Ratio and Relative Asymptotics of Polynomials Orthogonal on an Arc of the Unit Circle

M. Bello Hernández and G. López Lagomasino*

*Facultad de Matemáticas y Computación, Universidad de la Habana,
San Lázaro y L, Havana 4, Cuba*
E-mail: bello@matcom.uh.cu and E-mail: lago@dulcinea.uc3m.es

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Ratio and relative asymptotics are given for sequences of polynomials orthogonal with respect to measures supported on an arc of the unit circle, where their absolutely continuous component is positive almost everywhere. The results obtained extend to this setting known ones given by Rakhmanov and Máté, Nevai, and Totik for the case when the arc is the whole unit circle. Technically speaking, the main feature is the use of orthogonality with respect to varying measures.   © 1998 Academic Press

1. INTRODUCTION

1.   In recent years, growing attention has been paid to the study of the asymptotic behavior of sequences of polynomials which are orthogonal with respect to varying measures. This is not accidental. Such sequences arise naturally in the study of the convergence of sequences of rational functions which interpolate a given analytic functions along a table of interpolation points (see e.g. [9] and [12]). But perhaps their most attractive feature is that they become a powerful tool in solving problems where a fixed measure and orthogonality in the usual sense are involved. Some applications in this direction are asymptotics of orthogonal polynomials on unbounded intervals [11, 15, 19, 30, 33], (one-point) Padé [14] and Hermite–Padé approximations [2].

As far as the underlying idea is concerned, this paper resembles [15]. There, the question was the relative asymptotics of polynomials orthogonal with respect to measures on unbounded intervals. The problem was translated to varying measures on the unit circle Γ. Here, we are concerned with

the asymptotics of polynomials orthogonal on an arc γ of the unit circle; this problem too may be restated in terms of varying measures on Γ.

2. Undoubtedly, one of the results which motivated great interest and gave new impetus to the theory of orthogonal polynomials in the eighties is due to E. A. Rakhmanov. In a series of two papers (see [28, 29]), he proved the following.

Let σ be a finite positive Borel measure supported on the unit circle $\Gamma = [|z| = 1]$ and let $\{\varphi_n\}$, $\varphi_n(\zeta) = \alpha_n \zeta^n + \cdots, \alpha_n > 0$, be the corresponding sequence of orthonormal polynomials. Assume that $\sigma' > 0$ a.e. on Γ, then

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$$

and

$$\lim_{n \to \infty} \frac{\varphi_{n+1}(\zeta)}{\varphi_n(\zeta)} = \zeta,$$

uniformly on each compact subset of $\mathbb{C} \backslash [|\zeta| < 1]$.

For a short proof see [20], and for a miniature proof see [31] (also [25]).

It may be worth noting that Rakhmanov's interest in the ratio asymptotics of orthogonal polynomials came from a problem in rational approximation (see [8]). The same is true for extensions of Rakhmanov's Theorem which one of the authors of this paper has made to the case of varying measures (see [13, 14]). These extensions will be used here. From a technical point of view (though the statements are not equivalent), we obtain an analogue of Rakhmanov's result for a measure supported on an arc γ reducing the problem to the unit circle (but with varying measures). A prior attempt to extend Rakhmanov's result to measures supported on an arc of the unit circle was made by the other author in [1]. A different approach was employed without varying measures, but the method required the use of a Szegő-type condition at the end points of the arc.

Let E be a compact subset of the complex plane $\mathbb{C}$. By $C(E)$, we denote the logarithmic capacity of E (for the definition see [7, p. 310]). Let $\gamma = \{z = e^{i\theta}, \theta_1 \leqslant \theta \leqslant \theta_2, \theta_2 - \theta_1 \leqslant 2\pi\}$ be an arc of the unit circle. It is well known that $C(\gamma) = \sin(\theta_2 - \theta_1)/4$. In particular, if $\gamma = \Gamma$, then $C(\gamma) = 1$.

Given an arc γ, let $G(\zeta) = G(\gamma; \zeta)$ be the conformal mapping (of the unbounded connected component) of $\overline{\mathbb{C}} \backslash \gamma$ onto $\overline{\mathbb{C}} \backslash [|\omega| \leqslant 1]$ such that $G(\infty) = \infty$ and $G'(\infty) > 0$.

By Riemann's Theorem such a conformal representation exists and it is uniquely determined. When $\gamma = \Gamma$, we have that $G(\zeta) = \zeta$.

In Section 2, we prove the following extension of Rakhmanov's Theorem.

