Strong Asymptotic Behavior and Weak Convergence of Polynomials Orthogonal on an Arc of the Unit Circle

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M. Bello Hernández
Dpto. de Matemáticas y Computación
Universidad de la Rioja
Edificio J. L. Vives
Luis de Ulloa s/n 26004 Logroño
Spain
e-mail: mbello@dmc.unirioja.es

E. Miña Díaz
Instituto Superior de Ciencias y
Tecnologías Nucleares
Ave. Salvador Allende y Luaces
P. O. Box 6163, Havana 10600
Cuba
e-mail: erwin@rsrch.isctn.edu.cu

Asymptotics of polynomials orthogonal on an arc

Galley proofs should be sent to

M. Bello Hernández
Dpto. de Matemáticas y Computación
Universidad de la Rioja
Edificio J. L. Vives
Luis de Ulloa s/n 26004 Logroño
Spain
e-mail: mbello@dmc.unirioja.es

Abstract

Let σ be a finite positive Borel measure supported on an arc γ of the unit circle, such that $\sigma' > 0$ a.e. on γ. We obtain a theorem about the weak convergence of the corresponding sequence of orthonormal polynomials. Moreover, when σ satisfies Szegö's condition on the arc, we prove an analogue of Szegö's classical theorem on strong asymptotics of the orthogonal polynomials on the complement of γ, which completes to its full extent a result of N. I. Akhiezer. The key tool in the proofs is the use of orthogonality with respect to varying measures.

1 Introduction

The asymptotic properties of polynomials which are orthogonal with respect to varying measures have had important applications in different problems of approximation theory. Perhaps the most attractive applications are those which involve the solution of problems where orthogonality is considered in the usual sense, that is, with respect to a fixed measure. One such application can be found in a recent paper by M. Bello and G. López [3]. Translating the problems to varying measures, some results were obtained on ratio and relative asymptotics of orthogonal polynomials with respect to a fixed measure supported on a circular arc; these are similar to previous ones from the work of E. A. Rakhmanov and A. Maté, P. Nevai, and V. Totik relating to measures supported on the whole unit circle.

This paper can be considered as a continuation of [3] (see Remark 5 in this reference). Following the same techniques used therein, we obtain new asymptotic properties of sequences of orthogonal polynomials on an arc of the unit circle. First, in Section 3 we prove a theorem about weak convergence of such sequences assuming that the absolutely continuous component of the measure is positive almost everywhere. This theorem has analogous versions when the support of the measure is the whole unit circle or a real segment, both due to Maté, Nevai, and Totik (see Corollary 5.1 and Theorem 11.1 of [14]). Next, in Section 4, an analogue of Szegö's classical theorem for the unit circle is given (Theorem 12.1.1 in [17]). This result is complemented with some assertions similar to those in Chap. 1, paragraph 15 of [6]. For a circular arc, the form of the strong asymptotics was given in the Sixties by N. I. Akhiezer in his short note [1] (a rigorous and detailed discussion of this paper was given recently by Golinskii in [8]). For general Jordan arcs, the analogous results can be seen in the paper by V. A. Kaliaguine [10]. However, these papers only consider a particular class of measures, whereas we obtain a full extension of Szegö's theorem.

To explain in a more comprehensive manner the essentially new results of this present work, let us introduce some notations. Let E be a Borel subset of the complex plane $\mathbb{C}$. By$\mathcal{M}_E$, we denote the set of all finite positive Borel measures with infinite support on E. If E is a compact set and $\eta \in \mathcal{M}_E$, then

$$\int_E |\zeta|^n \, d\eta(\zeta) < +\infty, \qquad \zeta \in \mathbb{C}, \quad n = 0, 1, \ldots,$$

and we can construct a unique sequence $\{\varphi_n(\eta, \zeta)\}_{n=0}^{\infty}$ of orthonormal polynomials on E, defined by

$$\int_E \varphi_n(\zeta) \overline{\varphi_m(\zeta)} \, d\eta(\zeta) = \delta_{n,m}, \qquad n, m \geq 0, \tag{1}$$

where

$$\varphi_n(\zeta) = \varphi_n(\eta, \zeta) = \alpha_n \zeta^n + \cdots, \qquad \alpha_n = \alpha_n(\eta) > 0.$$

Let $\gamma = \left\{z = e^{i\vartheta} : \vartheta_1 \le \vartheta \le \vartheta_2,\ 0 \le \vartheta_2 - \vartheta_1 \le 2\pi\right\}$ be an arc of the unit circle Γ. When $\eta \in \mathcal{M}_\gamma$, condition (1) is equivalent to

$$\int_{\vartheta_1}^{\vartheta_2} \varphi_n\left(e^{i\vartheta}\right)\overline{\varphi_m\left(e^{i\vartheta}\right)}\,d\sigma(\vartheta) = \delta_{n,m}, \qquad \sigma \in \mathcal{M}_{[\vartheta_1,\vartheta_2]},$$

where $d\sigma(\vartheta) \overset{def}{=} d\eta\left(\zeta\right), \quad \zeta = e^{i\vartheta}, \quad \vartheta \in [\vartheta_1, \vartheta_2]$.

In order to avoid unnecessary complications in the succeeding discussion, we will restrict our attention to an arc γ symmetric with respect to $\mathbb{R}$ and such that $1 \notin \gamma$. Let

$$\gamma = \left\{\zeta = e^{i\vartheta} : \vartheta_1 \le \vartheta \le 2\pi - \vartheta_1,\ 0 < \vartheta_1 < \pi\right\} \tag{2}$$

be a symmetric arc and $G_\gamma(\zeta)$ the conformal mapping of $\overline{\mathbb{C}}\backslash\gamma$ onto $\overline{\mathbb{C}}\backslash\left\{|\xi| \le 1\right\}$ such that $G_\gamma(\infty) = \infty$ and $G'_\gamma(\infty) > 0$. The logarithmic capacity of γ is $C(\gamma) = \cos\frac{\vartheta_1}{2}$.

In [10], V. A. Kaliaguine obtained the strong asymptotics of orthogonal polynomials with respect to a measure of the type $\alpha+\beta$, where α is concentrated on a complex rectifiable arc E and is absolutely continuous with respect to the arc length measure on the arc, and β is a discrete measure with finite masses outside E. This result for the particular case when $E = \gamma$ and $\beta \equiv 0$ can be formulated in a more precise manner as follows:

Theorem 1. *Suppose that $\sigma \in \mathcal{M}_\gamma$, $d\sigma\left(\vartheta\right) = \sigma'\left(\vartheta\right)d\vartheta$, and the following Szegö type condition is satisfied*

$$\int_{\vartheta_1}^{2\pi-\vartheta_1} \log\left[\sigma'\left(\vartheta\right)\right] \frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\vartheta_1/2) - \cos^2(\vartheta/2)}}\,d\vartheta > -\infty, \tag{3}$$

then

$$\frac{\varphi_n(\zeta)}{G^n_\gamma(\zeta)} \underset{n}{\rightrightarrows} \frac{C(\gamma)G_\gamma\left(\zeta\right) - 1 - \sin(\vartheta_1/2)}{\sqrt{2\pi\left(1 + \sin(\vartheta_1/2)\right)}\left(\zeta - 1\right)}\,D_\gamma(t,\zeta), \qquad \zeta \in \overline{\mathbb{C}}\backslash\gamma, \tag{4}$$

where $D_\gamma(t,\zeta)$ denotes the Szegö function for the arc γ associated with

$$t(\vartheta) = \sigma'\left(\vartheta\right)\frac{\sqrt{\cos^2(\vartheta_1/2) - \cos^2(\vartheta/2)}}{\sin(\vartheta/2)}$$

(for definition, see Section 2.2).

Here, and in the following discussion, the notation $f_n(\zeta) \underset{n}{\rightrightarrows} f(\zeta)$, $\zeta \in \Omega$, stands for the uniform convergence of the sequence of functions $\{f_n\}$ to the function f on each compact subset of Ω.

This result is similar to Szegö's classical theorem for the unit circle [[17], Theorem 12.1.1], and extends the former version of this type due to N. I. Akhiezer. Formula (4) was first announced in his short note [1] as early as 1960, but for a very limited class of measures. A rigorous and detailed exposition of Akhiezer's note was published recently by L. Golinskii in [8].

