Convergence rate of Padé-type approximants for Stieltjes functions

M. Bello Hernández[a,*] F. Cala Rodríguez [b]
J. J. Guadalupe [a,*] G. López Lagomasino [c,†]

a: Dpto. de Matemáticas y Computación, Univ. de La Rioja, Logroño, Spain
b: Dpto. de Análisis Matemático, Univ. de La Laguna, Tenerife, Spain
c: Dpto. de Matemáticas, Univ. Carlos III de Madrid, Madrid, Spain

Abstract

For a wide class of Stieltjes functions we estimate the rate of convergence of Padé-type approximants when the number of fixed poles represents a fixed proportion with respect to the order of the rational approximant.

Keywords: Orthgogonal polynomials and Padé-type approximation.

1 Introduction

Let $\gamma > 1$, by f_γ we denote a continuous almost everywhere positive function on the real line such that

$$\lim_{|x|\to\infty} f_\gamma(x)|x|^{-\gamma} = 1. \tag{1}$$

In [9], E. A. Rakhmanov studied the asymptotic behavior of the sequence $h_n(d\rho_\gamma; .)$ of orthonormal polynomials with respect to

$$d\rho_\gamma(x) = exp\{-f_\gamma(x)\}\, dx,\ x \in \mathbb{R}. \tag{2}$$

(Within this class of measures, of particular interest are the so-called Freud weights

$$dw_\gamma(x) = exp\{-|x|^\gamma\}\, dx$$

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and their orthogonal polynomials.) He proved that

$$\lim_{n\to\infty} \frac{\log|h_n(d\rho_\gamma; z)|}{n^{1-\gamma^{-1}}} = D(\gamma)|\mathrm{Im}\, z|, \tag{3}$$

where this limit is uniform on compact subsets of $\mathbb{C}\backslash\mathbb{R}$,

$$D(\gamma) = \frac{\gamma}{\gamma - 1} \left[\frac{\Gamma((\gamma+1)/2)}{\Gamma(\gamma/2)}\right]^{(1/\gamma)},$$

and $\Gamma(.)$ denotes the Gamma function. Set

$$\hat{\rho}_\gamma(z) = \int \frac{d\rho_\gamma(x)}{z - x}\ .$$

Let π_n denote the $n - th$ diagonal Padé approximant with respect to $\hat{\rho}_\gamma$. From Rakhmanov's result it is not hard to deduce that

$$\lim_{n\to\infty} \frac{\log|\hat{\rho}_\gamma(z) - \pi_n(z)|}{n^{1-\gamma^{-1}}} \le -2D(\gamma)|\mathrm{Im}\, z|, \tag{4}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. We aim to obtain similar results when instead of Padé approximants, Padé-type approximants are used.
Let l_n^2 be a polynomial of degree $m(n)$ and $0 \le m(n) \le n$. We define the n-th Padé-type approximants of $\hat{\rho}_\gamma$ with fixed poles at the zeros of l_n^2 as the unique rational function

$$r_n = \frac{p_n}{q_n l_n^2}\ ,$$

where p_n and q_n are polynomials which satisfy

- deg $p_n \le n - 1$, deg $q_n \le n - m(n)$, $q_n \not\equiv 0$,
- $(q_n l_n^2 \hat{\rho}_\gamma - p_n)(z) = O(\frac{1}{z^{n-m(n)+1}}),\ \text{as } z = ix \to \infty,\ x > 0.$

It is easy to prove (see, for example, [4]) that

$$0 = \int x^\nu q_n(x)\ l_n^2(x)\ d\rho_\gamma(x),\ \ \nu = 0, \ldots, n - m(n) - 1 \tag{5}$$

and

$$(\hat{\rho}_\gamma - r_n)(z) = \frac{1}{(q_n l_n)^2(z)} \int \frac{(q_n l_n)^2(x)\ d\rho_\gamma(x)}{z - x}. \tag{6}$$

If $m(n) = 0$, then r_n is the $n - th$ diagonal Padé approximant with respect to $\hat{\rho}_\gamma$. If $m(n) = n$, all the poles of the rational approximant are fixed.