THEOREM 1. *Let σ be a finite positive Borel measure supported on an arc γ with $0 < \theta_2 - \theta_1 < 2\pi$ and let $\{\varphi_n\}$, $\varphi_n(\zeta) = \alpha_n \zeta^n + \cdots, \alpha_n > 0$, be the corresponding sequence of orthonormal polynomials. Assume that $\sigma' > 0$ a.e. on γ, then*

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \frac{1}{\mathrm{C}(\gamma)}. \tag{1}$$

Moreover $(\phi_n = \varphi_n / \alpha_n)$,

$$\lim_{n \to \infty} \frac{\phi_{n+1}(\zeta)}{\phi_n(\zeta)} = \mathrm{C}(\gamma)\, G(\zeta) \tag{2}$$

and

$$\lim_{n \to \infty} \frac{\varphi_{n+1}(\zeta)}{\varphi_n(\zeta)} = G(\zeta), \tag{3}$$

uniformly on each compact subset of $\mathbb{C} \backslash \gamma$.

The case when $\theta_2 - \theta_1 = 0$ degenerates into an arc with one point, and it is not interesting because we get a discrete measure. The situation $\theta_2 - \theta_1 = 2\pi$ corresponds to Rakhmanov's Theorem. There, convergence may be derived in (2) and (3) only on the unbounded component of $\mathbb{C} \backslash \Gamma$ because the zeros of the polynomials $\{\varphi_n\}$ may be everywhere dense in $[|\zeta| < 1]$. For $0 < \theta_2 - \theta_1 < 2\pi$ and $\sigma' > 0$ a.e. on γ, the zeros of $\{\varphi_n\}$ "concentrate" on γ in the sense that for each compact subset $K \subset \mathbb{C} \backslash \gamma$ there exists n_0 such that for $n \geqslant n_0$, φ_n has no zero lying on K and convergence extends to all $\mathbb{C} \backslash \gamma$.

3. Another subject of major interest (closely connected with ratio asymptotics) has been the extension of Szegő's Theory to measures σ on Γ for which $\log \sigma'$ is not integrable. The foundations were laid in Rakhmanov's paper [28] and Nevai's book [24]. As a positive reaction to [28] and [29] a new theory arose in which Szegő's condition $\log \sigma' \in L_1$ was substituted by the much weaker one $\sigma' > 0$ a.e. on Γ. The main contributions are due to Máté, Nevai, Rakhmanov, and Totik in [21–23, 31]. For the case of measures supported on unbounded intervals and varying measures see also [15].

Szegő's Theory may be interpreted as the comparison of the sequence of polynomials $\{\varphi_n\}$ orthonormal with respect to σ and the sequence $\{z^n\}$ which is orthonormal with respect to Lebesgue's measure. The new object is to compare two orthogonal polynomial systems when the corresponding measures do not satisfy Szegő's condition.

Let σ be a finite positive Borel measure on Γ. A second measure $h \, d\sigma$ is also considered, where h is a nonnegative integrable function with respect to σ. The corresponding sequences of orthonormal polynomials and their positive leading coefficients will be denoted $\{\varphi_n(\zeta)\}$, $\{\varphi_n(h; \zeta)\}$, $\{\alpha_n\}$, and $\{\alpha_n(h)\}$ respectively. We state the following result in the form it appears in [21] (see Theorem 3).

Assume that $\sigma' > 0$ a.e. on Γ and h is such that there exists a polynomial Q for which $Qh, Qh^{-1} \in L_\infty(\sigma)$. Then

$$\lim_{n \to \infty} \frac{\varphi_n(h; \zeta)}{\varphi_n(\zeta)} = D(h; \zeta),$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash [|\zeta| \leqslant 1]$, where

$$D(h; \zeta) = \exp \left\{ \frac{1}{4\pi} \int_\Gamma \log h(z) \frac{z + \zeta}{z - \zeta} |dz| \right\}.$$

In particular,

$$\lim_{n \to \infty} \frac{\alpha_n(h)}{\alpha_n} = \exp \left\{ \frac{-1}{4\pi} \int_\Gamma \log h(z) |dz| \right\}.$$

Whenever $\log h \in L_1$, $D(h; \zeta)$ is well defined for $|\zeta| > 1$ and almost everywhere on Γ. This is the so-called (exterior) Szegő function associated with h and it is characterized by the properties (see [3, Chap. 5] and [32, Chap. 10]).

— $D(h; \zeta) \in H_2$ in $\overline{\mathbb{C}} \backslash [|\zeta| \leqslant 1]$ and, therefore,

$$\lim_{r \to 1^+} D(h; r\zeta) = D(h; \zeta),$$

for almost every ζ in Γ,

— $D(h; \zeta) \neq 0$ for $|\zeta| > 1$, $D(h; \infty) > 0$, and

— $|D(h; \zeta)|^2 = 1/h(\zeta)$ almost everywhere on Γ.

The appropriate way of extending the notion of the Szegő function for an arc $\gamma \subset \Gamma$ and an h defined on γ is having in mind the defining properties described above.