Our contribution to this problem consists in showing that (4) is in fact valid for all $\sigma \in \mathcal{M}_\gamma$ not necessarily absolutely continuous and satisfying (3). Furthermore, we show that (3) is also a necessary condition in order that the sequence $\left\{\varphi_n / G_\gamma^n\right\}$ be bounded in at least one point of $\overline{\mathbb{C}} \backslash \gamma$.

It may be worth noting that our results are obtained by a method quite different from those followed by Akhiezer and Kaliaguine.

2 Auxiliary results

Before we can prove the theorems in the following sections, we need to establish several auxiliary results and notations.

1 Suppose that $\sigma \in \mathcal{M}_\Gamma$, $\Gamma = \{|\zeta| = 1\}$, and let $\{W_n\}_{n=0}^{\infty}$ be a sequence of polynomials such that, for each $n \geq 0$, $W_n(\zeta) = c_n \zeta^n + \cdots$, $c_n > 0$, and all its zeros $(w_{n,i})$, $1 \leq i \leq n$, lie in $\{|\zeta| \leq 1\}$. Let us set

$$d\sigma_n(\theta) = \frac{d\sigma(\theta)}{|W_n(\zeta)|^2}, \qquad n \geq 0, \quad \zeta = e^{i\theta}.$$

The following definition was introduced by G. López (see [12]).

Definition 1. *Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\sigma, \{W_n\}, k)$ is weakly admissible on Γ if:*

(i) $\int_0^{2\pi} d\sigma_n(\theta) < +\infty, \qquad n \geq 0;$

(ii) *In the case that $k < 0$,*

$$\int_0^{2\pi} \prod_{i=1}^{-k} |\zeta - w_{n,i}|^{-2}\, d\sigma(\theta) \leq M < +\infty, \quad n \geq -k,\ \zeta = e^{i\theta};$$

(iii) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty.$

Condition (**i**) guarantees that we can construct a table of polynomials $\{\varphi_{n,m}\}$, $m, n \geq 0$, such that for each fixed $n \geq 0$, the system $\{\varphi_{n,m}\}_{m=0}^{\infty}$, is orthonormal with respect to $d\sigma_n$. In other words, for each $n \geq 0$

$$\int_0^{2\pi} \varphi_{n,m}(\zeta)\overline{\varphi_{n,k}(\zeta)}\, d\sigma_n(\theta) = \delta_{m,k}, \qquad k,\, m \geq 0,\ \zeta = e^{i\theta}, \tag{5}$$

where

$$\varphi_{n,m}(\zeta) = \varphi_{n,m}(\sigma_n, \zeta) = \alpha_{n,m}\zeta^m + \cdots, \quad \alpha_{n,m} > 0.$$

The following result complements the main statement of G. López's extension of Szegö Theorem for orthogonal polynomials with respect to varying measures on the unit circle (see [13]).

Theorem 2. *Let $(\sigma, \{W_n\}, k)$ be weakly admissible on Γ. The following statements are then all equivalent:*

(a) $\log \sigma' \in L^1_\Gamma$, *that is,*

$$\int_0^{2\pi} \log[\sigma'(\theta)]\, d\theta > -\infty;$$

(b)

$$\frac{\varphi_{n,n+k}(\zeta)}{\zeta^k W_n(\zeta)} \underset{n}{\rightrightarrows} \frac{1}{\sqrt{2\pi}} D(\sigma', \zeta), \qquad \zeta \in \overline{\mathbb{C}} \setminus [|\zeta| \le 1],$$

where

$$D(\sigma', \zeta) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log[\sigma'(\theta)] \frac{z+\zeta}{z-\zeta}\, d\theta\right\}, \quad \zeta \in \overline{\mathbb{C}} \setminus \{|\zeta| \le 1\},\ z = e^{i\theta},$$

is the Szegö function for σ';

(c) *the sequence $\{\alpha_{n,n+k}/c_n\}$ converges to a finite number;*

(d) *there exists a subsequence $\left\{\varphi_{n,n+k}(\zeta)/(\zeta^k W_n(\zeta))\right\}$, $n \in \Lambda \subset \mathbb{N}$, bounded in at least one point of the region $\overline{\mathbb{C}} \setminus [|\zeta| \le 1]$.*

Proof. The proof of (**a**)⇒(**b**) is the contents of [13]. Assertions (**b**)⇒ (**c**)⇒(**d**) are trivial when $\zeta = \infty$. We now need to prove only that (**d**)⇒ (**a**). If P_n is a polynomial of degree equal to n, as usual we denote $P_n^*(\zeta) = \zeta^n \overline{P_n(1/\overline{\zeta})}$. Let us consider the subsequence of statement (**d**). In [13] it was proved that the sequence $\{W_n^*/\varphi_{n,n+k}^*\}$, $n \in \Lambda \subset \mathbb{N}$, is uniformly bounded on each compact subset of $\{|\zeta| < 1\}$ (more precisely, the entire sequence); therefore, by Montel's theorem, there is a subsequence $\{W_n^*/\varphi_{n,n+k}^*\}$, $n \in \Upsilon \subset \Lambda$, which is uniformly convergent to an analytic function S_Υ on each compact subset of the unit disk.

Since $W_n^*/\varphi_{n,n+k}^*$ is never zero in $\{|z| < 1\}$, one concludes from Hurwitz's theorem that either $S_\Upsilon \equiv 0$ or $S_\Upsilon \ne 0$ on $\{|\zeta| < 1\}$. But according to our assumption, there is a point $\zeta_0 \in \{|\zeta| < 1\}$ for which

$$\lim_{\substack{n\to\infty \\ n\in\Upsilon}} \frac{W_n^*(\zeta_0)}{\varphi_{n,n+k}^*(\zeta_0)} \ne 0.$$

Therefore, $S_\Upsilon(\zeta) \ne 0$. In [13] it was also proved that $S_\Upsilon \in H^2(|\zeta| < 1)$ and $|S_\Upsilon(e^{i\vartheta})|^2 \le \sigma'(\vartheta)$ almost everywhere on $[0, 2\pi]$. From this, it follows (see theorem 17.17 of [15]) that

$$-\infty < \int_0^{2\pi} \log\left|S_\Upsilon\left(e^{i\theta}\right)\right|^2 d\theta \le \int_0^{2\pi} \log[\sigma'(\theta)]\, d\theta,$$

which is just what we needed to obtain. ∎

Remark 1. Next, we would like to make several comments:

- If $|w_{n,i}| \leq r < 1$, $n \in \mathbb{N}$, $1 \leq i \leq n$, then $(\sigma, \{W_n\}, k)$ is always weakly admissible for all finite and positive Borel measure and all $k \in \mathbb{Z}$. This is the case we will have to consider.
- Theorem 2 can be expressed in terms of subsequences $\{\varphi_{n,n+k}/(\zeta^k W_n)\}$, $n \in \Lambda \subset \mathbb{N}$, for which $(\sigma, \{W_n\}, k)$, $n \in \Lambda$, is weakly admissible on Γ. In this case, condition **(iii)** must be changed to: $\lim_{n\in\Lambda} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty$. We will also need this.
- Observe that if we set $W_n(\zeta) = \zeta^n$, then we obtain the results corresponding to a fixed measure.

Corollary 1. *Let $(\sigma, \{W_n\}, k)$ be weakly admissible on Γ such that σ satisfies the Szegö condition*

$$\int_0^{2\pi} \log[\sigma'(\theta)]\, d\theta > -\infty,$$

then

$$\lim_{n\to\infty} \int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{z^k W_n(z)} - 1 \right|^2 d\theta = 0, \quad z = e^{i\theta} \tag{6}$$

and

$$\lim_{n\to\infty} \int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi^*_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{W^*_n(z)} - 1 \right|^2 d\theta = 0, \quad z = e^{i\theta}, \tag{7}$$

where

$$D\left((\sigma')^{-1}, z\right) = \exp\left\{ \frac{-1}{4\pi} \int_0^{2\pi} \log[\sigma'(\theta)] \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta \right\},$$

is the Szegö function for $(\sigma')^{-1}$, and

$$D((\sigma')^{-1}, z_+) = \lim_{r\to 1+} D((\sigma')^{-1}, r\, z).$$

Proof. On the one hand, Theorem 2 (**(a)**$\Rightarrow$**(b)**) give us

$$8\pi \leq \liminf_{n\to\infty} \int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{z^k W_n(z)} + 1 \right|^2 d\theta, \quad z = e^{i\theta}, \tag{8}$$

since $D((\sigma')^{-1}, .) \in H_2(\overline{\mathbb{C}} \backslash \{|z| \leq 1\})$ and $\left|D((\sigma')^{-1}, z_+)\right|^2 = \sigma'(\theta)$, $z = e^{i\theta}$.

