma_{k,n} f(x_{k,n}) + \sum_{k=1}^{n+1} \gamma_{k,n} f(y_{k,n}) + E_n(f), \tag{3}$$

where $\{x_{k,n}\}$ are the zeros of the orthogonal polynomial p_n. The nodes $\{y_{k,n}\}$ and weights $\{\sigma_{k,n}\}$, $\{\gamma_{k,n}\}$ are chosen so as to maximize the degree of exactness of the formula in the space of polynomials. It is easy to see that if for a given weight, $E_n(f) = 0$ for all polynomials of degree less than or equal to $3n+1$, then the nodes $y_{k,n}$ must be the zeros of the Stieltjes polynomial S_{n+1}. The reciprocal statement is also true if the zeros of the Stieltjes polynomials S_{n+1} happen to be simple and distinct from the zeros of p_n. In fact, there is equivalence between the construction of Stieltjes polynomials and Gauss–Kronrod quadrature formulas if multiple nodes are allowed (for details, see Section 5). Kronrod [10] was the first to consider this type of formula, taking as nodes the zeros of Legendre polynomials and the zeros of the corresponding Stieltjes polynomials. For further references and surveys on this topic, see [9], [12], and [7].

From the point of view of quadrature processes, it is important to know if the nodes are simple, their interlacing properties, and whether they are contained in the set where the function to be integrated is defined. Since S_n is orthogonal with respect to a sign changing function, equations (2) do not in general guarantee that the zeros of S_n lie in $[-1,1]$, that they are simple and distinct from the zeros of p_{n-1}, or even that they are real. However, for the ultraspherical weight function w_λ, $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $0 \le \lambda \le 2$, Szegö proved in [24] that these properties hold for all n. Positivity of the coefficients appearing in the quadrature formula and interlacing properties of the zeros have also been studied for the ultraspherical weights w_λ, $0 \le \lambda \le 1$, in [11] and [4], respectively. The same properties are analysed in [16] and [17] for weights of the type $\sqrt{1-x^2}w(x)$, where $\sqrt{1-x^2}w(x)$ is positive and twice continuously differentiable on $[-1,1]$. Estimates of the error in Gauss–Kronrod quadrature formulas have been given for classes of functions with different degrees of smoothness. For the case of analytic functions, see [6] and [15]. In connection with Lagrange interpolation, see also [8].

Thus, to some extent the study of Stieltjes polynomials has been marked so far by their applicability in Gauss–Kronrod quadrature. This has caused research to focus on weights for which quadrature is meaningful for classes of functions as large as possible. We have shifted the attention to the Stieltjes polynomials themselves and to the study of their asymptotic properties, regardless of their immediate use in quadrature. We aim to describe general classes of weights for which the corresponding Stieltjes polynomials have either nth root (weak), ratio, or strong asymptotic behaviour. Such results have direct application in the approximation of Markov functions by means of rational approximants with partially prescribed poles (Padé-type approximants in the terminology commonly used in recent years). Regarding such approximants, we refer to the papers [1]–[3] and the references therein. As a by-product of the results obtained in rational approximation, we give estimates of the rate of convergence of Gauss–Kronrod quadrature for functions which are analytic on a neighbourhood of the set of integration.

1.2 Definitions and Known Results Let μ be a finite, positive Borel measure on the real line $\mathbb{R}$ whose compact support $S(\mu)$ contains infinitely many points. Let $\mu' = d\mu/dx$ be the Radon–Nykodym derivative of μ with respect to the Lebesgue measure dx. Whenever we find it more convenient, we adopt the differential notation for a measure. The nth Stieltjes polynomial with respect to μ is defined by (1) and (2), with $\omega(x)\,dx$ replaced by $d\mu(x)$. That is, let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials such that for each $n \in \mathbb{N}$, S_n is defined as the monic polynomial of degree at most n verifying

$$\int_{S(\mu)} x^k\, S_n(x)\, p_{n-1}(x)\, d\mu(x) = 0, \qquad k = 0, 1, \dots, n-1,$$

where $p_{n-1} = \kappa_{n-1} z^{n-1} + \cdots$, $\kappa_{n-1} > 0$, is the $(n-1)$th orthonormal polynomial with respect to the measure μ. Finding S_n reduces to solving a system of n homogeneous equations in $n+1$ unknowns. Thus a non-trivial solution always exists. From the orthogonality relations satisfied by S_n, it is easy to conclude that $\deg S_n = n$. S_n is called the nth Stieltjes polynomial with respect to the measure $d\mu$. Unless otherwise stated, the set of integration is $S(\mu)$, in which case it will not be indicated. We refer to $s_n = \kappa_{n-1}\, S_n$ as the normalized nth Stieltjes polynomial. The introduction of this notation allows us to give several formulas a closed form; of course, this has nothing to do with attempting to orthonormalize the Stieltjes polynomials.

The largest class of measures with which we deal is that of regular measures. This class of measures was introduced in recent years and has been extensively studied. The excellent monograph [21] by H. Stahl and V. Totik is dedicated to the study of these measures and their orthogonal polynomials. For the precise definition and different equivalent forms of its expression, see page 61 of that treatise. The regularity of the measure μ, for which we write $\mu \in \mathbf{Reg}$, is equivalent to either one of the following two limit relations (see Theorem 3.1.1 in [21]),

$$\lim_{n\to\infty} \kappa_n^{1/n} = \frac{1}{\operatorname{cap} S(\mu)}, \tag{4}$$

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = \exp\{g_\Omega(z,\infty)\}, \tag{5}$$

uniformly on compact subsets of $\mathbb{C}\setminus \operatorname{Co}(S(\mu))$, where $\operatorname{Co}(S(\mu))$ denotes the convex hull of $S(\mu)$, $\operatorname{cap} S(\mu)$ stands for the logarithmic capacity of $S(\mu)$, and $g_\Omega(z,\infty)$ is the (generalized) Green function with singularity at infinity relative to the region $\Omega = \overline{\mathbb{C}} \setminus S(\mu)$ (cf. Section 1.2 and Appendix A.5 in [21] for the definition). We assume that $\operatorname{cap} S(\mu) > 0$, which is equivalent to $g_\Omega(z,\infty) \not\equiv +\infty$.

The Blumenthal–Nevai class of measures is also important in the theory of orthogonal polynomials and related subjects. Let

$$x\, p_n(x) = a_{n+1}p_{n+1} + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geq 1,$$

be the recurrence relation satisfied by the sequence $\{p_n\}_{n\in\mathbb{N}}$ of orthonormal polynomials. We say that $\mu \in M(a,b)$ if $\lim_{n\to\infty} b_n = b$ and $\lim_{n\to\infty} a_n = a/2$. In this case, it is known that $S(\mu) = [b-a, b+a] \cup e$, where e is an at most denumerable set whose only possible accumulation points are $b \pm a$. We assume that $a \neq 0$, so that $[b-a,\ b+a]$ does not reduce to a point. In this case,

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = \Psi\left(\frac{z-b}{a}\right) \tag{6}$$

uniformly on compact subsets of $\mathbb{C} \setminus S(\mu)$, where $\Psi(z) = z + \sqrt{z^2-1}$. The square root is taken to be positive for $z > 1$. This function is the conformal mapping of $\overline{\mathbb{C}} \setminus [-1,1]$ onto $\{|w| > 1\}$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Because of these properties, $\log|\Psi((z-b)/a)|$ is the Green function with singularity at infinity relative to the region $\mathbb{C} \setminus [b-a, b+a]$. If $\mu \in M(a,b)$, then in addition to (6), we have

$$\lim_{n\to\infty} \int_{S(\mu)} f(x)\, p_n^2(x)\, d\mu(x) = \frac{1}{\pi} \int_{b-a}^{b+a} f(x) \frac{dx}{\sqrt{a^2-(x-b)^2}} \tag{7}$$

for every bounded Borel-measurable function f on $S(\mu)$, continuous on $[b-a, b+a]$. For more details on this class of measures and its properties, see the book [13] by P. Nevai. A well-known sufficient condition for $\mu \in M(a,b)$ due to E. A. Rakhmanov is that $S(\mu) = [b-a, b+a]$ and $\mu' > 0$ almost everywhere on $[b-a,\ b+a]$ (see [19] for a proof).