Let h be a nonnegative measurable function given on an arc $\gamma \subset \Gamma$ such that $\log h$ is integrable with respect to the Lebesgue measure on γ. We define $D(h; \zeta) = D_\gamma(h; \zeta)$ as the unique function which satisfies the conditions:

(i) $D(h; \zeta) \in H_2$ in $\overline{\mathbb{C}} \backslash \gamma$ and, therefore,

$$\lim_{r \to 1^+} D(h; r\zeta) = D(h; \zeta_+), \qquad \lim_{r \to 1^-} D(h; r\zeta) = D(h; \zeta_-),$$

for almost every $\zeta \in \gamma$,

(ii) $D(h; \zeta) \neq 0$ for $\zeta \in \overline{\mathbb{C}} \backslash \gamma$, $D(h; \infty) > 0$, and

(iii) $|D(h; \zeta_+)|^2 = |D(h; \zeta_-)|^2 = 1/h(\zeta)$ almost everywhere on γ.

The construction of this function and its uniqueness is easy to reduce by conformal mapping to the case of the unit circle Γ (see Section 3).

In Section 3, we prove the following extension of the theorem of Máté–Nevai–Totik stated above.

THEOREM 2.   *Let σ be a finite positive Borel measure supported on an arc γ, $0 < \theta_2 - \theta_1 < 2\pi$, with $\sigma' > 0$ a.e. on γ, and let h be such that there exists a polynomial Q for which $Qh, Qh^{-1} \in L_\infty(\sigma)$. Then*

$$\lim_{n \to \infty} \frac{\varphi_n(h; \zeta)}{\varphi_n(\zeta)} = D_\gamma(h; \zeta), \tag{4}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash \gamma$. In particular,

$$\lim_{n \to \infty} \frac{\alpha_n(h)}{\alpha_n} = D_\gamma(h; \infty). \tag{5}$$

In the following, we maintain the notations introduced above.

2. RAKHMANOV ON THE ARC

Consider the automorphism of $\overline{\mathbb{C}}$, $\zeta = (\tau + i)/(\tau - i)$. It takes the real line $\mathbb{R}$ onto the unit circle Γ. The inverse $\tau = i((\zeta + 1)/(\zeta - 1))$ does the opposite. We write $\zeta = z$ when $|\zeta| = 1$, and $\tau = t$ when $\tau \in \mathbb{R}$.

Let σ be a finite measure on Γ whose support, $\text{supp}(\sigma)$, contains infinitely many points and let Θ_n be an algebraic polynomial of degree n orthogonal to all polynomials of lower degree with respect to σ (for the time being no normalization is imposed on Θ_n). That is,

$$0 = \int \bar{z}^\nu \Theta_n(z) \, d\sigma(z), \qquad \nu = 0, ..., n-1.$$

Along with σ, we will consider the following measures on $\mathbb{R}$; $d\mu(t) = d\sigma((t+i)/(t-i))$ and $d\mu_n(t) = d\mu(t)/(1+t^2)^n$. The mth monic orthogonal polynomial with respect to $d\mu_n$ is denoted by $L_{n,m}(\tau)$. It satisfies

$$0 = \int t^\nu L_{n,m}(t)\, d\mu_n(t), \qquad \nu = 0, ..., m-1.$$

The corresponding orthonormal polynomial is $l_{n,m}(\tau) = \kappa_{n,m} L_{n,m}(\tau)$, where

$$\frac{1}{\kappa_{n,m}} = \|L_{n,m}\|_{\mu_n} = \left(\int |L_{n,m}(t)|^2\, d\mu_n(t) \right)^{1/2}.$$

Finally, we consider the mth kernel function relative to $d\mu_n$,

$$K_{n,m}(\tau, w) = \sum_{k=0}^{m-1} l_{n,k}(\tau)\, l_{n,k}(w). \tag{6}$$

Since n remains fixed in the sum (as does the measure, see [32, pg. 43])

$$K_{n,m}(\tau, w) = \frac{\kappa_{n,m-1}}{\kappa_{n,m}} \frac{l_{n,m}(\tau)\, l_{n,m-1}(w) - l_{n,m}(w)\, l_{n,m-1}(\tau)}{\tau - w}. \tag{7}$$

LEMMA 1. *With the notations above, we have*

$$\mathscr{L}_n(\tau) = (\tau - i)^n\, \Theta_n\left(\frac{\tau+i}{\tau-i} \right) = \frac{\Theta_n(1)}{\kappa_{n,n} l_{n,n}(-i)} K_{n,n+1}(\tau, -i).$$

Proof. For all $\nu = 0, ..., n-1$,

$$0 = \int_\Gamma z^{-\nu} \Theta_n(z)\, d\sigma(z) = \int_\mathbb{R} \left(\frac{t-i}{t+i} \right)^\nu \Theta_n\left(\frac{t+i}{t-i} \right) d\mu(t)$$

$$= \int_\mathbb{R} (t+i)^{n-\nu-1} (t-i)^\nu\, \mathscr{L}_n(t)(t+i)\, d\mu_n(t).$$