On the other hand, using parallelogram law, $\left|D((\sigma')^{-1}, z_+)\right|^2 = \sigma'(\theta)$, $z =$

$e^{i\theta}$, and orthonormality property of $\varphi_{n,n+m}$ we have

$$\int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{z^k W_n(z)} - 1 \right|^2 d\theta$$

$$+ \int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{z^k W_n(z)} + 1 \right|^2 d\theta$$

$$= 2 \left[\int_0^{2\pi} \left| \frac{\sqrt{2\pi}\varphi_{n,n+k}(z) D((\sigma')^{-1}, z_+)}{z^k W_n(z)} \right|^2 d\theta + \int_0^{2\pi} d\theta \right] \leq 8\pi.$$

Thus, these inequalities prove us (6). Lastly, it is obviuos that (6) is equivalent to (7). ■

2 Let $\varphi(\tau) = \tau + \sqrt{\tau^2 - 1}$ (the root is taken so that $|\varphi(\tau)| > 1$) be the conformal mapping of $\overline{\mathbb{C}} \setminus [-1, 1]$ onto $\overline{\mathbb{C}} \setminus \{|\xi| \leq 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. Let us also consider the automorphisms of $\overline{\mathbb{C}} : \zeta = (\tau + i)/(\tau - i)$ and its inverse $\tau = i(\zeta + 1)/(\zeta - 1)$. The latter takes the unit circle onto the extended real axis $\overline{\mathbb{R}}$.

Let

$$\nu = \nu(\zeta) = \varphi\left(\frac{i}{c}\frac{\zeta + 1}{\zeta - 1}\right), \quad c = \cot\frac{\vartheta_1}{2},$$

be the conformal mapping from $\overline{\mathbb{C}} \setminus \gamma$ onto $\overline{\mathbb{C}} \setminus \{|\xi| \leq 1\}$ associated with $\varphi(.)$. Let h be a weight on γ satisfying the Szegö condition

$$\int_{\vartheta_1}^{2\pi - \vartheta_1} \log[h(\vartheta)] \frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\vartheta_1/2) - \cos^2(\vartheta/2)}} d\vartheta > -\infty, \tag{9}$$

Szegö's function, $D_\gamma(h, \zeta)$, associated with the domain $\overline{\mathbb{C}} \setminus \gamma$ and weight h is defined by the following identity

$$D_\gamma(h, \zeta) = \frac{D(h, \nu(\zeta)) |D(h, \varphi(i/c))|}{D(h, \varphi(i/c))}, \qquad \zeta \in \overline{\mathbb{C}} \setminus \gamma; \tag{10}$$

where

$$D(h, z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \log[h(\vartheta)] \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right\}, \quad (\vartheta = 2\,\mathrm{arccot}\,(c\cos\theta)),$$

is the Szegö function for the unit circle and weight $h(2\mathrm{arccot}\,(c\cos\theta))$, $\theta \in [0, 2\pi]$, and $z \in \overline{\mathbb{C}} \setminus \{|z| \leq 1\}$.

Taking into account the properties of the Szegö function for the unit disk (see [[5], Chap. 5] and [[17], Chap. 10]) it is not hard to prove that $D_\gamma(h, \zeta)$ satisfies the following properties:

1. $D_\gamma(h,\zeta) \in H^2(\overline{\mathbb{C}}\backslash\gamma)$ and therefore

$$\lim_{r\to 1+} D_\gamma(h, r\zeta) = D_\gamma(h,\zeta_+) \quad \text{and} \quad \lim_{r\to 1-} D_\gamma(h, r\zeta) = D_\gamma(h,\zeta_-)$$

exist for almost every $\zeta \in \gamma$;

2. $D_\gamma(h,\zeta) \neq 0$ for all $\zeta \in \overline{\mathbb{C}}\backslash\gamma$ and $D_\gamma(h,\infty) > 0$;

3. $|D_\gamma(h,\zeta_+)|^2 = |D_\gamma(h,\zeta_-)|^2 = h^{-1}(\zeta)$ almost everywhere on γ.

4. If h_1 and h_2 are weight functions on γ satisfying (9), then the following multiplicative property holds

$$D_\gamma(h_1 h_2,\zeta) = D_\gamma(h_1,\zeta) D_\gamma(h_2,\zeta).$$

3 The automorphism $\tau = i(\zeta+1)/(\zeta-1)$ takes this arc onto the segment $[-c,c]$. We write $\zeta = z$ when $|\zeta| = 1$, and $\tau = t$ when $\tau \in \mathbb{R}$. Let us introduce the following notations:

If $\sigma \in \mathcal{M}_\gamma$, we put

$$d\mu(t) = d\sigma\left(\frac{t+i}{t-i}\right) \text{ and } d\mu_n(t) = \frac{d\mu(t)}{(1+t^2)^n}, \quad n \in \mathbb{N}, \quad t \in [-c,c].$$

We denote by $l_{n,m}(\tau)$ the m-th orthonormal polynomial with positive leading coefficient $k_{n,m}$ relative to $d\mu_n$.

As before, $\varphi_n(\zeta) = \varphi_n(\sigma,\zeta)$ denotes the n-th orthonormal polynomial with respect to $d\sigma$ on γ and α_n is its leading coefficient.

The next lemma is a reformulation of relations (11) and (12) of Lemma 2 in [3].

Lemma 1. *With the notations above, we have:*

$$\begin{aligned}\mathcal{L}_n(\tau) &\overset{def}{=} (\tau-i)^n \varphi_n\left(\frac{\tau+i}{\tau-i}\right) \\ &= \frac{\varphi_n(1)}{k_{n,n+1}}\left[\frac{l_{n,n+1}(\tau) - (l_{n,n+1}(-i)\, l_{n,n}(\tau)/l_{n,n}(-i))}{\tau+i}\right],\end{aligned} \tag{11}$$

and

$$\frac{\varphi_n(1)}{i^n k_{n,n}} = \frac{l_{n,n}(-i)}{\alpha_n 2^n}.$$

Using these expressions, Lemma 3 of [3] may be rewritten as follows

Lemma 2. *Assume that $d\sigma \in \mathcal{M}_\gamma$ and $\sigma' > 0$ almost everywhere on γ. Then*

(a)

$$\frac{\mathcal{L}_n(\tau)\, l_{n,n}(i)}{i^n l_{n,n}(\tau)\, |l_{n,n}(i)|} \underset{n}{\rightrightarrows} \sqrt{\frac{c}{2\,|\varphi(i/c)|}}\,\frac{\varphi(\tau/c) - \varphi(-i/c)}{\tau+i}, \quad \tau \in \overline{\mathbb{C}}\backslash[-c,c],$$

and

(b)

$$\lim_{n\to\infty}\left|\frac{\varphi_n(1)}{k_{n,n}}\right| = \sqrt{\frac{2}{c\,|\varphi(i/c)|}}.$$

Lemma 3. *We have*

$$\left|(z-1)^n\, l_{n,n}\left(i\frac{z+1}{z-1}\right)\right| \le 2^n\sqrt{\sum_{j=0}^{n}|\varphi_j(z)|^2}. \tag{12}$$

Proof. This fact was essentially proved in [9]. Here, we will limit ourselves to making brief comments to facilitate the reader's understanding. The new objective is to find a suitable expression for $l_{n,n}$ starting from the polynomials $\{\varphi_n\}$. That is to say, we will try to find an inverse formula for (11). Now, carrying $d\mu_n$ over γ and following similar steps to those used to derive Formula (8) in Lemma 1 of [3], it follows that for $v = 0, 2, \ldots, n-1$,

$$\int_\gamma \overline{z}^v\,(z-1)^n\,\mathcal{H}_{n,n}\left(i\frac{z+1}{z-1}\right)\frac{d\mu_n\left(i(z+1)/(z-1)\right)}{|z-1|^{2n}} = 0, \tag{13}$$