In recent years (see, for example, [1], [2], [3], [4] and [6]) the rate of convergence of Padé-type and multipoint Padé-type approximants has been studied when the measure defining the function has compact support. We will show that results of type (4) take place for Padé-type approximants when the support of the measure is unbounded. To this end, we will restrict the type of polynomials which carry as their zeros the fixed poles of the Padé-type approximants. In the sequel, l_n denotes the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $dw_\beta(x)$ introduced above. Unless otherwise stated, we take $\gamma > \beta > 1$.
We prove

Theorem 1 *Let l_n denote the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $dw_\beta(x)$ where $1 < \beta < \gamma$. Let r_n denote the n-th Padé-type approximant of $\hat{\rho}_\gamma$ with fixed poles at the zeros of l_n^2 and assume that*

$$\lim_{n\to\infty} \frac{m(n)}{n} = \theta \in [0,1) .$$

Then

$$\limsup_{n\to\infty} \frac{\log|\hat{\rho}_\gamma(z) - r_n(z)|}{n^{1-\gamma^{-1}}} \le -2(1-\theta)^{1-\gamma^{-1}} D(\gamma)|Im\ z| , \tag{7}$$

where convergence takes place uniformly on each compact subset of $\mathbb{C} \setminus \mathbb{R}$.

The paper is divided as follows. In Section 2, we give some auxiliary results. Section 3 is devoted to the proof of the theorem stated above and some comments.

2 Auxiliary results

Let $d\rho$ be a finite positive Borel measure on $\mathbb{R}$, with an infinite number of points in its support and finite moments. Denote

$$K_j(d\rho, z) = \sup_{p\in\Pi_j,\ p\not\equiv 0} \frac{|p^2(z)|}{\int |p^2(x)| d\rho(x)}, \tag{8}$$

where Π_j is the set of all polynomials of degree $\le j$.
If $d\rho = l_n^2\ d\rho_\gamma$ we denote

$$K_{n,j}(z) = K_j(d\rho, z).$$

Lemma 2.1 *There exist constants $D > 0$ and $\alpha \in \mathbb{R}$ such that*

$$Dn^\alpha K_j(d\tilde{\rho}_\gamma, z) \le K_{n,j}(z) \le K_j(l_n^2 d\rho_\gamma|_{(-n^{1/\gamma}, n^{1/\gamma})}, z), \tag{9}$$

where $d\tilde{\rho}_\gamma(x) = \exp\{-(f_\gamma(x) - |x|^\beta)\}dx$, $1 < \beta < \gamma$, and $l_n^2 d\rho_\gamma|_{(-n^{1/\gamma}, n^{1/\gamma})}$ is the restriction of the measure $l_n^2 d\rho_\gamma$ to $(-n^{1/\gamma}, n^{1/\gamma})$.

Proof: From (8) the inequality on the right side of (9) follows directly. On the other hand from **Corollary 1.4** in [7] there exist constants $D_1 > 0$ and $\alpha_1 \in \mathbb{R}$ such that

$$l_n^2(x)\exp\{-|x|^\beta\} \le D_1(m(n)+1)^{\alpha_1},\ x \in \mathbb{R}.$$

Since $0 \le m(n) \le n$, we obtain

$$l_n^2(x)\exp(-|x|^\beta) \le D_2 n^{\alpha_1}.$$

Thus, if $p \in \Pi_j$ and $p \not\equiv 0$, we get

$$\frac{|p^2(z)|}{\int |p^2(x)| l_n^2(x)\ d\rho_\gamma(x)} \ge \frac{|p^2(z)|}{D_2 n^{\alpha_1} \int |p^2(x)| d\tilde{\rho}_\gamma(x)}$$

and the proof is concluded.■

Lemma 2.2 *Let K be a compact subset of $\mathbb{C}\setminus\mathbb{R}$, $1 < \beta < \gamma$, and $\lim \frac{m(n)}{n} = \theta$. Then*

$$\liminf_{n\to\infty} \frac{\log|q_n|(z)}{n^{1-1/\gamma}} \ge (1-\theta)^{1-1/\gamma} D(\gamma)|Im\ z|$$

uniformly on K, where q_n is the $(n-m(n))$-th orthonormal polynomial with respect to $l_n^2\ d\rho_\gamma$ and l_n denotes the orthonormal polynomial of degree $m(n)/2$ with respect to the Freud measure $dw_\beta(x)$.