Finally, we consider the Szegö class of measures. For simplicity in notation, we restrict our attention here to measures supported on $[-1,1]$. We say that $\mu \in \mathbf{S}$ if $S(\mu) = [-1,1]$ and $\log \mu'(x)/\sqrt{1-x^2} \in L^1[-1,1]$. In this case,

$$\lim_{n\to\infty} \frac{p_n(z)}{[\Psi(z)]^n} = \frac{1}{\sqrt{2\pi}} S_\mu(\Psi(z)), \tag{8}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$, where

$$S_\mu(z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \log(\mu'(\cos\theta)\,|\sin(\theta)|)\, \frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta \right\}, \quad |z| \neq 1.$$

Relations of type (8) are called exterior strong asymptotic formulas.

As for strong asymptotics on the support of the measure, it is necessary to place more restrictions on the measure μ to obtain some results. Thus, for instance, suppose that $d\mu(x) = w(x)\,dx$ and the function $f(\theta) = w(\cos\theta)\,|\sin\theta|$ satisfies the Lipschitz–Dini condition

$$|f(\theta+\delta) - f(\theta)| < M\,(\log\delta)^{-L-1},$$

where M and L are fixed positive numbers. Then we have (see Theorem 12.1.4 in [23]), uniformly on $-1 \le x \le 1$,

$$(1-x^2)^{1/4}\sqrt{w(x)}p_n(x) = \sqrt{2/\pi}\cos\{n\theta + \gamma(\theta)\} + \mathcal{O}\{(\log n)^{-L}\}, \tag{9}$$

where $x = \cos\theta$, $\exp\{i\gamma(\theta)\} = S_\mu(e^{i\theta})/|S_\mu(e^{i\theta})|$, and $S_\mu(e^{i\theta}) := \lim_{r\to 1^-} S_\mu(re^{i\theta})$.

1.3 Statement of Main Results As mentioned above, the main object of this paper is the study of the asymptotic behaviour of Stieltjes polynomials. In this direction, not much is known so far. Most of the results to the present are formulas of type (9), which allow one to obtain reasonably accurate information on the location and asymptotic distribution of the zeros of Stieltjes polynomials. Ehrich, [4] and [5], proves relations similar to (9) for Stieltjes polynomials with respect to the ultraspherical weights w_λ, $0 \le \lambda \le 2$. Previously, Peherstorfer had given in [16] a representation of the limit of the Stieltjes polynomials with respect to the weight $\sqrt{1-x^2}w(x)$ in terms of the series expansion of S_w at $z=0$ provided that $\sqrt{1-x^2}w(x)$ is positive and twice continuously differentiable on $[-1,1]$ (see also [17]). It is quite surprising that formulas for the exterior asymptotic behaviour of Stieltjes polynomials are only known for the class of weights considered in [17]. In that work, the author asks whether such a relation takes place under weaker assumptions. As we shall see, the only restriction is that the measure satisfy

Szegö's condition ($\mu \in \mathbf{S}$). Regarding such other types of asymptotics as nth root or ratio asymptotics, to the best of our knowledge, the results we present are the first available.

Set

$$E = \left\{ z \in \mathbb{C} : g_\Omega(z, \infty) \leq \max_{w \in \mathrm{Co}(S(\mu))} g_\Omega(w, \infty) \right\}.$$

We write $S(\mu) = \mathrm{ess}[b - a, b + a]$ if the support of the measure μ has the same structure as in the Blumenthal–Nevai class $M(a, b)$, that is, consists of the interval $[b - a, b + a]$ and an at most denumerable set which accumulates only at the points $b \pm a$.

The functions of second kind with respect to μ are given by

$$g_n(z) = \int \frac{p_n(x)}{z - x} d\mu(x), \quad z \in \Omega = \overline{\mathbb{C}} \setminus S(\mu).$$

These functions are analytic in Ω and $g_n(\infty) = 0$. Because of the orthogonality relations satisfied by p_n with respect to the measure μ, $z = \infty$ is a zero of g_n of multiplicity $n + 1$.

We have

Theorem 1. *The following assertions hold.*

(a) *If* $\mu \in \mathbf{Reg}$ *and* $\mathrm{cap}\, S(\mu) > 0$, *then*

$$\lim_{n \to \infty} s_{n+1}(z)\, g_n(z) = 1, \tag{10}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$. *In addition, the set of accumulation points of the zeros of* $\{S_{n+1}\}_{n \in \mathbf{N}}$ *is contained in* E *and*

$$\lim_{n \to \infty} |s_{n+1}(z)|^{1/n} = \exp\{g_\Omega(z, \infty)\}, \tag{11}$$

uniformly on each compact subset of $\mathbb{C} \setminus E$. *Moreover, if* $S(\mu) = \mathrm{ess}[b - a, b + a]$, *then relations* (10) *and* (11) *hold uniformly on each compact subset of* $\overline{\mathbb{C}} \setminus S(\mu)$ *and* $\mathbb{C} \setminus S(\mu)$ *respectively; and the set of accumulation points of the zeros of the* $\{S_n\}_{n \in \mathbf{N}}$ *is contained in* $S(\mu)$.

(b) *If* $\mu \in M(a, b)$ *with* $a \neq 0$, *then*

$$\lim_{n \to \infty} \frac{s_{n+1}(z)}{s_n(z)} = \Psi\left(\frac{z - b}{a}\right) \quad \text{and} \quad \lim_{n \to \infty} \frac{p_n(z)}{s_{n+1}(z)} = \frac{1}{\sqrt{(z - b)^2 - a^2}}, \tag{12}$$

uniformly on each compact subset of $\mathbb{C} \setminus S(\mu)$.

(c) *If* $\mu \in \mathbf{S}$, *then*

$$\lim_{n \to \infty} \frac{s_{n+1}(z)}{[\Psi(z)]^n} = \sqrt{\frac{z^2 - 1}{2\pi}}\, S_\mu(\Psi(z)), \tag{13}$$

uniformly on each compact subset of $\mathbb{C} \setminus [-1, 1]$, *where* $S_\mu(z)$ *is as in* (8).