Since $(t+i)^{n-\nu-1} (t-i)^\nu$, $\nu = 0, 1, ..., n-1$, forms a basis in the space of all polynomials of degree at most $n-1$, we conclude that

$$0 = \int_\mathbb{R} t^\nu \mathscr{L}_n(t)(t+i)\, d\mu_n(t), \qquad \nu = 0, ..., n-1. \tag{8}$$

From (8) it follows that

$$(\tau + i)\, \mathscr{L}_n(\tau) = C_{n,1} L_{n,n+1}(\tau) + C_{n,2} L_{n,n}(\tau), \tag{9}$$

where $C_{n,1}$, $C_{n,2}$ are constants and $C_{n,1}$ is the leading coefficient of $\mathscr{L}_n(\tau)$. Thus

$$C_{n,1} = \lim_{\tau \to \infty} \frac{\mathscr{L}_n(\tau)}{\tau^n} = \lim_{\tau \to \infty} \left(\frac{\tau - i}{\tau}\right)^n \Theta_n \left(\frac{\tau + i}{\tau - i}\right) = \Theta_n(1).$$

Substituting in (9) and taking $\tau \to -i$, we find

$$C_{n,2} = -\Theta_n(1) \frac{L_{n,n+1}(-i)}{L_{n,n}(-i)}.$$

Note that $L_{n,n}(-i) \neq 0$, since all its zeros must be in $\mathbb{R}$.

With these values for $C_{n,1}$ and $C_{n,2}$ in (9), we obtain by using (7)

$$\begin{aligned}
\mathscr{L}_n(\tau) &= \frac{\Theta_n(1)}{L_{n,n}(-i)} \left[\frac{L_{n,n+1}(\tau) L_{n,n}(-i) - L_{n,n+1}(-i) L_{n,n}(\tau)}{\tau - (-i)} \right] \\
&= \frac{\Theta_n(1)}{\kappa_{n,n} l_{n,n}(-i)} \frac{\kappa_{n,n}}{\kappa_{n,n+1}} \left[\frac{l_{n,n+1}(\tau) l_{n,n}(-i) - l_{n,n+1}(-i) l_{n,n}(\tau)}{\tau - (-i)} \right] \\
&= \frac{\Theta_n(1)}{\kappa_{n,n} l_{n,n}(-i)} K_{n,n+1}(\tau, -i). \quad \blacksquare
\end{aligned}$$

Let us consider this formula for the cases when $\Theta_n = \phi_n$ if the nth monic orthogonal polynomial and, for $\Theta_n = \varphi_n = \alpha_n \phi_n$, $\alpha_n > 0$, the nth orthonormal polynomial with respect to σ.

LEMMA 2. *We have*

$$(\tau - i)^n \phi_n \left(\frac{\tau + i}{\tau - i}\right) = (2i)^n \frac{K_{n,n+1}(\tau, -i)}{K_{n,n+1}(i, -i)}, \tag{10}$$

$$(\tau - i)^n \varphi_n \left(\frac{\tau + i}{\tau - i}\right) = i^n \frac{K_{n,n+1}(\tau, -i)}{\sqrt{K_{n,n+1}(i, -i)}}, \tag{11}$$

and

$$\alpha_n = \frac{\sqrt{K_{n,n+1}(i, -i)}}{2^n}. \tag{12}$$

Proof. From Lemma 1

$$(\tau - i)^n \phi_n \left(\frac{\tau + i}{\tau - i}\right) = \frac{\phi_n(1)}{\kappa_{n,n} l_{n,n}(-i)} K_{n,n+1}(\tau, -i).$$

Taking the limit $\tau \to i$, we obtain

$$(2i)^n = \frac{\phi_n(1)}{\kappa_{n,n} l_{n,n}(-i)} K_{n,n+1}(i,-i).$$

Thus, we have (10). Formula (6) shows that $K_{n,n+1}(i,-i) > 0$. Indeed,

$$K_{n,n+1}(i,-i) = \sum_{k=0}^{n} l_{n,k}(i)\, l_{n,k}(-i) = \sum_{k=0}^{n} |l_{n,k}(i)|^2 > 0. \qquad (13)$$

To obtain the analogous formula for φ_n we must multiply both sides of (10) by $\alpha_n = \|\phi_n\|_\sigma^{-1}$. Let us calculate this quantity using (10)

$$\begin{aligned}
\frac{1}{\alpha_n^2} = \|\phi_n\|_\sigma^2 &= \int_\Gamma |\phi_n(z)|^2 \, d\sigma(z) = 2^{2n} \int_{\mathbb{R}} \left| \frac{K_{n,n+1}(t,-i)}{K_{n,n+1}(i,-i)} \right|^2 d\mu_n(t) \\
&= \frac{2^{2n}}{K_{n,n+1}^2(i,-i)} \int_{\mathbb{R}} K_{n,n+1}(t,-i)\, \overline{K_{n,n+1}(t,-i)} \, d\mu_n(t) \\
&= \frac{2^{2n}}{K_{n,n+1}^2(i,-i)} \int_{\mathbb{R}} K_{n,n+1}(t,-i)\, K_{n,n+1}(t,i) \, d\mu_n(t) \\
&= \frac{2^{2n}}{K_{n,n+1}(i,-i)}.
\end{aligned}$$