$\mathcal{H}_{n,n}(\zeta) = l_{n,n}(\zeta)/l_{n,n}(i)$. This is Formula (1) in Section 4 of [12], applied to our case. Since

$$d\mu_n\left(i\frac{z+1}{z-1}\right) = \frac{d\mu\left(i(z+1)/(z-1)\right)}{\left|1+\left(i(z+1)/(z-1)\right)^2\right|^n} = \frac{|z-1|^{2n}\,d\sigma(z)}{4^n}, \quad z\in\gamma,$$

and $2^n\varphi_m(z)$ is the m-th orthonormal polynomial with respect to $d\sigma/4^n$, we can develop Formula (13) as was done in [9, Section 4, see Lemma 9] to obtain

$$(z-1)^n\, l_{n,n}\left(i\frac{z+1}{z-1}\right) = \frac{(2i)^n K_n(z,1)}{\sqrt{K_n(1,1)}},$$

where (see e.g. [4, Section 1.1-4)])

$$K_n(z,1) = \frac{\varphi_n^*(z)\overline{\varphi_n^*(1)} - z\varphi_n(z)\overline{\varphi_n(1)}}{1-z} = \sum_{j=0}^{n}\varphi_j(z)\overline{\varphi_j(1)}.$$

Therefore,

$$\left|(z-1)^n\, l_{n,n}\left(i\frac{z+1}{z-1}\right)\right| = \frac{2^n}{\sqrt{\sum_{j=0}^{n}|\varphi_j(1)|^2}}\left|\sum_{j=0}^{n}\overline{\varphi_j(1)}\varphi_j(z)\right|, \tag{14}$$

and (12) follows immediately from (14) by using the Cauchy-Schwartz inequality. ■

4 Now we define a measure $d\tilde{\sigma} \in \mathcal{M}_\Gamma$ through the equality

$$\tilde{\sigma}(\theta) = \begin{cases} \mu(-c) - \mu(c\cos\theta), & 0 \le \theta \le \pi \\ \mu(c\cos\theta) - \mu(-c), & \pi \le \theta \le 2\pi, \end{cases} .$$

associated with $d\mu(t) = d\sigma\left(\frac{t+i}{t-i}\right)$, and $d\sigma \in M_\gamma$.

Lemma 4. *Let $d\sigma$ and $d\tilde{\sigma}$ be as above, then*

$$\log \tilde{\sigma}' \in L^1_\Gamma \Leftrightarrow \log \sigma' \in L^1_{d\Theta_\gamma} \Leftrightarrow \int_{\vartheta_1}^{2\pi-\vartheta_1} \log\left[\sigma'(\vartheta)\right] d\Theta_\gamma(\vartheta) > -\infty,$$

where

$$d\Theta_\gamma(\vartheta) = \frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\vartheta_1/2) - \cos^2(\vartheta/2)}}\, d\vartheta, \qquad \vartheta \in [\vartheta_1, 2\pi - \vartheta_1]. \tag{15}$$

Proof. The measure $d\tilde{\sigma}(\theta)$ is symmetric with respect to π on the segment $[0, 2\pi]$, and therefore its derivative $\tilde{\sigma}'(\theta)$ is also. Let us consider the distribution function $\tilde{\sigma}(\theta)$. If $\theta \in (0, \pi)$, then

$$\begin{aligned} \tilde{\sigma}(\theta) - \tilde{\sigma}(0+) &= d\tilde{\sigma}\{(0,\theta]\} = d\mu\{[c\cos\theta, c)\} = d\sigma\{(\vartheta_1, 2\mathrm{arccot}(c\cos\theta)]\} \\ &= \sigma(2\mathrm{arccot}(c\cos\theta)) - \sigma(\vartheta_1+). \end{aligned}$$

Thus,

$$\tilde{\sigma}'(\theta) = \sigma'(2\mathrm{arccot}(c\cos\theta)) \frac{2c|\sin\theta|}{1 + c^2\cos^2\theta}, \tag{16}$$

almost everywhere on $[0, 2\pi]$, and

$$\int_0^{2\pi} \log\left[\tilde{\sigma}'(\theta)\right] d\theta = 2\int_{\vartheta_1}^{2\pi-\vartheta_1} \log\left[\sigma'(\vartheta)\, w(\vartheta)\right] \frac{d\vartheta}{w(\vartheta)},$$

with

$$w(\vartheta) = \frac{2\sin(\vartheta/2)\sqrt{\cos^2(\vartheta_1/2) - \cos^2(\vartheta/2)}}{\sin(\vartheta_1/2)}.$$

Then, the first equivalence follows from

$$\int_{\vartheta_1}^{2\pi-\vartheta_1} \left|\log\left[w(\vartheta)\right]\right| \frac{d\vartheta}{w(\vartheta)} < +\infty, \quad 0 < m \le \frac{2\sin^2(\vartheta/2)}{\sin(\vartheta_1/2)} \le M < +\infty,$$

$\vartheta \in [\vartheta_1, 2\pi - \vartheta_1]$, and the second from the inequality $\log x < x, \quad x > 0$. ■

Let us consider the positive trigonometric polynomial $\left(1 + c^2\cos^2\theta\right)^n$, $\theta \in [0, 2\pi]$, and set

$$\mathcal{W}_{2n}(u) \overset{def}{=} \left(u - \frac{1}{\varphi(i/c)}\right)^n \left(u - \frac{1}{\overline{\varphi(i/c)}}\right)^n. \tag{17}$$

It is easy to check that

$$\left|\mathcal{W}_{2n}(u)\right|^2=\left(\frac{2}{c\left|\varphi(i/c)\right|}\right)^{2n}\left(1+c^2\cos^2\theta\right)^n,\quad u=e^{i\theta}. \tag{18}$$

We denote

$$d\tilde{\sigma}_{2n}(u)=\frac{d\tilde{\sigma}(u)}{\left|\mathcal{W}_{2n}(u)\right|^2}=\left(\frac{c\left|\varphi(i/c)\right|}{2}\right)^{2n}\frac{d\mu(c\cos\theta)}{\left(1+c^2\cos^2\theta\right)^n},\quad u=e^{i\theta},$$

and let $\tilde{\varphi}_{2n,2n}(\xi)$ be the $2n$-th orthonormal polynomial with respect to $d\tilde{\sigma}_{2n}$. This polynomial has real coefficients (see [3, Lemma 1.3, Chap. 5]). Since $\left[2/\left(c\left|\varphi(i/c)\right|\right)\right]^n l_{n,n}(cx)$ is the n-th orthonormal polynomial with respect to measure $\left(c\left|\varphi(i/c)\right|/2\right)^{2n} d\mu(cx)/\left(1+c^2x^2\right)^n$, $x\in[-1,1]$, then $l_{n,n}$ and $\tilde{\varphi}_{2n,2n}$ are related (see (50) in [3]) by

$$\left(\frac{2}{c\left|\varphi(i/c)\right|}\right)^n l_{n,n}(c\chi)=\frac{\tilde{\varphi}_{2n,2n}(\xi)+\tilde{\varphi}^*_{2n,2n}(\xi)}{\xi^n\sqrt{1+\tilde{\phi}_{2n,2n}(0)}},\quad \chi=\frac{1}{2}\left(\xi+\frac{1}{\xi}\right), \tag{19}$$

where $\tilde{\phi}_{2n,2n}$ denotes the monic orthogonal polynomial corresponding to $\tilde{\varphi}_{2n,2n}$.

Finally, we wish to point out that for every system $\{\varphi_{n,m}\}$ defined by (5), the following relations hold. They are simple reformulations of known results (notice that in all of them, n is fixed and so is the measure). For all $n,m\geq 0$

$$\left|\phi_{n,m}(0)\right|<1,\quad \phi_{n,m}\overset{def}{=}\alpha_{n,m}^{-1}\varphi_{n,m}, \tag{20}$$

$$\left|\frac{\varphi^*_{n,m}(\zeta)}{\varphi_{n,m}(\zeta)}\right|\begin{cases}<&1,\quad |\zeta|>1,\\=&1,\quad |\zeta|=1,\\>&1,\quad |\zeta|<1,\end{cases} \tag{21}$$

$$\alpha_{n,m}\varphi_{n,m+1}(\zeta)=\alpha_{n,m+1}\zeta\varphi_{n,m}(\zeta)+\varphi_{n,m+1}(0)\,\varphi^*_{n,m}(\zeta), \tag{22}$$

$$\alpha_{n,m}\varphi^*_{n,m+1}(\zeta)=\alpha_{n,m+1}\varphi^*_{n,m}(\zeta)+\overline{\varphi_{n,m+1}(0)}\zeta\varphi_{n,m}(\zeta), \tag{23}$$

$$\alpha^2_{n,m+1}-\alpha^2_{n,m}=\left|\varphi_{n,m+1}(0)\right|^2, \tag{24}$$

For the proof of (22), (23), and (24), see [[6], Section. 1.1] and [[5], Chap. 5, Theorem 1.8].