The paper is organized as follows. The next section is essentially dedicated to proving an integral relation between Stieltjes polynomials and functions of the second kind, which plays an important role in the subsequent arguments. The following section is dedicated to the study of the asymptotic properties of Stieltjes polynomials and to proving our main results stated above. In Section 4, we apply our result on nth root asymptotics to obtain convergence of a certain type of Padé-type approximants to Markov functions. In turn, this result is applied in Section 5 to estimate the speed of convergence of the Gauss–Kronrod quadrature formula when integrating functions which are analytic on a neighbourhood of the support of the measure. The final section contains an example illustrating the type of difficulties one encounters with the location of the zeros and the asymptotic properties of the Stieltjes polynomials when the support of the measure contains more than one interval.

2 Some lemmas

The following lemma provides some useful relations.

Lemma 1. *We have*

$$g_n(z)\,p_n(z) = \int \frac{p_n^2(x)}{z-x}\,d\mu(x), \qquad z \in \Omega; \tag{14}$$

and, for any polynomial ℓ_n of degree less than or equal to n,

$$\ell_n(z) \int \frac{s_n(x)}{z-x} p_{n-1}(x)\,d\mu(x) = \int \frac{\ell_n(x)s_n(x)}{z-x} p_{n-1}(x)\,d\mu(x), \qquad z \in \Omega. \tag{15}$$

Let K be a compact subset of $\mathbb{C} \setminus \mathrm{Co}(S(\mu))$; then there exist positive constants M_1, M_2, independent of n, satisfying

$$M_1 \le |g_n(z)\,p_n(z)| \le M_2, \qquad z \in K. \tag{16}$$

In particular, $p_n g_n$ has no zeros on $\mathbb{C}\setminus\mathrm{Co}(S(\mu))$. Moreover, if $S(\mu) = \mathrm{ess}[b-a, b+a]$, $a < a'$, and K is a compact subset of $\mathbb{C} \setminus (S(\mu) \cup [b-a', b+a'])$, then there exists n_0 such that for all $n \ge n_0$, g_n has no zeros on $\mathbb{C} \setminus [b-a', b+a']$ and (16) *holds uniformly on K.*

Proof. Relation (14) is well-known (see, e.g., Theorem 6.1.8 in [21]). It follows directly from the orthogonality properties of p_n. To prove (15), notice that from the orthogonality relations satisfied by s_n, we have

$$\int \frac{\ell_n(z) - \ell_n(x)}{z-x} s_n(x) p_{n-1}(x)\,d\mu(x) = 0,$$

which is equivalent to (15).

The general statement concerning (16) is also well-known. It is an immediate consequence of (14). The upper bound is obvious. For the lower bound, notice that

$$g_n(z)\,p_n(z) = \int \frac{(\Re(z) - x)p_n^2(x)}{|z - x|^2} d\mu(x) + i\Im(z) \int \frac{p_n^2(x)}{|z - x|^2} d\mu(x), \quad z \in \Omega,$$

which in the first place makes it obvious that $g_n p_n$ has no zero in $\mathbb{C} \setminus \mathrm{Co}(S(\mu))$. On the other hand, it is easy to bound from below in absolute value by a positive constant on a compact subset of $\mathbb{C} \setminus \mathrm{Co}(S(\mu))$.

Now assume that $S(\mu) = \mathrm{ess}[b - a, b + a]$, and $[b - a', b + a']$ and K are as in the second part of the statement relative to (16). In this case, the set $\mathrm{Co}(S(\mu)) \setminus (S(\mu) \cup [b - a', b + a'])$ is made up of at most a finite number of non-intersecting open intervals (let us assume at least one; otherwise, we would have nothing to prove). It is easy to show that on the closure of any bounded connected component of $\mathbb{R} \setminus S(\mu)$, p_n can have at most one zero (otherwise, one could construct a polynomial ℓ of degree $\leq n - 2$ such that ℓp_n would have a constant sign on the support of the measure, contradicting the orthogonality relations satisfied by p_n). On the other hand, each mass point of μ attracts at least one zero of p_n (see Theorem 6.1.1 [23]). Therefore, for all sufficiently large n, each one of the non-intersecting open intervals which compose $\mathrm{Co}(S(\mu)) \setminus (S(\mu) \cup [b - a', b + a'])$ contains exactly one zero of p_n which lies beyond a prescribed sufficiently small distance from K. Using this, the upper bound in (16) on K is immediate on account of (14). Thus we have that the family $\{p_n g_n\}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus (S(\mu) \cup [b - a', b + a'])$. Suppose that on the compact set K we had chosen before, $|p_n g_n|$ is not uniformly bounded from below by a positive constant. Take a convergent subsequence of $\{p_n g_n\}$ whose limit function has a zero at $z_0 \in K$. The limit function cannot be identically equal to zero, because that would contradict the lower bound which was shown to hold on compact subsets of the complement of the convex hull of the support. Therefore, z_0 is an isolated zero. Choose a neighborhood V of z_0 at a positive distance from $S(\mu) \cup [b - a', b + a']$. By Hurwitz' Theorem, we conclude that there is a subsequence of indices Δ such that for each $n \in \Delta$, $p_n g_n$ has at least one zero in V. Such zeros must be contained in the real line, since (as was proved earlier) $p_n g_n$ does not have zeros outside the real line for any n. Let us show that they cannot be on the real line either for all sufficiently large n. Having proved this, we arrive at a contradiction, which implies that on K the sequence $\{|p_n g_n|\}$ is uniformly bounded from below on K by a positive constant as needed.

First of all, using the arguments employed above, we know that for all sufficiently large n we can guarantee that p_n has no zero on V. Let us prove that for all sufficiently large n, g_n does not vanish on $\mathrm{Co}(S(\mu)) \setminus [b-a', b+a']$ and thus on $\mathbb{C} \setminus [b-a', b+a']$. To this end, notice that

$$(17) \qquad (p_n g_n)'(z) = \int \frac{p_n^2(x)}{z-x} d\mu(x) < 0, \quad z \in \mathbb{R} \setminus S(\mu).$$

Therefore the function $p_n g_n$ has at most one simple zero in each of the open intervals which give the connected components of $\mathrm{Co}(S(\mu)) \setminus S(\mu)$. On those intervals, we saw that p_n has exactly one zero for all sufficiently large n; therefore, for such n, the functions g_n cannot have any zeros. With this we conclude the proof. □

Now, let us obtain some integral expression connecting the Stieltjes polynomials and functions of the second kind.

Lemma 2. *We have*

$$(18) \quad s_n(z) - \frac{1}{g_{n-1}(z)} = \frac{1}{g_{n-1}(z)} \int \frac{s_n(x)}{z-x} p_{n-1}(x)\, d\mu(x), \quad z \in \overline{\mathbb{C}} \setminus \mathrm{Co}(S(\mu)),$$

and

$$(19) \qquad s_n(z)\, g_{n-1}(z) = 1 + \frac{g_{n-1}(z)}{2\pi i} \int_\gamma \frac{d\zeta}{g_{n-1}(\zeta)\,(\zeta - z)},$$

where γ is any positively oriented closed smooth curve which surrounds $\mathrm{Co}(S(\mu))$ *such that z is contained in the unbounded component of $\overline{\mathbb{C}} \setminus \gamma$. If $S(\mu)$* $= \mathrm{ess}[b-a, b+a]$, *then we can take γ in* (19) *as any smooth contour surrounding* $[b-a, b+a]$, *and the formula remains valid for all sufficiently large n.*