Thus, we obtain (12) and (11). In the last two steps, we used that $\overline{K_{n,m}(t,-i)} = K_{n,m}(t,i)$ (see (6)) and the reproducing property of the kernel function

$$\int_{\mathbb{R}} K_{n,m}(t,w)\, A_m(t)\, d\mu_n(t) = A_m(w),$$

for any polynomial A_m of degree $\leqslant m-1$. ∎

Note that in Lemmas 1 and 2 we did not require any additional conditions on the finite measure σ (except that it have infinitely many points in its support). Analogous formulas, when you start from a (fixed) measure on $\mathbb{R}$ and carry it over to Γ, may be seen in Lemma 9 of [15].

If $z = 1$ belongs to supp(σ), then supp(μ) is unbounded; otherwise, supp(μ) is bounded. In the following, we restrict our attention to the case when supp(σ) = γ is an arc different from the whole unit circle. For our purpose, without loss of generality, we may assume that $1 \notin \gamma$ and γ is symmetric with respect to $\mathbb{R}$.

Let $e^{\pm i\theta}$, $\theta \in (0, \pi)$, be the end points of γ ($\theta = \pi$ is not possible because supp(σ) contains infinitely many points). From the expression of μ it follows that

$$\text{supp}(\mu) = \left[-\cot\frac{\theta}{2}, \cot\frac{\theta}{2} \right]$$

(if we would have only assumed that supp(σ) $\subset \gamma$, then supp(μ) $\subset [-\cot(\theta/2), \cot(\theta/2)]$). In the following, $c = \cot(\theta/2)$ and φ denotes the conformal mapping of $\overline{\mathbb{C}} \setminus [-1, 1]$ onto the complement of the unit disk such that $\varphi(\infty) = \infty$, $\varphi'(\infty) > 0$.

LEMMA 3.   *Assume that* supp(σ) $= \gamma$, *where γ is an arc as described above and $\sigma' > 0$ a.e. on γ with respect to the Lebesgue measure. Then*

$$\lim_{n \to \infty} \frac{(\tau - i)^n \, \phi_n((\tau + i)/(\tau - i)) \, l_{n,n}(i)}{(2i)^n \, l_{n,n}(\tau)} = \frac{1}{|\varphi(i/c)|} \frac{\varphi(\tau/c) - \varphi(-i/c)}{\tau + i} \quad (14)$$

and

$$\lim_{n \to \infty} \frac{(\tau - i)^n \, \varphi_n((\tau + i)/(\tau - i)) \, l_{n,n}(i)}{i^n l_{n,n}(\tau) \, |l_{n,n}(i)|} = \sqrt{\frac{c}{2 \, |\varphi(i/c)|}} \frac{\varphi(\tau/c) - \varphi(-i/c)}{\tau + i}, \quad (15)$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus [-c, c]$. *In particular,*

$$\lim_{n \to \infty} \frac{2^n \alpha_n}{|l_{n,n}(i)|} = \sqrt{\frac{c}{2} \, |\varphi(i/c)|} . \quad (16)$$

Proof.   From Theorem 7 in [17] (see also Theorem 1 in [13]), we have

$$\lim_{n \to \infty} \frac{L_{n,n+j+1}(\tau)}{L_{n,n+j}(\tau)} = \frac{c}{2} \lim_{n \to \infty} \frac{l_{n,n+j+1}(\tau)}{l_{n,n+j}(\tau)} = \frac{c}{2} \varphi\left(\frac{\tau}{c}\right), \quad (17)$$

uniformly on each compact subset of $\mathbb{C} \setminus [-c, c]$, for each fixed integer j. In particular,

$$\lim_{n \to \infty} \frac{\kappa_{n,n+j+1}}{\kappa_{n,n+j}} = \frac{2}{c}. \quad (18)$$

Formula (10) gives

$$\frac{(\tau - i)^n \phi_n((\tau + i)/(\tau - i)) \, l_{n,n}(i)}{(2i)^n \, l_{n,n}(\tau)}$$

$$= \frac{2i}{\tau + i} \frac{(l_{n,n+1}(\tau))/(l_{n,n}(\tau)) - (l_{n,n+1}(-i))/(l_{n,n}(-i))}{(l_{n,n+1}(i))/(l_{n,n}(i)) - (l_{n,n+1}(-i))/(l_{n,n}(-i))}$$

$$= \frac{(l_{n,n+1}(\tau))/(l_{n,n}(\tau)) - (l_{n,n+1}(-i))/(l_{n,n}(-i))}{(\tau + i) \, \mathscr{I}(l_{n,n+1}(i))/(l_{n,n}(i))}.$$

Using (17), Eq. (14) immediately follows. Note that $\mathscr{I}\varphi(i/c) = |\varphi(i/c)|$ because it is a purely imaginary number in the upper half plane.