3 Weak convergence on the arc

Theorem 3. *Let γ be an arc of the unit circle described by (2) and let $d\Theta_\gamma$ be the measure on γ defined by (15). Suppose that $d\sigma\in\mathcal{M}_\gamma$ and that $\sigma'>0$ almost everywhere on γ. Then, for every bounded Borel-measurable function f on γ, we have*

$$\lim_{n\to\infty}\int_{\vartheta_1}^{2\pi-\vartheta_1} f\left(e^{i\vartheta}\right)\left|\varphi_n\left(e^{i\vartheta}\right)\right|^2\sigma'(\vartheta)\,d\vartheta=\frac{1}{2\pi}\int_{\vartheta_1}^{2\pi-\vartheta_1} f\left(e^{i\vartheta}\right)\,d\Theta_\gamma(\vartheta)$$

and

$$\lim_{n\to\infty}\int_{\vartheta_1}^{2\pi-\vartheta_1} f\left(e^{i\vartheta}\right)\left|\varphi_n\left(e^{i\vartheta}\right)\right|^2 d\sigma(\vartheta)=\frac{1}{2\pi}\int_{\vartheta_1}^{2\pi-\vartheta_1} f\left(e^{i\vartheta}\right)\, d\Theta_\gamma(\vartheta)\ .$$

Proof. We will only prove the first limit, since the second is obtained in an identical manner. We have

$$\begin{aligned} I_n &\overset{def}{=} \int_\gamma f(z)\left|\varphi_n(z)\right|^2\sigma'(z)\,|dz| \\ &= \int_{-c}^{c} f\left(\frac{t+i}{t-i}\right)\left|\varphi_n\left(\frac{t+i}{t-i}\right)\right|^2 \frac{2\sigma'((t+i)/(t-i))}{1+t^2}\,dt \\ &= \int_{-c}^{c} f\left(\frac{t+i}{t-i}\right)\left|(t-i)^n\varphi_n\left(\frac{t+i}{t-i}\right)\right|^2 \frac{\mu'(t)\,dt}{(1+t^2)^n}. \end{aligned}$$

Applying (11) on the latter integral, we obtain

$$\begin{aligned} I_n &= \left|\frac{\varphi_n(1)}{k_{n,n+1}}\right|^2\int_{-c}^{c} g(t)\left|l_{n,n+1}(t)-\frac{l_{n,n+1}(-i)}{l_{n,n}(-i)}l_{n,n}(t)\right|^2\frac{\mu'(t)\,dt}{(1+t^2)^n} \\ &= \left|\frac{\varphi_n(1)}{k_{n,n+1}}\right|^2\left\{\int_{-c}^{c} g(t)\,l_{n,n+1}^2(t)\frac{\mu'(t)\,dt}{(1+t^2)^n}\right. \\ &\quad -2\mathrm{Re}\left(\frac{l_{n,n+1}(-i)}{l_{n,n}(-i)}\right)\int_{-c}^{c} g(t)\,l_{n,n}(t)\,l_{n,n+1}(t)\frac{\mu'(t)\,dt}{(1+t^2)^n} \\ &\quad \left.+\left|\frac{l_{n,n+1}(-i)}{l_{n,n}(-i)}\right|^2\int_{-c}^{c} g(t)\,l_{n,n+1}^2(t)\frac{\mu'(t)\,dt}{(1+t^2)^n}\right\}, \end{aligned} \tag{25}$$

where $g(t)=f((t+i)/(t-i))/(1+t^2)$.

According to Theorem 7 and Theorem 9 in [11], we have that for all $k\in\mathbb{Z}$ and $m\in\mathbb{N}$,

$$\lim_{n\to\infty}\int_{-c}^{c} g(t)\,l_{n,n+k}(t)\,l_{n,n+k+m}(t)\frac{\mu'(t)\,dt}{(1+t^2)^n}=\frac{1}{\pi}\int_{-c}^{c} g(t)\,T_m\left(\frac{t}{c}\right)\frac{dt}{\sqrt{c^2-t^2}}\ ,$$

$$\lim_{n\to\infty}\frac{l_{n,n+k+1}(-i)}{l_{n,n+k}(-i)}=\varphi\left(\frac{-i}{c}\right)\quad\text{and}\quad\lim_{n\to\infty}\frac{k_{n,n+k+1}}{k_{n,n+k}}=\frac{2}{c}\ ,$$

where $T_m(t)$ denotes the m-th Chebyshev polynomial; i.e., $T_m(\cos\theta)=\cos m\theta$.

Finally, since $\varphi(-i/c)$ is a purely imaginary number, taking the limit as $n\to\infty$ in (25) and keeping in mind the last three limit relations given above, together with Lemma 2 (b), we obtain

$$\begin{aligned} \lim_{n\to\infty} I_n &= \frac{c}{2\pi}\left(\left|\varphi(i/c)\right|^{-1}+\left|\varphi\left(\frac{i}{c}\right)\right|\right)\int_{-c}^{c} g(t)\frac{dt}{\sqrt{c^2-t^2}} \\ &= \frac{1}{\pi\sin(\vartheta_1/2)}\int_{\vartheta_1}^{2\pi-\vartheta_1}\frac{f\left(e^{i\vartheta}\right)d\vartheta}{2\sqrt{c^2-\cot^2(\vartheta/2)}} \\ &= \frac{1}{2\pi}\int_{\vartheta_1}^{2\pi-\vartheta_1}\frac{f\left(e^{i\vartheta}\right)\sin(\vartheta/2)\,d\vartheta}{\sqrt{\cos^2(\vartheta_1/2)-\cos^2(\vartheta/2)}}\ , \end{aligned}$$

which concludes the proof. ■

Remark 2. From Theorem 2.2.1 of [16], we have:

- If $d\nu_{\varphi_n}$ denotes the positive measure that has a mass equal to one at every zero of φ_n, then under the assumptions of Theorem 3 with respect to $d\sigma$,

$$\lim_{n\to\infty}\frac{1}{n}\int g\,d\nu_{\varphi_n}=\frac{1}{2\pi}\int_\gamma g\,d\Theta_\gamma, \tag{26}$$

  where g is any continuous function on $\mathbb{C}$ with compact support.

- It is not hard to prove that (26) also holds when the weak conditions

$$|\phi_n(0)|\to a,\ \frac{\phi_{n+1}(0)}{\phi_n(0)}\to b,\ 0<a<1$$

  hold, where $\phi_n(z)=\frac{\varphi_n(z)}{\alpha_n}$. In this case, $\gamma=\{e^{i\theta}:\vartheta_1\le\theta-\arg b\le 2\pi-\vartheta_1\}$ with $\sin\frac{\vartheta_1}{2}=a$ (see also [2]).

4 Szegö's theorem for an arc

1 In this paragraph we give an analogous result to Theorem 2 for an arc.

Theorem 4. *Let γ be an arc of the unit circle and assume that $\sigma\in\mathcal{M}_\gamma$. Then, the following statements are equivalent:*

(a) $\log\sigma'\in L^1_{d\Theta_\gamma}$*; that is,*

$$\int_{\vartheta_1}^{2\pi-\vartheta_1}\log\left[\sigma'(\vartheta)\right]\frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\vartheta_1/2)-\cos^2(\vartheta/2)}}\,d\vartheta>-\infty;$$

(b)

$$\frac{\varphi_n(d\sigma,\zeta)}{G^n_\gamma(\zeta)}\underset{n}{\rightrightarrows}\Psi_\gamma(\zeta),\quad \zeta\in\overline{\mathbb{C}}\setminus\gamma,$$

where $\Psi_\gamma(\zeta)$ is an analytic function on $\overline{\mathbb{C}}\setminus\gamma$;

(c) *the sequence $\{C^n(\gamma)\alpha_n(d\sigma)\}_{n=0}^\infty$ converges to a finite number;*

(d) *the sequence $\left\{\varphi_n(d\sigma,\zeta)/G^n_\gamma(\zeta)\right\}$ is bounded in at least one point of $\overline{\mathbb{C}}\setminus\gamma$.*

Proof. Let us carry out the constructions of Section 2.4 for the measure σ of this theorem. Since the zeros of $\mathcal{W}_{2n}$ are two fixed points of $\{|\xi|<1\}$, we have that $(\tilde\sigma,\{\mathcal{W}_{2n}\},k)$ is weakly admissible on Γ for any integer k.