Proof: Let t_k be the k-th orthonormal polynomial with respect to $d\rho$. From the general theory of orthogonal polynomials, we know (see [5]) that

$$K_j(d\rho, z) = \sum_{k=0}^{j} |t_k(z)|^2 \ge |t_j(z)|^2,\ z \in \mathbb{C} \tag{10}$$

and

$$K_{j-1}(d\rho, z) = \frac{\tau_{j-1}}{\tau_j} \frac{t_j(z)t_{j-1}(\overline{z}) - t_j(\overline{z})t_{j-1}(z)}{z - \overline{z}},$$

where $z \in \mathbb{C}\setminus\mathbb{R}$ and τ_j is the leading coefficient of t_j. Thus, with the aid of (10), we obtain

$$\begin{aligned} K_{j-1}(d\rho, z) &= \frac{\tau_{j-1}}{\tau_j} \frac{Im(t_j(z)\overline{t_{j-1}(z)})}{\text{Im } z} \\ &\le \frac{\tau_{j-1}}{\tau_j} \frac{|t_j t_{j-1}(z)|}{|\text{Im } z|} \\ &\le \frac{\tau_{j-1}}{\tau_j} \frac{|t_j(z)| K_{j-1}^{1/2}(z)}{|\text{Im } z|}. \end{aligned}$$

This inequality yields

$$K_{j-1}(d\rho, z) \le \frac{\tau_{j-1}^2}{\tau_j^2}\frac{|t_j(z)|^2}{|\text{Im } z|^2}, \tag{11}$$

therefore,

$$K_j(d\rho, z) = K_{j-1} + |t_j(z)|^2 \le \left[\frac{\tau_{j-1}^2}{\tau_j^2 |\text{Im}(z)|^2} + 1\right] |t_j(z)|^2. \tag{12}$$

On the other hand,

$$\begin{aligned} \frac{1}{\tau_j^2} &= \inf_{P = z^j + \cdots} \int |P^2(x)| \; d\rho(x) \\ &\le \int |x \frac{t_{j-1}(x)}{\tau_{j-1}}|^2 \; d\rho(x), \end{aligned}$$

or what is the same

$$\frac{\tau_{j-1}^2}{\tau_j^2} \le \int |x t_{j-1}|^2 \; d\rho(x).$$

If $d\rho(x)$ satisfies (2), there exist constants $D_1,\ D_2,\ D_3 > 0$ such that for all $k \in \mathbb{N}$ and $p \in \Pi_k$, we have (see **Theorem 2.6** in [8])

$$\int |p^2(x)| \; d\rho(x) \le D_2 \int_{-D_1 k^{1/\gamma}}^{D_1 k^{1/\gamma}} |p(x)|^2 \; d\rho(x),$$

in particular,

$$\begin{aligned} \frac{\tau_{j-1}^2}{\tau_j^2} &\le D_2 \int_{-D_1 j^{1/\gamma}}^{D_1 j^{1/\gamma}} |x t_{j-1}(x)|^2 \; d\rho(x), \\ &\le D_3 j^{2/\gamma}. \end{aligned} \tag{13}$$

Take $d\rho(x) = d\tilde{\rho}_\gamma(x) = \exp\{-(f_\gamma(x) - |x|^\beta)\}dx$. Since $1 < \beta < \gamma$, the function $f_\gamma(x) - |x|^\beta$satisfies (1). Using (3), (10), (12), and (13) one obtains

$$\lim_{n\to\infty} \frac{\log K_n(d\tilde{\rho}_\gamma, z)}{n^{1-1/\gamma}} = 2D(\gamma)|Im(z)|. \tag{14}$$