In order to derive (15) and (16), let us use (12). From that formula, we have

$$\frac{2^n \alpha_n}{|l_{n,n}(i)|} = \sqrt{\frac{\kappa_{n,n}}{\kappa_{n,n+1}} \frac{(l_{n,n+1}(i))/(l_{n,n}(i)) - (l_{n,n+1}(-i))/(l_{n,n}(-i))}{2i}}$$

$$= \sqrt{\frac{\kappa_{n,n}}{\kappa_{n,n+1}}} \, \mathscr{I}\left(\frac{l_{n,n+1}(i)}{l_{n,n}(i)}\right).$$

By using (17) and (18), we arrive at (16). Formula (15) is the product of (14) and (16). ∎

Note that the function on the right-hand side of (14) is never zero in $\overline{\mathbb{C}} \setminus [-c, c]$. Since $l_{n,n}(\tau)$ has no zeros in this set, one concludes from Hurwitz's Theorem that given a compact set $K \subset \overline{\mathbb{C}} \setminus [-c, c]$, $(\tau - i)^n \phi_n((\tau + i)/(\tau - i))$ has no zeros on K, for all sufficiently large n. In other words, if $\operatorname{supp}(\sigma) \subset \gamma$ and $\sigma' > 0$ a.e. on γ, then γ "attracts" all the zeros of the orthogonal polynomials with respect to σ in such a way that no point in $\overline{\mathbb{C}} \setminus \gamma$ may be an accumulation point of zeros, or a zero of infinitely many polynomials. Therefore, it makes sense to study the asymptotics of ratios of such polynomials not only on $[\,|\zeta| > 1\,]$ (as usual) but on all $\mathbb{C} \setminus \gamma$. We do this in the following without further reference to what we have just pointed out.

LEMMA 4. *Under the assumptions of Lemma* 3, *we have*

$$\lim_{n \to \infty} \frac{\phi_{n+1}((\tau + i)/(\tau - i))}{\phi_n((\tau + i)/(\tau - i))} \frac{l_{n,n+1}(\tau)}{l_{n+1,n+1}(\tau)} \frac{l_{n+1,n+1}(i)}{l_{n,n+1}(i)} = \frac{2i}{\tau - i} \frac{\varphi(\tau/c)}{\varphi(i/c)}, \qquad (19)$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus ([-c, c] \cup \{i\})$, *and*

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} \left| \frac{l_{n,n+1}(i)}{l_{n+1,n+1}(i)} \right| = \frac{1}{2} \left| \varphi\left(\frac{i}{c}\right) \right|. \qquad (20)$$

Proof. Formulas (14) and (16) applied to consecutive sequences of indexes indicate that

$$\lim_{n \to \infty} \frac{\phi_{n+1}((\tau+i)/(\tau-i))}{\phi_n((\tau+i)/(\tau-i))} \frac{l_{n,n}(\tau)}{l_{n+1,n+1}(\tau)} \frac{l_{n+1,n+1}(i)}{l_{n,n}(i)} = \frac{2i}{\tau-i},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash ([-c, c] \cup \{i\}]$, and

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} \left| \frac{l_{n,n}(i)}{l_{n+1,n+1}(i)} \right| = \frac{1}{2}.$$

Using (17) one sees that these relations are equivalent to (19) and (20) respectively. ∎

In order to conclude the proof of Theorem 1, we must investigate (should the limit exist)

$$\lim_{n \to \infty} \frac{l_{n+1,n+1}(\tau)}{l_{n,n+1}(\tau)}. \tag{21}$$

As opposed to (17), here we have ratios of polynomials of equal degree but orthogonal with respect to different measures. Nonetheless, $d\mu_n = (1+t^2) d\mu_{n+1}$; therefore, we have a problem of relative asymptotics. While $l_{n+1,n+1}$ is orthonormal with respect to $d\mu_{n+1}$, $l_{n,n+1}$ is orthonormal with respect to $g \, d\mu_{n+1}$, with $g(t) = (1+t^2)$. This problem was studied in [15] on the unit circle for sufficiently general "perturbing" functions and varying measures. In order to avoid, at this point, complicating matters with generality, let us calculate (21) for this particular case. For monic polynomials and rational perturbing functions, as we have here, the corresponding result was stated (without proof) in Theorem 10 of [17]. Unfortunately, there is a typo in the expression of the formula (see (3.2) of [18] where the formula is proved and appears correctly, but only the case of fixed measures is considered). On the other hand, we need the asymptotics of the leading coefficients of the orthonormal polynomials as well.