According to (11) and (17), we have

$$\frac{\left[2/\left(c\left|\varphi\left(i/c\right)\right|\right)\right]^{n}\varphi^{n}\left(\tau/c\right)\left(\tau-i\right)^{n}\varphi_{n}\left(\left(\tau+i\right)/\left(\tau-i\right)\right)}{\left[\varphi(\tau/c)-\varphi^{-1}(i/c)\right]^{n}\left[\varphi(\tau/c)-\overline{\varphi^{-1}(i/c)}\right]^{n}}$$
$$=\frac{\left[2/\left(c\left|\varphi\left(i/c\right)\right|\right)\right]^{n}\varphi^{n}\left(\tau/c\right)\mathcal{L}_{n}\left(\tau\right)}{\mathcal{W}_{2n}\left(\varphi(\tau/c)\right)}. \tag{27}$$

With the use of the explicit expression $\varphi^{-1}(\xi)=\left(\xi+\xi^{-1}\right)/2$, and noting that $\varphi(i/c)=-\varphi(-i/c)=\overline{\varphi(-i/c)}$, a simple computation quickly shows that

$$\frac{\left[\varphi(\tau/c)\varphi(i/c)-1\right]\left[\varphi(\tau/c)\overline{\varphi(i/c)}-1\right]c}{2\left(\tau-i\right)\varphi(\tau/c)\left|\varphi(i/c)\right|}=i\frac{\varphi(\tau/c)\overline{\varphi(i/c)}-1}{\varphi(\tau/c)-\varphi(i/c)}, \tag{28}$$

and hence, (27) is equivalent to

$$\frac{\varphi_{n}\left(\left(\tau+i\right)/\left(\tau-i\right)\right)}{\left[i\left(\varphi(\tau/c)\overline{\varphi(i/c)}-1\right)/\left(\varphi(\tau/c)-\varphi(i/c)\right)\right]^{n}}$$
$$=\frac{\mathcal{L}_{n}\left(\tau\right)l_{n,n}\left(i\right)}{i^{n}l_{n,n}\left(\tau\right)\left|l_{n,n}\left(i\right)\right|}\frac{i^{n}\left|l_{n,n}\left(i\right)\right|\left[2/\left(c\left|\varphi\left(i/c\right)\right|\right)\right]^{n}\varphi^{n}\left(\tau/c\right)l_{n,n}\left(\tau\right)}{l_{n,n}\left(i\right)\mathcal{W}_{2n}\left(\varphi\left(\tau/c\right)\right)}. \tag{29}$$

From (19), we obtain

$$\left(\frac{2}{c\left|\varphi\left(i/c\right)\right|}\right)^{n}\frac{\varphi^{n}\left(\tau/c\right)l_{n,n}(\tau)}{\mathcal{W}_{2n}\left(\varphi\left(\tau/c\right)\right)}=\frac{1+\tilde{\varphi}^{*}_{2n,2n}(\varphi\left(\tau/c\right))/\tilde{\varphi}_{2n,2n}(\varphi\left(\tau/c\right))}{\sqrt{1+\tilde{\phi}_{2n,2n}\left(0\right)}}$$
$$\times\frac{\tilde{\varphi}_{2n,2n}(\varphi\left(\tau/c\right))}{\mathcal{W}_{2n}\left(\varphi\left(\tau/c\right)\right)}. \tag{30}$$

Let us assume that Statement (**a**) of Theorem 4 holds. It is then obvious that $\sigma'>0$ almost everywhere on γ and therefore $\tilde{\sigma}'>0$ almost everywhere on Γ (see (16)). Hence, the following relations hold (see Theorem 3 of [11]):

$$\lim_{n\to\infty}\tilde{\phi}_{2n,2n}\left(0\right)=0, \tag{31}$$

and

$$\frac{\tilde{\varphi}^{*}_{2n,2n}\left(\xi\right)}{\tilde{\varphi}_{2n,2n}\left(\xi\right)}\underset{n}{\rightrightarrows}0,\quad\xi\in\overline{\mathbb{C}}\setminus\left[\left|\xi\right|\leq1\right]. \tag{32}$$

Futhermore, Lemma 4 shows that $\log\tilde{\sigma}'\in L^{1}_{\Gamma}$; therefore, from **(a)**$\Rightarrow$ **(b)** in Theorem 2 and (30)-(31)-(32), we have

$$\left(\frac{2}{c\left|\varphi\left(i/c\right)\right|}\right)^{n}\frac{\varphi^{n}\left(\tau/c\right)l_{n,n}(\tau)}{\mathcal{W}_{2n}\left(\varphi\left(\tau/c\right)\right)}\underset{n}{\rightrightarrows}\frac{1}{\sqrt{2\pi}}D\left(\tilde{\sigma}',\varphi\left(\tau/c\right)\right),\quad\tau\in\overline{\mathbb{C}}\setminus\left[-c,c\right],$$

and from this, it follows that

$$\lim_{n\to\infty}\frac{i^{n}\left|l_{n,n}(i)\right|}{l_{n,n}(i)}=\frac{\left|D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)\right|}{D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)}.$$

Using these relations together with Lemma 2 **(a)**, from (29), we obtain

$$\frac{\varphi_n\left((\tau+i)/(\tau-i)\right)}{\left[i\left(\varphi(\tau/c)\overline{\varphi(i/c)}-1\right)/\left(\varphi(\tau/c)-\varphi(i/c)\right)\right]^n} \underset{n}{\rightrightarrows} \sqrt{\frac{c}{2\left|\varphi\left(i/c\right)\right|}}\,\frac{\varphi\left(\tau/c\right)-\varphi\left(-i/c\right)}{\tau+i}\,\frac{D\left(\tilde{\sigma}',\varphi\left(\tau/c\right)\right)\left|D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)\right|}{\sqrt{2\pi}\,D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)}, \tag{33}$$

$\tau\in\overline{\mathbb{C}}\setminus[-c,c]$.

It is very easy to check that $\Phi_\gamma(\zeta)\overset{def}{=}\Phi\left(i(\zeta+1)/(\zeta-1)\right)$, $\zeta\in\overline{\mathbb{C}}\setminus\gamma$, with

$$\Phi\left(\tau\right)=i\frac{\varphi\left(\tau/c\right)\overline{\varphi(i/c)}-1}{\varphi\left(\tau/c\right)-\varphi(i/c)}\,,$$

is a conformal mapping of $\overline{\mathbb{C}}\setminus\gamma$ onto $\overline{\mathbb{C}}\setminus\{|\xi|\le 1\}$ such that

$$\Phi_\gamma(\infty)=\infty \quad\text{and}\quad \Phi'_\gamma(\infty)=\frac{1}{C\left(\gamma\right)}>0.$$

In other words,

$$G_\gamma(\zeta)=\Phi_\gamma(\zeta). \tag{34}$$

Because of this equality, (33) is equivalent to

$$\frac{\varphi_n\left(\zeta\right)}{G^n_\gamma\left(\zeta\right)} \underset{n}{\rightrightarrows} \sqrt{\frac{c}{2\left|\varphi\left(i/c\right)\right|}}\,\frac{\varphi\left((i/c)(\zeta+1)/(\zeta-1)\right)-\varphi\left(-i/c\right)}{i(\zeta+1)/(\zeta-1)+i}$$

$$\times\frac{1}{\sqrt{2\pi}}D\left(\tilde{\sigma}',\varphi\left((i/c)(\zeta+1)/(\zeta-1)\right)\right)\frac{\left|D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)\right|}{D\left(\tilde{\sigma}',\varphi\left(i/c\right)\right)}, \tag{35}$$

$\zeta\in\overline{\mathbb{C}}\setminus\gamma$, and from this, (**a**)⇒(**b**) follows because the right hand side of (35) is an analytic function on $\overline{\mathbb{C}}\setminus\gamma$.