Proof. From the orthogonality relations of p_{n-1} with respect to the measure μ, we obtain

$$\int \frac{S_n(z) - S_n(x)}{z-x} p_{n-1}(x)\, d\mu(x) = \int \frac{z^n - x^n}{z-x} p_{n-1}(x)\, d\mu(x)$$
$$= \int x^{n-1}\, p_{n-1}(z)\, d\mu(x) = \frac{1}{\kappa_{n-1}}.$$

Rewriting this equality, we find that

$$g_{n-1}(z)\, s_n(z) = 1 + \int \frac{s_n(x)}{z-x} p_{n-1}(x)\, d\mu(x),$$

which is equivalent to the first formula of the lemma. From (18), (14) and (15) (used with $\ell_n(x) = x\, p_{n-1}(x)$), it follows that

$$s_n(z) - \frac{1}{g_{n-1}(z)} = \left(\int \frac{p_{n-1}^2(x)}{1 - x/z} d\mu(x) \right)^{-1} \int \frac{x\, s_n(x)}{z-x} p_{n-1}^2(x)\, d\mu(x).$$

Therefore, this function is analytic in $\overline{\mathbb{C}} \setminus \mathrm{Co}(S(\mu))$ and has a zero of order at least 1 at infinity. Using Cauchy's integral formula with a curve γ as indicated above, we obtain

$$s_n(z) - \frac{1}{g_{n-1}(z)} = \frac{1}{2\pi i} \int_\gamma \frac{s_n(\zeta) - 1/g_{n-1}(\zeta)}{(z-\zeta)} \, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{g_{n-1}(\zeta)\,(\zeta - z)},$$

which is (19) in different form. When $S(\mu) = \mathrm{ess}[b-a, b+a]$, take any smooth contour that surrounds $[b-a, b+a]$. Choose $a' > a$ such that $[b-a', b+a']$ does not intersect the contour. According to Lemma 1, for all sufficiently large n, g_n does not have zeros on $\mathbb{C} \setminus [b-a', b+a']$. Therefore, reasoning as before, we can obtain (19) using this γ. The proof is complete. □

In his letter to Hermite [22], Stieltjes considers the function g_n of the second kind with respect to the Legendre weight. He notices that such a function has a zero at infinity of degree $n+1$ and concludes that

$$\frac{1}{g_n(z)} = E_{n+1} + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad n \in \mathbb{N},$$

where E_{n+1} is a polynomial of degree $n+1$. Using Cauchy's integral formula he obtains an integral representation of the polynomial E_{n+1} (different from (18) and (19)) which allows him to prove that it satisfies full orthogonality relations with respect to $p_n(x)\,dx$. Therefore, E_{n+1} is S_{n+1} up to a multiplicative constant. We have preferred to take (2) as the starting-point for the Stieltjes polynomials.

3 Asymptotics of Stieltjes polynomials

Recall that

$$E = \left\{ z \in \mathbb{C} \,:\, g_\Omega(z,\infty) \le \max_{w \in \mathrm{Co}(S(\mu))} g_\Omega(w,\infty) \right\}.$$

The set E contains the convex hull of $S(\mu)$ and has, in general, non-empty interior. Moreover, E coincides with $S(\mu)$ (and thus has empty interior) if and only if $S(\mu)$ is connected (an interval). It is well-known that the Green function $g_\Omega(z,\infty)$ tends to zero, except at most on a set of capacity zero, as z goes to $S(\mu)$.

Let f be a bounded function defined on K. As usual,

$$\|f\|_K = \sup\{|f(z)| \,:\, z \in K\}.$$

The following theorem provides a stronger version of (10).

Theorem 2. *Let* $\mu \in \mathbf{Reg}$ *and* $\operatorname{cap} S(\mu) > 0$. *Then*

$$(20)\quad \limsup_{n\to\infty} \| s_{n+1}\, g_n - 1\|_K^{1/n} \le \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K \,\|\exp\{g_\Omega(\cdot,\infty)\}\|_{\mathrm{Co}(S(\mu))},$$

where K *is any compact subset of* $\overline{\mathbb{C}}\setminus E$. *Moreover, if additionally we suppose that* $S(\mu) = \operatorname{ess}[b-a, b+a]$, *then*

$$(21)\qquad \limsup_{n\to\infty} \| s_{n+1}\, g_n - 1\|_K^{1/n} \le \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K,$$

with K *any compact subset of* $\overline{\mathbb{C}}\setminus S(\mu)$.

Proof. Fix a compact set $K \subset \mathbb{C}\setminus E$. Let r be a real number, $r > \|g_\Omega(\cdot,\infty)\|_{\mathrm{Co}(S(\mu))}$, such that K lies in the unbounded component of $\mathbb{C}\setminus\gamma_r$, where $\gamma_r = \{\zeta \in \mathbb{C} : g_\Omega(\zeta,\infty) = r\}$. Obviously, γ_r surrounds $\mathrm{Co}(S(\mu))$. From (19), applied integrating over γ_r, we have

$$\| s_{n+1}\, g_n - 1\|_K \le C\, \frac{\|g_n\|_K}{\inf_{\zeta\in\gamma_r} |g_n(\zeta)|},$$

where C is a positive constant depending on the length of γ_r and the distance between γ_r and K. Therefore,

$$(22)\qquad \limsup_{n\to\infty} \| s_{n+1}\, g_n - 1\|_K^{1/n} \le \frac{\limsup_{n\to\infty} \|g_n\|_K^{1/n}}{\lim_{n\to\infty} \inf_{\zeta\in\gamma_r} |g_n(\zeta)|^{1/n}}.$$

By (16) and (5), we have

$$(23)\qquad \lim_{n\to\infty} |g_n(z)|^{1/n} = \exp\{-g_\Omega(z,\infty)\},$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathrm{Co}(S(\mu))$. From (23), it follows that

$$(24)\qquad \limsup_{n\to\infty} \|g_n\|_K^{1/n} = \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K$$

and

$$(25)\qquad \lim_{n\to\infty} \inf_{\zeta\in\gamma_r} |g_n(\zeta)|^{1/n} = \exp\{-r\}.$$

Relations (24) and (25) together with (22) give

$$\limsup_{n\to\infty} \| s_{n+1}\, g_n - 1\|_K^{1/n} \le \exp\{r\}\, \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K.$$

The left hand of this inequality does not depend on r; therefore, we can make r tend to $\|g_\Omega(\cdot,\infty)\|_{\mathrm{Co}(S(\mu))}$, obtaining (20) for compact subsets of $\mathbb{C}\setminus E$. The function under the norm sign on the left hand of (20) is analytic on $\overline{\mathbb{C}}\setminus S(\mu)$ and, in particular, on $\overline{\mathbb{C}} \setminus E$; therefore, by use of the Maximum Principle, the result is easily extended to compact subsets of $\overline{\mathbb{C}} \setminus E$.

The proof of (21) is analogous to that of (20), taking advantage of the special structure of $S(\mu)$. To avoid repetitions, we simply outline the main ingredients. We start out again with a fixed compact subset K, which is now contained in $\mathbb{C} \setminus S(\mu)$. Take $r > 0$ sufficiently small so that K lies entirely on the unbounded component of the complement of γ_r, taking care that γ_r does not intersect any of the mass points which $S(\mu)$ has outside of $[b-a, b+a]$. According to Lemma 1, we know that for all sufficiently large n, g_n has no zero on or outside γ_r. Therefore, according to Lemma 2, (19) remains valid integrating over this γ_r; and we deduce a bound analogous to (22). Now, (5) holds uniformly on compact subsets of $\mathbb{C} \setminus S(\mu)$ (see Theorem 3.1.1 and Corollary 1.1.5 in [21]) since the zeros of p_n are bounded away from K. Using this and (16), we obtain (23) on each compact subset of $\mathbb{C} \setminus S(\mu)$; we can then proceed as before, with the advantage that we can make r approach zero. With this we conclude the proof. □

With the aid of this theorem, we are able to prove our main result stated in the introduction.