LEMMA 5. *Under the assumptions of Lemma* 3, *we have*

$$\frac{2}{c \, |\varphi(i/)|} \lim_{n \to \infty} \frac{l_{n+1,n+1}(\tau)}{l_{n,n+1}(\tau)} = \lim_{n \to \infty} \frac{L_{n+1,n+1}(\tau)}{L_{n,n+1}(\tau)}$$

$$= \frac{2(\tau-i)}{c\varphi(\tau/c) \, \overline{\varphi(i/c)}} \frac{\varphi(\tau/c) \, \overline{\varphi(i/c)} - 1}{\varphi(\tau/c) - \varphi(i/c)}, \tag{22}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash [-c, c]$. *In particular,*

$$\lim_{n \to \infty} \frac{\kappa_{n+1, n+1}}{\kappa_{n, n+1}} = \frac{c}{2} \left| \varphi \left(\frac{i}{c} \right) \right|. \tag{23}$$

Proof. Since

$$0 = \int_{\mathbb{R}} t^{v} L_{n, n+1}(t)(1 + t^2) \, d\mu_{n+1}(t), \qquad v = 0, ..., n,$$

then

$$(1 + \tau^2) \, L_{n, n+1}(\tau) = L_{n+1, n+3}(\tau) + \lambda_{n, 1} L_{n+1, n+2}(\tau) + \lambda_{n, 2} L_{n+1, n+1}(\tau), \tag{24}$$

or what is the same

$$(1 + \tau^2) \frac{L_{n, n+1}(\tau)}{L_{n+1, n+1}(\tau)} = \frac{L_{n+1, n+3}(\tau)}{L_{n+1, n+1}(\tau)} + \lambda_{n, 1} \frac{L_{n+1, n+2}(\tau)}{L_{n+1, n+1}(\tau)} + \lambda_{n, 2}. \tag{25}$$

Evaluating (25) at $\pm i$, we obtain the system of equations

$$\lambda_{n, 1} \frac{L_{n+1, n+2}(i)}{L_{n+1, n+1}(i)} + \lambda_{n, 2} = -\frac{L_{n+1, n+3}(i)}{L_{n+1, n+1}(i)},$$

$$\lambda_{n, 1} \frac{L_{n+1, n+2}(-i)}{L_{n+1, n+1}(-i)} + \lambda_{n, 2} = -\frac{L_{n+1, n+3}(-i)}{L_{n+1, n+1}(-i)}.$$

The determinant of this system has for a limit (use (17)) $(c/2)[\varphi(i/c) - \varphi(-i/c)] \neq 0$. Therefore, it is different from zero for all sufficiently large n (in fact, it is different from zero for all n because of (7) and (13)). Solving the system and taking limits, we obtain

$$\lim_{n \to \infty} \lambda_{n, 1} = -\frac{c}{2} \left[\varphi \left(\frac{i}{c} \right) + \varphi \left(\frac{-i}{c} \right) \right] \tag{26}$$

and

$$\lim_{n \to \infty} \lambda_{n, 2} = \left(\frac{c}{2} \right)^2 \varphi \left(\frac{i}{c} \right) \varphi \left(\frac{-i}{c} \right) = \left(\frac{c}{2} \right)^2 \left| \varphi \left(\frac{i}{c} \right) \right|^2. \tag{27}$$

Using (17) once more, and (25)–(27), we have

$$\lim_{n \to \infty} \frac{L_{n, n+1}(\tau)}{L_{n+1, n+1}(\tau)} = \left(\frac{c}{2} \right)^2 \left[\frac{\varphi(\tau/c) - \varphi(i/c)}{\tau - i} \right] \left[\frac{\varphi(\tau/c) - \varphi(-i/c)}{\tau + i} \right], \tag{28}$$

uniformly on each compact subset of $\overline{\mathbb{C}}\backslash[-c, c]$. It is easy to check that

$$\left[\frac{\varphi(w) - \varphi(w_0)}{2(w - w_0)}\right]\left[1 - \frac{1}{\varphi(w)\,\varphi(w_0)}\right] = 1, \qquad w, w_0 \in \overline{\mathbb{C}}\backslash[-1, 1].$$

Using this relation to substitute the second factor to the right of (28) and using that $\varphi(-i/c) = \overline{\varphi(i/c)}$, we arrive at the second equality in (22).