Implications (**b**)⇒(**c**)⇒(**d**) are immediately obvious from the fact that

$$\frac{\varphi_n}{G^n_\gamma}(\infty)=C^n(\gamma)\alpha_n$$

(which follows from (34)). It only remains to prove that (**d**)⇒(**a**). Let us assume that (**d**) holds; that is, there exists $\zeta_0\in\overline{\mathbb{C}}\setminus\gamma$ for which there is a constant m_{ζ_0} such that for all $n\ge 0$

$$\left|\frac{\varphi_n\left(\zeta_0\right)}{G^n_\gamma\left(\zeta_0\right)}\right|\le m_{\zeta_0},$$

Since $\left|G_\gamma\left(\zeta_0\right)\right|>1$, Lemma 3 and the last inequality imply that for all $n\ge 0$,

$$\left|\frac{\left(\zeta_0-1\right)^n l_{n,n}\left(i(\zeta_0+1)/(\zeta_0-1)\right)}{(2i)^n\,G^n_\gamma\left(\zeta_0\right)}\right|\le\sqrt{\sum_{j=0}^{n}\left|\frac{\varphi_j\left(\zeta_0\right)}{G^j_\gamma\left(\zeta_0\right)}\right|^2\frac{1}{\left|G^{n-j}_\gamma\left(\zeta_0\right)\right|^2}}$$

$$\le m_{\zeta_0}\sqrt{\sum_{j=0}^{n}\frac{1}{\left|G_\gamma\left(\zeta_0\right)\right|^{2j}}}\le\frac{m_{\zeta_0}\left|G_\gamma\left(\zeta_0\right)\right|}{\sqrt{\left|G_\gamma\left(\zeta_0\right)\right|^2-1}}=\mathcal{N}_{\zeta_0}<+\infty.$$

From (34) and (28), we find $\tau_0 = i(\zeta_0+1)/(\zeta_0-1) \in \overline{\mathbb{C}}\setminus[-c,c]$ such that

$$\left|\frac{\left[2/(c\left|\varphi\left(i/c\right)\right|)\right]^n \varphi^n(\tau_0/c) l_{n,n}\left(\tau_0\right)}{\left[\varphi(\tau_0/c)-\varphi^{-1}(i/c)\right]^n \left[\varphi(\tau_0/c)-\overline{\varphi^{-1}(i/c)}\right]^n}\right| \leq \mathcal{N}_{\zeta_0}, \qquad n \geq 0.$$

From (30) it follows that there exists $\xi_0 \in \overline{\mathbb{C}}\setminus[|\xi|\leq 1]$ exists for which

$$\left|\frac{\tilde{\varphi}_{2n,2n}(\xi_0)+\tilde{\varphi}^*_{2n,2n}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)\sqrt{1+\tilde{\phi}_{2n,2n}\left(0\right)}}\right| \leq \mathcal{N}_{\zeta_0}, \qquad n \geq 0. \tag{36}$$

Since $\tilde{\varphi}_{2n,2n}(0)$ is a real number, it is easy to obtain the following two equalities from (22)-(24):

$$\tilde{\varphi}_{2n,2n}(\xi)+\tilde{\varphi}^*_{2n,2n}(\xi) = \left(\xi\tilde{\varphi}_{2n,2n-1}(\xi)+\tilde{\varphi}^*_{2n,2n-1}(\xi)\right)\frac{\alpha_{2n,2n}\left(1+\tilde{\phi}_{2n,2n}\left(0\right)\right)}{\alpha_{2n,2n-1}},$$

and

$$\frac{\alpha_{2n,2n}\left(1+\tilde{\phi}_{2n,2n}\left(0\right)\right)}{\alpha_{2n,2n-1}} = \frac{\sqrt{1+\tilde{\phi}_{2n,2n}\left(0\right)}}{\sqrt{1-\tilde{\phi}_{2n,2n}\left(0\right)}}.$$

With these, (36) becomes

$$\left|\frac{\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0)+\tilde{\varphi}^*_{2n,2n-1}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)\sqrt{1-\tilde{\phi}_{2n,2n}\left(0\right)}}\right| \leq \mathcal{N}_{\zeta_0}, \qquad n \geq 1.$$

From (20) and (21), we find that

$$\begin{aligned}
&\left|\frac{\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0)+\tilde{\varphi}^*_{2n,2n-1}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)\sqrt{1-\tilde{\phi}_{2n,2n}\left(0\right)}}\right| \\
&\quad = \left|\frac{\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)}\right| \left|\frac{1+\tilde{\varphi}^*_{2n,2n-1}(\xi_0)/(\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0))}{\sqrt{1-\tilde{\phi}_{2n,2n}\left(0\right)}}\right| \\
&\quad \geq \frac{1}{\sqrt{2}}\left|\frac{\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)}\right|\left(1-\frac{1}{|\xi_0|}\right).
\end{aligned}$$

Combining the two above inequalities, we conclude that

$$\left|\frac{\xi_0\tilde{\varphi}_{2n,2n-1}(\xi_0)}{\mathcal{W}_{2n}(\xi_0)}\right| \leq \frac{\sqrt{2}\,|\xi_0|\,\mathcal{N}_{\zeta_0}}{|\xi_0|-1}, \qquad n \geq 1.$$

Together with Theorem 2 ((**d**)⇒(**a**), $k=-1$), this estimate guarantees that $\log\tilde{\sigma}' \in L^1_{\Gamma}$. Therefore, according to Lemma 4, the condition that $\log\sigma' \in L^1_{d\Theta_\gamma}$ is also satisfied.

■

2 In this paragraph, let us reconsider asymptotic formula (35), in order to reduce it to the most symmetrical expression given by Akhiezer [1].

Lemma 5. *The Szegö function for arc γ and weight $\rho(\vartheta) = 2\sin^{-1}(\vartheta_1/2)\sin^2(\vartheta/2)$ is*

$$F_\gamma(\zeta) = -\frac{4\sqrt{2\sin^3(\vartheta_1/2)}\,(1+\sin(\vartheta_1/2))\zeta}{\cos^2(\vartheta_1/2)\left[\varphi^2\left((i/c)(\zeta+1)/(\zeta-1)\right) - \varphi^2\left(i/c\right)\right](\zeta-1)^2}\,, \quad \zeta \in \overline{\mathbb{C}}\backslash\gamma.$$

Proof. If $\vartheta = 2\text{arccot}\,(c\cos\theta)$ we have

$$\rho(e^{i\vartheta}) \stackrel{def}{=} \rho(\vartheta) = 2\sin^{-1}(\vartheta_1/2)\sin^2(\vartheta/2) = \frac{2\sin^{-1}(\vartheta_1/2)}{1+c^2\cos\theta}, \tag{37}$$

and combining this relation with (17), and (18), we obtain

$$\rho(e^{i\vartheta}) = 2\sin^{-1}(\vartheta_1/2)\left(\frac{c|\varphi(i/c)|}{2}\right)^{-2}\left|\left(z-\frac{1}{\varphi(i/c)}\right)\left(z-\frac{1}{\overline{\varphi(i/c)}}\right)\right|^{-2}, \; z=e^{i\theta}. \tag{38}$$

Now, if we use the multiplicative property of the Szegö function and the following very well known formula (see [5])

$$D(|e^{i\theta}-z_0|^2, z) = \frac{z}{z-z_0}, \quad \text{for } |z_0| \le 1,$$

we obtain

$$\begin{aligned} D(\rho(e^{i\vartheta}); z) &= 2^{-1/2}\sin^{1/2}(\vartheta_1/2)\frac{c|\varphi(i/c)|}{2}\frac{\left(z-\frac{1}{\varphi(i/c)}\right)\left(z-\frac{1}{\overline{\varphi(i/c)}}\right)}{z^2} \\ &= 2^{-1/2}\sin^{1/2}(\vartheta_1/2)\frac{c}{2\,|\varphi(i/c)|}\frac{(z\varphi(i/c)-1)\left(\overline{\varphi(i/c)}z-1\right)}{z^2} \end{aligned}$$

Then the proof is completed by combining (28) with (10), and noting the fact that $\zeta = (\varphi(\tau))^{\pm 1}$ are the solutions of $\tau = \frac{1}{2}(\zeta+\zeta^{-1})$. ■

Theorem 5. *Let γ be the arc of the unit circle described by (2). Suppose that $d\sigma \in \mathcal{M}_\gamma$ and that*

$$\int_{\vartheta_1}^{2\pi-\vartheta_1} \log\left[\sigma'(\vartheta)\right]\, d\Theta_\gamma(\vartheta) > -\infty, \tag{39}$$

then,

$$\begin{aligned} &\varphi_n(d\sigma,\zeta)\Bigg/\left(\frac{1+\zeta+\sqrt{(1+\zeta)^2-4C^2(\gamma)\zeta}}{2C(\gamma)}\right)^n \\ &\underset{n}{\rightrightarrows} \frac{\sqrt{(1+\zeta)^2-4C^2(\gamma)\zeta}+\zeta-1-2\sin(\vartheta_1/2)}{\sqrt{2\pi\,(1+\sin(\vartheta_1/2))}\,(\zeta-1)} D_\gamma(t,\zeta), \end{aligned} \tag{40}$$

$t = \left(\Theta_\gamma'\right)^{-1}\sigma', \quad \zeta \in \overline{\mathbb{C}}\backslash\gamma.$

Proof. From Theorem 4 ((**a**)⇒(**b**)) and the following compact analytic expression for $G_\gamma(\zeta)$ (see [[8], pag. 233] or [[3], Lemma 6-(iv)])

$$G_\gamma(\zeta) = \frac{1+\zeta+\sqrt{(1+\zeta)^2 - 4C^2(\gamma)\zeta}}{2C(\gamma)}, \tag{41}$$

where the root is taken such that $G_\gamma(0) = C^{-1}(\gamma)$, it is enough to show that the right hand sides of (35) and (40) are equal.