Proof of Theorem 1. (a) Since

$$\| \exp\{-g_\Omega(z,\infty)\}\|_K \; \|\exp\{g_\Omega(z,\infty)\}\|_{\mathrm{Co}(S(\mu))} < 1$$

by the harmonicity of $g_\Omega(z,\infty)$ on $\overline{\mathbb{C}} \setminus E$, relation (10) follows immediately from (20). The statement concerning the zeros of $\{S_{n+1}\}_{n\in\mathbb{N}}$ is a direct consequence of (10) and Hurwitz' Theorem since the function 1 has no zeros on $\overline{\mathbb{C}} \setminus E$. Finally, (10) and (23) yield (11). The case when $S(\mu) = \mathrm{ess}[b-a, b+a]$ is proved analogously, using (21) in place of (20).

(b) First notice that it is sufficient to consider the case when $a = 1$ and $b = 0$: the general case may be reduced to this by means of an affine change of variables. Secondly, according to (21), under the present conditions we know that (10) holds uniformly on each compact subset of $\overline{\mathbb{C}} \setminus S(\mu)$. Finally, from (14) and (7), we have

$$\lim_{n\to\infty} g_n(z)\, p_n(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{dx}{(z-x)\sqrt{1-x^2}} = \frac{1}{\sqrt{z^2-1}}, \tag{26}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus S(\mu)$. Putting these things together and using (6), we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{s_{n+1}(z)}{s_n(z)} &= \lim_{n\to\infty} \frac{s_{n+1}(z)\, g_n(z)}{s_n(z)\, g_{n-1}(z)} \times \lim_{n\to\infty} \frac{p_{n-1}(z)\, g_{n-1}(z)}{p_n(z)\, g_n(z)} \times \lim_{n\to\infty} \frac{p_n(z)}{p_{n-1}(z)} \\ &= z + \sqrt{z^2-1},\end{aligned}$$

where all limits hold uniformly on each compact subset K of $\mathbb{C} \setminus S(\mu)$. With this, we have proved the first part of (12). From (26) and (10), we immediately obtain the second relation.

(c) From (10), (8) and (26), we have

$$\lim_{n\to\infty} \frac{s_{n+1}(z)}{[\Psi(z)]^n} = \lim_{n\to\infty} s_{n+1}(z)\, g_n(z) \times \lim_{n\to\infty} \frac{p_n(z)}{[\Psi(z)]^n} \times \lim_{n\to\infty} \frac{1}{g_n(z)\, p_n(z)}$$
$$= \sqrt{\frac{z^2-1}{2\pi}}\, S_\mu(\Psi(z)),$$

with uniform convergence on any compact subset K of $\mathbb{C} \setminus [-1, 1]$, which proves (13). □

Comparing these results with (5), (6) and (8), one observes that, for these important classes of measures, there are points in common between the asymptotic behaviour of Stieltjes polynomials and of orthonormal polynomials; this is specially so when $S(\mu) = \text{ess}[b-a, b+a]$. When the support of the measure already contains two whole intervals, some differences do arise, as the example in Section 6 illustrates. That example also reveals that Theorem 2 is, in some sense, sharp.

With respect to the location of the zeros, it is known (cf. [14] and [18]) that, in general, Stieltjes polynomials may have complex zeros. Despite this fact, statement (a) of Theorem 1 shows that the zeros can only accumulate on E (on $S(\mu)$ if $S(\mu) = \text{ess}[b-a, b+a]$) when $\mu \in \mathbf{Reg}$. We complement this assertion in the next result. In order to state it properly, it is necessary to give some additional definitions. It is well-known (see [20], Section 3.3) that among all probability measures σ with support in $S(\mu)$ there exists a probability measure λ (which is unique if $\operatorname{cap} S(\mu) > 0$) with support in $S(\mu)$, called the extremal or equilibrium measure of $S(\mu)$, minimizing the energy

$$\mathcal{I}(\sigma) = \int\int \log \frac{1}{|z-t|}\, d\sigma(z)\, d\sigma(t).$$

Let $P(\sigma; z) = -\int \log|z-t|\, d\sigma(t)$ be the potential of the measure σ. There exists a constant F, the equilibrium constant of $S(\mu)$, such that

$$\begin{aligned} &P(\lambda; z) \le F, \quad z \in \mathbb{C}, \\ &P(\lambda; z) = F, \quad z \in S(\mu) \setminus A \quad \text{with} \quad \operatorname{cap} A = 0. \end{aligned}$$

It can be shown that the property above characterizes the equilibrium measure and that the equilibrium constant F is precisely the minimal energy $\mathcal{I}(\lambda)$. We also recall that $\operatorname{cap} S(\mu) = \exp\{-F\}$. If $\operatorname{cap} S(\mu) > 0$, the equilibrium measure of $S(\mu)$

is closely related to the Green function relative to the region $\overline{\mathbb{C}} \setminus S(\mu)$ by means of the formula

$$g_\Omega(z,\infty) = F - P(\lambda; z), \quad z \in \mathbb{C} \setminus S(\mu). \tag{27}$$

Let ρ_n and ρ be finite Borel measures on $\overline{\mathbb{C}}$. By $\rho_n \xrightarrow{*} \rho$, $n \to \infty$, we denote the weak* convergence of ρ_n to ρ as n tends to infinity. This means that for every continuous function f on $\overline{\mathbb{C}}$,

$$\lim_{n\to\infty} \int f(x)\, d\rho_n(x) = \int f(x)\, d\rho(x).$$

For a given polynomial T, we denote by Λ_T the normalized zero counting measure of T. That is

$$\Lambda_T = \frac{1}{\deg T} \sum_{\xi:\, T(\xi)=0} \delta_\xi .$$

The sum is taken over all the zeros of T, and δ_ξ denotes the Dirac measure concentrated at ξ.

Theorem 3. *Suppose that* $S(\mu) = \mathrm{ess}[b-a, b+a], a > 0,$ *and* $\mu \in$ **Reg**. *Then*

$$\Lambda_{S_{n+1}} \xrightarrow{*} \frac{dx}{\pi\sqrt{a^2 - (x-b)^2}}, \quad n \to \infty.$$

Proof. Set $\Lambda_{S_{n+1}} \equiv \Lambda_n$. In this case, it is known (see [20], Corollary 5.2.4) that $\operatorname{cap} S(\mu) = a/2$, that $g_\Omega(z,\infty) \equiv \log|\Psi((z-b)/a)|$, and the equilibrium measure λ is $dx/(\pi\sqrt{a^2-(x-b)^2})$.