This result appears in [9], **Lemma 4**.
For $d\rho(x) = l_n^2(x) d\rho_\gamma(x)$ and $j = n - m(n)$, (12) gives

$$K_{n,n-m(n)}(z) \le \left[\frac{\tau_{n,n-m(n)-1}^2}{\tau_{n,n-m(n)}^2} + 1\right] |q_n(z)|^2, \tag{15}$$

where q_n is the $(n-m(n))$-th orthonormal polynomial with respect to $|l_n|^2\, d\rho_\gamma$ and $\tau_{n,n-m(n)}$ its leading coefficient. Notice that, infinite-finite range L_2 estimates give as above

$$\frac{\tau^2_{n-m(n)-1}}{\tau^2_{n-m(n)}} \le D_4 n^{2/\gamma}. \tag{16}$$

From the first inequality in (9), (14), (15) and (16), we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log|q_n(z)|}{n^{1-1/\gamma}} &\ge \lim_n \left(\frac{n-m(n)}{n}\right)^{1-1/\gamma} \frac{\log|K_{n-m(n)}(d\tilde{\rho}_\gamma, z)|}{2(n-m(n))^{1-1/\gamma}} \\ &= (1-\theta)^{1-1/\gamma} D(\gamma)|Im(z)|\end{aligned}$$

and the proof is finished.■

3 Proof of Theorem 1

Let K be a compact subset of $\mathbb{C}\setminus\mathbb{R}$, then there exists $D_1 = D_1(K) > 0$ such that

$$|z-x| \ge D_1, \quad z\in K, x\in\mathbb{R}.$$

Using (6) and the orthonormality of q_n, we get

$$\begin{aligned}|(\hat{\rho}_\gamma - r_n)(z)| &= \left|\frac{1}{(q_n l_n)^2(z)}\int \frac{(q_n l_n)^2(x)}{z-x}\, d\rho_\gamma\right| \\ &\le \frac{1}{D_1|(q_n l_n)^2(z)|}.\end{aligned}$$

Now, from **Lemma 2.2** and (3) as applied to the sequence $\{l_n\}$, we obtain (7).■

Corollary 1 *Under the assumptions of* **Theorem 1**

$$r_n \to \hat{\rho}_\gamma$$

, uniformly on each compact set of $\mathbb{C}\setminus\mathbb{R}$.

Proof: It is immediate from the fact that the right hand of (7) is continuous and negative on $\mathbb{C}\setminus\mathbb{R}$.■

Remark 1 *In the case when $\theta = 1$ and $1<\beta=\gamma$ it is possible to construct examples where there is divergence. For example, taking $m(n) = n$ and $f_\gamma(x) = |x|^\gamma - |x|^{\gamma'}$ with $\gamma' < \gamma$ sufficiently close to γ. For this reason we do not discuss this limiting situation.*

Remark 2 *When $1<\gamma<\beta$ and $m(n) = n$ there is always divergence.*

References

[1] A. Ambroladze, H. Wallin: Padé type approximants of Markov and meromorphic functions, J. Approx. Theory 88 (1997), 354-369.

[2] A. Ambroladze, H. Wallin: Convergence of rational interpolants with preassingned poles, J. Approx. Theory 89 (1997), 238-256.

[3] F. Cala, G. López: From Padé toPadé-type approximants. Exact rate of convergence, Proc. Second Int.Conf. Approx. and Opt. in the Caribbean. Ed. by M. Florenzano et. al., Peter Series in Approx. and Optimization, Vol. 8 (1995), 155-163.

[4] F. Cala, G. López: Multipoint Padé-type approximants. Exact rate of convergence. Constr. Approx. 14 (1998), No. 2, 259-272.

[5] G. Freud: Orthogonal Polynomials, Pergamon Press (1966), Budapest.

[6] A.A. Gonchar: On convergence of Padé approximants for some classes of meromorphic function, Mat. Sb. 97(139)(1975),4; English Transl. in Math. USSR Sb. 26 (1975) 555-575.

[7] A.L. Levin, D.S. Lubinsky: Christoffel functions, Orthogonal Polynomials, and Nevai's Conjecture for Freud Weights, Constr.Approx. 8(1992), 463-535.

[8] H.N. Mhaskar, E.B. Saff: Where does the L_p norm of a weighted polynomial live? Trans. Amer. Math. Soc., 303 (1987), 109-124.

[9] E.A. Rakhmanov: On the asymptotic properties of polynomials orthogonal on the real axis, Mat. Sb. 119(161)(1982), 161-203; English Transl. in Math. USSR Sb. 47(1984), 155-193.