A simple calculation gives us

$$\frac{\sin(\vartheta_1/2)}{2\sin(\vartheta/2)\sqrt{\cos^2(\vartheta_1/2)-\cos^2(\vartheta/2)}} = \frac{1+c^2\cos^2\theta}{2c\,|\sin\theta|}, \quad \vartheta = 2\mathrm{arccot}\,(c\cos\theta)\,, \tag{42}$$

$\theta \in [0, 2\pi]$, thus from (16)

$$\tilde{\sigma}'(\theta) = t(\vartheta)\, 2\sin^{-1}(\vartheta_1/2)\sin^2(\vartheta/2), \tag{43}$$

where

$$t(\vartheta) = \sigma'(\vartheta)\left(\Theta'_\gamma(\vartheta)\right)^{-1} = \frac{\sigma'(\vartheta)\sqrt{\cos^2(\vartheta_1/2)-\cos^2(\vartheta/2)}}{\sin(\vartheta/2)}.$$

Using the multiplicative property of the Szegö function and composing (10), Lemma 5, and (43) on (35), we obtain

$$\Psi_\gamma(\zeta) = \frac{2i\sin(\vartheta_1/2)\sqrt{1+\sin(\vartheta_1/2)}}{\cos(\vartheta_1/2)\left[\varphi((i/c)(\zeta+1)/(\zeta-1)) - \varphi(i/c)\right](\zeta-1)}\frac{1}{\sqrt{2\pi}}D_\gamma(t,\zeta)\,. \tag{44}$$

Through (34) and (41), (44) becomes

$$\Psi_\gamma(\zeta) = \frac{C(\gamma)G_\gamma(\zeta) - 1 - \sin(\vartheta_1/2)}{\sqrt{2\pi\,(1+\sin(\vartheta_1/2))}\,(\zeta-1)}D_\gamma(t,\zeta)\,,$$

and this completes the proof. ■

In particular, evaluating at $\zeta = \infty$ (see (10)) we obtain

Corollary 2. *Under Assumption (39), the asymptotic behavior of the leading coefficients is given by*

$$\lim_{n\to\infty} C^n(\gamma)\alpha_n(d\sigma) = \frac{1}{\sqrt{2\pi\,(1+\sin(\vartheta_1/2))}} \times \exp\left\{\frac{-1}{4\pi}\int_{\vartheta_1}^{2\pi-\vartheta_1} \log\left[t(\vartheta)\right]\, d\Theta_\gamma(\vartheta)\right\}.$$

Corollary 3. *(see [10]) Under Assumption (39), we have*

$$\lim_{n\to\infty}\int_\gamma \left|\varphi_n(\zeta) - \left[\Psi_\gamma(\zeta_+)\,G^n_\gamma(\zeta_+) + \Psi_\gamma(\zeta_-)\,G^n_\gamma(\zeta_-)\right]\right|^2 \sigma'(\zeta)|d\zeta| = 0. \tag{45}$$

Proof. Denoting by I_n the integral under the limit in (45), it is obvious that

$$0 \le I_n \le 1 - 2\mathbf{Re}\int_\gamma \varphi_n(\zeta)\overline{H_n(\zeta)}\sigma'(\zeta)|d\zeta| + \int_\gamma |H_n(\zeta)|^2\sigma'(\zeta)|d\zeta|,$$

where $H_n(\zeta) = \Psi_\gamma(\zeta_+)\, G_\gamma^n(\zeta_+) + \Psi_\gamma(\zeta_-)\, G_\gamma^n(\zeta_-)$. On the other hand, a simple calculation give us

$$\frac{|G_\gamma'(\zeta_\pm)|}{|\Psi_\gamma(\zeta_\pm)|^2} = 2\pi\sigma'(\zeta), \quad \zeta \in \gamma. \tag{46}$$

Thus, using Theorem 4 we obtain

$$\begin{aligned}&\limsup_{n\to\infty} -2\mathbf{Re}\int_\gamma \varphi_n(\zeta)\overline{H_n(\zeta)}\sigma'(\zeta)|d\zeta| \\ &\qquad = \limsup_{n\to\infty} \frac{-2}{2\pi}\mathbf{Re}\oint_\gamma \frac{\varphi_n(\zeta)}{G_\gamma(\zeta)\Psi_\gamma(\zeta)}|G_\gamma'(\zeta)d\zeta| \le -2.\end{aligned}$$

Finally, using again (46) and the Riemann-Lebesgue Lemma, we have

$$\begin{aligned}\int_\gamma |H_n(\zeta)|^2\sigma'(\zeta)|d\zeta| &= \oint_\gamma |\Psi_\gamma(\zeta)\, G_\gamma^n(\zeta)|^2\sigma'(\zeta)|d\zeta| \\ &\quad +2\mathbf{Re}\int_\gamma \Psi_\gamma(\zeta_+)\, G_\gamma^n(\zeta_+)\overline{\Psi_\gamma(\zeta_-)\, G_\gamma^n(\zeta_-)}\sigma'(\zeta)|d\zeta| \\ &= \frac{1}{2\pi}\oint_\gamma |G_\gamma'(\zeta)||d\zeta| + o(1) = 1 + o(1), \quad \text{as } n\to\infty.\end{aligned}$$

■

Remark 3.
- It is not hard to prove that if there exists a function $\Psi_\gamma \in H_2(\sigma)$ (see definition in [4]) such that (45) holds then we have that **(a)-(d)** of Theorem 4 hold.
- If $\gamma = \{z = e^{i\vartheta} : \vartheta_1 \le \vartheta \le \vartheta_2,\ 0 \le \vartheta_2 - \vartheta_1 \le 2\pi\}$ is an arbitrary arc, then $\hat\gamma = e^{i\vartheta_0}\gamma$ is the symmetric arc obtained from γ by a rotation of angle $\vartheta_0 = (2\pi - \vartheta_1 - \vartheta_2)/2$. Let us set $c_0 = e^{i\vartheta_0}$. Let us consider the measure $d\hat\sigma(\hat\vartheta) = d\sigma(\hat\vartheta - \vartheta_0)$, $\hat\vartheta \in [\vartheta_1 + \vartheta_0, \vartheta_2 + \vartheta_0]$.

  The general case is obtained immediately from the following easily provable statements:

  1. $\hat\sigma'(\hat\vartheta) = \sigma'(\hat\vartheta - \vartheta_0)$; $\log\hat\sigma' \in L^1_{d\Theta_{\hat\gamma}} \Leftrightarrow \log\sigma' \in L^1_{d\Theta_\gamma}$;
  2. $\varphi_n(\sigma,\zeta) = c_0^{-n}\varphi_n(\hat\sigma, c_0\zeta)$ $\alpha_n(d\sigma) = \alpha_n(d\hat\sigma)$;
  3. $G_\gamma(\zeta) = c_0^{-1}G_{\hat\gamma}(c_0\zeta)$, $C(\gamma) = C(\hat\gamma) = \sin\frac{\vartheta_2-\vartheta_1}{2}$;
  4. $d\Theta_\gamma(\vartheta) = \frac{\sin((\vartheta+\vartheta_0)/2)}{\sqrt{\cos^2((\vartheta_1+\vartheta_0)/2)-\cos^2((\vartheta+\vartheta_0)/2)}}\,d\vartheta, \qquad \vartheta \in [\vartheta_1, \vartheta_2]$;
  5. $\Psi_\gamma(\zeta) = \frac{\sqrt{(1+c_0\zeta)^2-4C^2(\gamma)c_0\zeta}+c_0\zeta-1-2\sin((\vartheta_1+\vartheta_0)/2)}{\sqrt{2\pi(1+\sin((\vartheta_1+\vartheta_0)/2))}(c_0\zeta-1)}D_\gamma(t, c_0\zeta)$.

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