All the measures Λ_n are probability measures. Let $\Delta \subset \mathbb{N}$ be a subsequence of indices such that

$$\Lambda_n \xrightarrow{*} \Lambda, \quad n \in \Delta, \quad n \to \infty. \tag{28}$$

It is sufficient to prove that $\Lambda \equiv \lambda$ for any such sequence Δ of indices. According to Theorem 1, in this case, the support of Λ is contained in the set $S(\mu)$. Taking (11), (27) and (4) into account, we have

$$\lim_{n\in\Delta} P(\Lambda_n; z) = \lim_{n\in\Delta} \frac{-1}{n+1} \log|S_{n+1}| = P(\lambda; z), \quad z \in \mathbb{C} \setminus S(\mu).$$

On the other hand, from (28), one obtains

$$\lim_{n\in\Delta} P(\Lambda_n; z) = P(\Lambda; z), \quad z \in \mathbb{C} \setminus S(\mu).$$

Thus, $P(\Lambda; z) = P(\lambda; z)$ except at most on a set of Lebesgue measure zero in the complex plane; therefore, from Theorem 3.7.4 in [20], we obtain that $\Lambda \equiv \lambda$, as we wanted to prove. □

Using basically the same arguments, one can show that the balayage onto ∂E of any convergent subsequence of $\{\Lambda_{S_{n+1}}\}_{n\in\mathbb{N}}$ is the balayage onto ∂E of the corresponding equilibrium measure λ.

4 Padé-type approximation

The first part of Theorem 1 may be applied to estimate the rate of convergence of a certain sequence of interpolating rational functions to Markov functions when part of the poles are fixed at the zeros of the orthogonal polynomials of the given measure. Set

$$\widehat{\mu}(z) = c + \int \frac{d\mu(x)}{z-x}, \quad z \in \overline{\mathbb{C}} \setminus S(\mu), \quad c \in \mathbb{R}.$$

Let p_n be the nth orthonormal polynomial with respect to μ. It is easy to verify that there exists a unique rational function $R_n = L_n/(Q_n p_n)$, such that L_n and Q_n satisfy

- $\deg Q_n \leq n+1, \deg L_n \leq 2n+1$, and $Q_n \not\equiv 0$.
- $Q_n(z)\, p_n(z)\, \widehat{\mu}(z) - L_n(z) = \mathcal{O}(1/z^{n+2}), \quad z \to \infty\,.$

From the definition, it follows immediately, using the Cauchy and Fubini Theorems, that Q_n satisfies

$$\int x^k Q_n(x) p_n(x) d\mu(x) = 0, \qquad k = 0, 1, \ldots, n.$$

Therefore, taking Q_n to be monic, we have $Q_n = S_{n+1}$. Another immediate consequence of the definition and Cauchy's integral formula (taking into account that $Q_n = S_{n+1}$) is

$$\widehat{\mu}(z) - R_n(z) = \frac{1}{(s_{n+1}^2\, p_n)(z)} \int \frac{(s_{n+1}^2\, p_n)(x)}{z-x}\, d\mu(x), \quad z \in \overline{\mathbb{C}} \setminus S(\mu). \tag{29}$$

Using the remainder formula above and the nth root asymptotic behaviour of the polynomials p_n and s_{n+1}, we prove

Theorem 4. *Let $\mu \in$ **Reg** and $\operatorname{cap} S(\mu) > 0$. Then, on each compact subset K of $\overline{\mathbb{C}} \setminus E$, we have*

(30)

$$\limsup_{n\to\infty} \|\, \widehat{\mu}(z) - R_n(z)\|_K^{1/3n} \leq \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K \,\|\exp\{g_\Omega(\cdot,\infty)\}\|_{\mathrm{Co}(S(\mu))}.$$

If, additionally, we suppose that $S(\mu) = [b-a, b+a], a > 0$, then

$$\limsup_{n\to\infty} \|\, \widehat{\mu}(z) - R_n(z)\|_K^{1/3n} \leq \|\exp\{-g_\Omega(\cdot,\infty)\}\|_K, \tag{31}$$

where K is any compact subset of $\overline{\mathbb{C}} \setminus S(\mu)$.

Proof. Fix a compact set $K \subset \mathbb{C} \setminus E$. Let r be a real number, $r > \|g_\Omega(\cdot,\infty)\|_{\mathrm{Co}(S(\mu))}$, such that K lies in the unbounded component of $\mathbb{C} \setminus \gamma_r$, where $\gamma_r = \{\zeta \in \mathbb{C} : g_\Omega(\zeta,\infty) = r\}$. For short, let us denote $1/(s_{n+1}^2(z)\, p_n(z))$ by $h_n(z)$. From (29), we have that

$$\widehat{\mu}(z) - R_n(z) = h_n(z) \int \frac{1}{h_n(x)} \frac{d\mu(x)}{z-x}, \quad z \in K.$$

Since for each $z \in K$, $1/((z-x)\, h_n(x))$ is analytic in an open neighbourhood of the bounded component of $\mathbb{C} \setminus \gamma_r$, we may use Cauchy's integral formula to obtain

$$\widehat{\mu}(z) - R_n(z) = h_n(z) \int \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{h_n(\zeta)\,(\zeta - x)} \frac{d\zeta}{z-\zeta} \right) d\mu(x)$$

$$= \frac{h_n(z)}{2\pi i} \int_{\gamma_r} \left(\int \frac{d\mu(x)}{\zeta - x} \right) \frac{d\zeta}{h_n(\zeta)\,(z-\zeta)} = \frac{h_n(z)}{2\pi i} \int_{\gamma_r} \frac{1}{h_n(\zeta)} \frac{\widehat{\mu}(\zeta)}{z-\zeta}\, d\zeta,$$

where Fubini's Theorem has been used in the second equality. Hence

$$\| \widehat{\mu}(z) - R_n(z) \|_K \le C \, \frac{\|h_n\|_K}{\inf_{\zeta\in\gamma_r} |h_n(\zeta)|},$$

where C is a positive constant depending on the length of γ_r and the distance between γ_r and K. Therefore,

$$\limsup_{n\to\infty} \| \widehat{\mu}(z) - R_n(z) \|_K^{1/3n} \le \frac{\limsup_{n\to\infty} \|h_n\|_K^{1/3n}}{\lim_{n\to\infty} \inf_{\zeta\in\gamma_r} |h_n(\zeta)|^{1/3n}}. \tag{32}$$

From (11) and (5), we obtain

$$\lim_{n\to\infty} |h_n(z)|^{1/3n} = \exp\{-g_\Omega(z,\infty)\}, \tag{33}$$

uniformly on compact subsets of $\mathbb{C} \setminus E$. By use of (33), we obtain

(34)

$$\limsup_{n\to\infty} \|h_n\|_K^{1/3n} = \|\exp\{-g_\Omega(z,\infty)\}\|_K \quad \text{and} \quad \lim_{n\to\infty} \inf_{\zeta\in\gamma_r} |h_n(\zeta)|^{1/3n} = \exp\{-r\}.$$

Relations (34) together with (32) give

$$\limsup_{n\to\infty} \| \widehat{\mu}(z) - R_n(z) \|_K^{1/3n} \le \exp\{r\} \; \|\exp\{-g_\Omega(z,\infty)\}\|_K.$$

The left hand of this inequality does not depend on r; therefore, we can make r tend to $\|g_\Omega(\cdot,\infty)\|_{\mathrm{Co}(S(\mu))}$, obtaining (30) for compact subsets of $\mathbb{C} \setminus E$. Since the function under the norm on the left hand of (30) is analytic on a neighbourhood of infinity, from the Maximum Principle it is obvious that (30) is also true for any $K \subset \overline{\mathbb{C}} \setminus E$. Formula (31) is a direct consequence of (30) when the support is an interval. □

So far, most papers dealing with Padé-type approximants of Markov-type functions take the distinct fixed poles to have even order (cf. [2] and [3]). This is done in order to ensure that the polynomials, whose zeros are the free poles of the rational approximant, be orthogonal with respect to a positive measure. This simplifies matters considerably, as it forces the free poles to fall within the convex hull of the support of the measure. The question arises whether this restriction, due to the method used in the proofs, can be dropped or weakened. Theorem 4 is a first step in that direction.

5 Gauss-Kronrod quadrature

We first introduce an extended Gauss–Kronrod quadrature formula. Let us consider the partial fraction decomposition of the approximant R_n,

$$R_n(z) = \sum_{i=1}^{N} \sum_{j=0}^{M_i} \frac{j!\, a_{i,j,n}}{(z - z_{n,i})^{j+1}}.$$

Here N denotes the total number of distinct poles of R_n. The points $z_{n,i}$ are the zeros of $s_{n+1}\, p_n$. Though the zeros of p_n are simple they may coincide with zeros of s_{n+1}; therefore, for given $z_{n,i}$, any value of M_i is possible (of course $M_i \le n+1$). Obviously, $N = N(n)$ and $M_i = M_i(n)$; but in order to simplify the notation, we omit the explicit reference to this dependence.

Let f be an analytic function on a neighbourhood V of the compact set E. Set (35)

$$I(f) = \int f(x)\, d\mu(x), \quad I_n(f) = \sum_{i=1}^{N} \sum_{j=0}^{M_i} a_{i,j,n}\, f^{(j)}(z_{n,i}), \quad E_n(f) = I(f) - I_n(f).$$

If $\mu \in \mathbf{Reg}$ and $\operatorname{cap} S(\mu) > 0$, from (a) of Theorem 1, we know that for $n \ge n_0(V)$ all the zeros of s_{n+1} are contained in V and the expressions above make sense. In the sequel, we only consider sufficiently large n's. Notice that if the zeros $z_{n,i}$ are all simple (which is not known in general), then I_n is the usual Gauss–Kronrod quadrature formula. This fact is made explicit by the following lemma, where we study the degree of exactness of the quadrature formula just introduced in the space of polynomials.

Lemma 3. *There exists $N \in \mathbb{N}$ such that for each $n \geq N$, we have*

$$I(h) = I_n(h),$$

where h is any polynomial of degree less than or equal to $3n + 1$.

Proof. Let V be a neighbourhood of E. Let γ be an analytic Jordan curve such that V lies in the bounded component of $\mathbb{C} \setminus \gamma$. For $n \geq N$, all the zeros of S_{n+1} belong to V and, therefore, $\widehat{\mu} - R_n$ is holomorphic in $\overline{\mathbb{C}} \setminus \overline{V}$. From (29), we know that

$$\widehat{\mu}(z) - R_n(z) = \mathcal{O}(1/z^{3n+3}), \quad z \to \infty.$$

Then, if h is any polynomial of degree less than or equal to $3n + 1$, $h(\widehat{\mu} - R_n)$ has a zero at infinity of multiplicity at least two. Therefore, we can use Cauchy's Theorem, Fubini's Theorem and Cauchy's integral formula to obtain, for $n \geq N$,

$$\begin{aligned} 0 &= \int_\gamma h(\zeta)(\widehat{\mu} - R_n)(\zeta)\, d\zeta \\ &= \int_\gamma h(\zeta) \left(\int \frac{d\mu(x)}{\zeta - x} \right) d\zeta - \sum_{i=1}^{N} \sum_{j=0}^{M_i} a_{i,j,n}\, j! \int_\gamma \frac{h(\zeta)}{\zeta - z_{n,i}}\, d\zeta \\ &= \int \left(\int_\gamma \frac{h(\zeta)}{\zeta - x}\, d\zeta \right) d\mu(x) - 2\pi i \sum_{i=1}^{N} \sum_{j=0}^{M_i} a_{i,j,n} h^{(j)}(z_{n,i}) \\ &= 2\pi i \left[I(h) - I_n(h) \right]. \end{aligned}$$

□

Finally, we estimate the error of this extended Gauss–Kronrod quadrature formula for analytic functions. In the following statement, $E_n(f)$ should be understood in the sense of (35). In case that all the zeros of $S_{n+1}p_n$ are simple, it coincides with (3).

Theorem 5. *Let f be an analytic function on a neighbourhood V of E. Let $\mu \in$ **Reg** and $\operatorname{cap} S(\mu) > 0$. Then*

$$\limsup_{n\to\infty} |E_n(f)|^{1/3n} \leq \|\exp\{-g_\Omega(\cdot,\infty)\}\|_{\partial V} \|\exp\{g_\Omega(\cdot,\infty)\}\|_{\mathrm{Co}(S(\mu))}, \tag{36}$$

where ∂V is the set of boundary points of V. If $S(\mu) = [b - a, b + a]$, then

$$\limsup_{n\to\infty} |E_n(f)|^{1/3n} \leq \|\exp\{-g_\Omega(\cdot,\infty)\}\|_{\partial V} \tag{37}$$

for any function f analytic on a neighbourhood V of $[b - a, b + a]$.

Proof. Let W be a neighbourhood of E with $\overline{W} \subset V$. There exists a natural number $n_0(W)$ such that for each $n \in \mathbb{N}$ with $n \geq n_0(W)$, the polynomial S_{n+1} has all its zeros contained in the open set W.

Let γ be an analytic Jordan curve contained in V such that W lies in the bounded component of $\mathbb{C} \setminus \gamma$. Using the Fubini and Cauchy Theorems, we have

$$I(f) - I_n(f) = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\widehat{\mu} - R_n)(\zeta)\, d\zeta.$$

From this equality and (30), we obtain

$$\limsup_{n\to\infty} |I(f) - I_n(f)|^{1/3n} \leq \| \exp\{-g_\Omega(\cdot,\infty)\}\|_\gamma \| \exp\{g_\Omega(\cdot,\infty)\}\|_{\mathrm{Co}(S(\mu))}.$$

We can choose γ as close to ∂V as we want, so (36) immediately follows. Obviously, (37) is a direct consequence of (36). □

Notice that, under the conditions of Theorem 5, we have that $\lim_{n\to\infty} E_n(f) = 0$ with geometric rate of order $3n$. The closer ∂V is to E, the slower $E_n(f)$ tends to 0. We wish to point out that Theorem 5 ensures the convergence of the Gauss–Kronrod quadrature formula for analytic functions regardless of the signs which the coefficients $a_{i,j,n}$ may have. This approach allows us to obtain estimates for the rate of convergence of Gauss–Kronrod quadrature formulas for a very general class of measures as compared with the measures considered in [6] and [15]. Also, the order of convergence which we give is better than that which follows from Theorem 1 in [6]. As regards [15], it is more difficult to compare the order of convergence because of the different nature of the estimates.

Finally, we remark that (31) and (37) can also be proved imposing on the support of the measure the weaker condition $S(\mu) = \mathrm{ess}[b-a, b+a]$.

6 Example

The next example illustrates the nature of the difficulties one encounters in trying to improve the results when $S(\mu)$ contains more than one interval. In fact, it shows that, in general, in the class of regular measures one cannot obtain asymptotics on a set larger than $\mathbb{C} \setminus E$, or get estimates of the rate of convergence better than that expressed on the right hand of (20).

Recall that $p_n\, g_n$ has at most one simple zero in each of the open intervals which give the connected components of $\mathbb{R} \setminus S(\mu)$ (see (17)).

Set $d\mu(x) = w(x)\,dx$, where $w(x)$ is an even function defined on $[-\beta, -\alpha] \cup [\alpha, \beta]$, $\beta > \alpha > 0$. This measure is symmetric with respect to the origin. Therefore, p_{2n+1} is an odd function, and it must have a zero at $z = 0$;

thus, according to what was said above, g_{2n+1} does not vanish in $(-\alpha,\alpha)$. On the other hand, p_{2n} is even; thus p_{2n}/x is odd, from which it follows that $g_{2n}(0) = -\int p_{2n}(x)/x\, d\mu(x) = 0$.

Let $r > 0$ and set $\gamma_r = \{\zeta \in \mathbb{C} : g_\Omega(z,\infty) = r\}$. Since g_{2n+1}^{-1} is analytic in $\mathbb{C}\setminus S(\mu)$, we can prove (19) with $\gamma = \gamma_r$ reasoning as in Lemma 2. With this formula on γ_r, following the same proof as in Theorem 2, it is easy to obtain that

$$\limsup_{n\to\infty} \|s_{2n+2}\, g_{2n+1} - 1\|_K^{1/2n} \le \|\exp\{-g_\Omega(z,\infty)\}\|_K,$$

on each compact subset $K \subset \overline{\mathbb{C}}\setminus S(\mu)$.

From symmetry, it is not difficult to see that $g_\Omega(0,\infty) = \max_{\zeta\in[-\beta,\beta]} g_\Omega(\zeta,\infty)$. Take $a \in \mathbb{C}\setminus S(\mu)$, with $g_\Omega(a,\infty) < g_\Omega(0,\infty)$. Let $0 < r < g_\Omega(a,\infty)$ and let γ be a positively oriented circle centered at $z = 0$ such that a, γ_r and $S(\mu)$ lie in the unbounded connected component of the complement of γ. Since g_{2n}^{-1} has a simple pole at $z = 0$, following the arguments used in proving (19) and using the Residue Theorem, one has

$$(38)\quad \begin{aligned} s_{2n+1}(a)\, g_{2n}(a) &= 1 + \frac{g_{2n}(a)}{2\pi i}\int_{\gamma_r}\frac{d\zeta}{g_{2n}(\zeta)\,(\zeta-a)} + \frac{g_{2n}(a)}{2\pi i}\int_{\gamma}\frac{d\zeta}{g_{2n}(\zeta)\,(\zeta-a)} \\ &= 1 + \frac{g_{2n}(a)}{2\pi i}\int_{\gamma_r}\frac{d\zeta}{g_{2n}(\zeta)\,(\zeta-a)} + \frac{-g_{2n}(a)}{a\, g_{2n}'(0)}. \end{aligned}$$

For the integral on the right-hand side, it is easy to deduce that

$$(39)\quad \limsup_{n\to\infty}\left|\frac{g_{2n}(a)}{2\pi i}\int_{\gamma_r}\frac{d\zeta}{g_{2n}(\zeta)\,(\zeta-a)}\right|^{1/2n} \le \exp\{r - g_\Omega(a,\infty)\} < 1.$$

For the third term in (38) (see (17) and take into account that $g_{2n}(0) = 0$), we have

$$\frac{-g_{2n}(a)}{a\, g_{2n}'(0)} = \frac{g_{2n}(a)\, p_{2n}(0)}{a\int p_{2n}^2(x)\, x^{-2}\, d\mu(x)}.$$

Since

$$\frac{1}{\beta^2} \le \left|\int \frac{p_{2n}^2(x)}{x^2}\, d\mu(x)\right| \le \frac{1}{\alpha^2}$$

and $\lim_{n\to\infty} |p_{2n}(z)|^{1/2n} = \exp\{g_\Omega(z,\infty)\}$ uniformly on compact subsets of $\mathbb{C}\setminus S(\mu)$ (because p_{2n} has no zeros in $(-\alpha,\alpha)$), it follows that

$$(40)\quad \lim_{n\to\infty}\left|\frac{g_{2n}(a)}{a\, g_{2n}'(0)}\right|^{1/2n} = \exp\{g_\Omega(0,\infty) - g_\Omega(a,\infty)\} > 1.$$

Therefore, taking account of (38), (39), and (40), we obtain

$$\lim_{n\to\infty} |s_{2n+1}(a)\, g_{2n}(a) - 1|^{1/2n} = \exp\{g_\Omega(0,\infty) - g_\Omega(a,\infty)\}.$$

Hence, at the point a, $s_{2n+1}(a)\, g_{2n}(a)$ does not converge to 1. Moreover, it diverges with geometric rate.

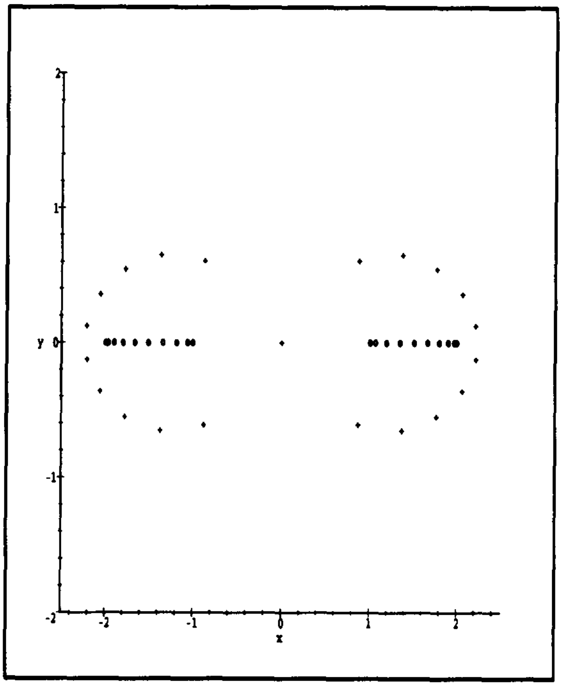

Figure 1. Zeros of S_{20} and S_{21} for $w \equiv 1$, $\alpha = 1$, $\beta = 2$.

We have considered the particular case $w \equiv 1$, $\alpha = 1$, $\beta = 2$. Numerical experiments show that the zeros of the Stieltjes polynomials for this measure have an interesting behaviour; while the zeros of S_{2n} sit on $[-2,-1] \cup [1,2]$, those of S_{2n+1} draw the level curve $\{\zeta \in \mathbb{C} : g_\Omega(\zeta,\infty) = g_\Omega(0,\infty)\}$. Figure 1 shows the zeros of S_n for $n = 20, 21$ (the small circles are the zeros of S_{20} and the crosses the zeros of S_{21}). As this example shows, the only drawback in extending the results of this paper to compact subsets closer to $S(\mu)$ is the existence of zeros of g_n in $\mathrm{Co}(S(\mu)) \setminus S(\mu)$. If we know for some reason that the functions of second kind (or some subsequence) have no zeros on $\mathrm{Co}(S(\mu)) \setminus S(\mu)$, then we can extend (20) to any compact subset of $\overline{\mathbb{C}} \setminus S(\mu)$ (for the corresponding subsequence).

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