In order to prove (23), we proceed as follows. Note that $\deg[L_{n,n+1} - L_{n+1,n+1}] < n+1$; thus,

$$\int (L_{n,n+1} - L_{n+1,n+1})(t)\,L_{n,n+1}(t)\,d\mu_n(t) = 0.$$

Therefore, from (24), it follows that

$$\frac{1}{\kappa_{n,n+1}^2} = \int L_{n,n+1}^2(t)\,d\mu_n(t)$$

$$= \int (1 + t^2)\,L_{n,n+1}(t)\,L_{n+1,n+1}(t)\,d\mu_{n+1}(t)$$

$$= \lambda_{n,2}\int L_{n+1,n+1}^2(t)\,d\mu_{n+1}(t) = \frac{\lambda_{n,2}}{\kappa_{n+1,n+1}^2}.$$

This relation and (27) imply (23), which together with the second equality in (22) gives the first one. ∎

Now, let us proceed with the

Proof of Theorem 1. Let us assume that the arc γ is symmetric with respect to $\mathbb{R}$ and $1 \notin \gamma$. Then, according to (20) and (22),

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \frac{1}{2}\left|\varphi\left(\frac{i}{c}\right)\right| \lim_{n \to \infty}\left|\frac{l_{n+1,n+1}(i)}{l_{n,n+1}(i)}\right|$$

$$= \frac{c}{4}\left|\varphi\left(\frac{i}{c}\right)\right|^2 \lim_{n \to \infty}\left|\frac{L_{n+1,n+1}(i)}{L_{n,n+1}(i)}\right|. \tag{29}$$

For our purpose, it is easier to use the expression for the last limit derived from (28) (instead of (22)). That quantity is the absolute value of

$$\lim_{n \to \infty} \frac{L_{n+1,n+1}(i)}{L_{n,n+1}(i)} = \left[\frac{c}{4}\,\varphi'\left(\frac{i}{c}\right)\mathscr{I}\varphi\left(\frac{i}{c}\right)\right]^{-1}. \tag{30}$$

Since

$$\varphi'(w) = \frac{\varphi(w)}{\sqrt{w^2 - 1}} \qquad \text{and} \qquad \left| \mathscr{I}\varphi\left(\frac{i}{c}\right) \right| = \left| \varphi\left(\frac{i}{c}\right) \right|,$$

because $\varphi(i/c)$ is purely imaginary, from (29) and (30) it follows that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \left| \sqrt{\left(\frac{i}{c}\right)^2 - 1} \right| = \sqrt{1 + \tan^2 \frac{\theta}{2}} = \sec \frac{\theta}{2}.$$

Thus, we have proved the existence of limit in (1) and some may identify that quantity with $C(\gamma)^{-1}$. But if you do not, we shall get to that.

Formulas (19), (22), and (30) give

$$\lim_{n \to \infty} \frac{\phi_{n+1}((\tau + i)/(\tau - i))}{\phi_n((\tau + i)/(\tau - i))} = \frac{2i}{\tau - i} \frac{\varphi(\tau/c)}{\varphi(i/c)} \frac{2(\tau - i)}{c\varphi(\tau/c) \, \overline{\varphi(i/c)}}$$

$$\times \frac{\varphi(\tau/c) \, \overline{\varphi(i/c)} - 1}{\varphi(\tau/c) - \varphi(i/c)} \frac{c\varphi(i/c) \, \mathscr{I}\varphi(i/c)}{4\sqrt{(i/c)^2 - 1}}$$

$$= \frac{i\mathscr{I}\varphi(i/c)}{\overline{\varphi(i/c)} \sqrt{(i/c)^2 - 1}} \frac{\varphi(\tau/c) \, \overline{\varphi(i/c)} - 1}{\varphi(\tau/c) - \varphi(i/c)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash ([-c, c] \cup \{i\})$. Since $\varphi(i/c)$ is a purely imaginary number, then $i\mathscr{I}\varphi(i/c) = \varphi(i/c)$ and $\overline{\varphi(i/c)} = -\varphi(i/c)$. On the other hand, $\sqrt{(i/c)^2 - 1} = i \sec(\theta/2)$. Therefore,

$$\lim_{n \to \infty} \frac{\phi_{n+1}(\xi)}{\phi_n(\zeta)} = \cos \frac{\theta}{2} \Phi\left(i \frac{\zeta + 1}{\zeta - 1}\right), \tag{31}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash \gamma$, where

$$\Phi(\tau) = i \frac{\varphi(\tau/c) \, \overline{\varphi(i/c)} - 1}{\varphi(\tau/c) - \varphi(i/c)}.$$

By construction, $\Phi(i(\zeta + 1)/(\zeta - 1))$ is obviously a conformal representation of $\overline{\mathbb{C}} \backslash \gamma$ onto the complement of the unit disk such that ∞ goes to ∞. Let us check that its derivative at infinity is positive. In fact,

$$\lim_{\zeta \to \infty} \frac{1}{\zeta} \Phi\left(i\frac{\zeta+1}{\zeta-1}\right) = \lim_{\tau \to i} \frac{ic}{\tau+i} \frac{(\tau/c)-(i/c)}{\varphi(\tau/c)-\varphi(i/c)}\left[\varphi\left(\frac{\tau}{c}\right)\overline{\varphi\left(\frac{i}{c}\right)}-1\right]$$

$$= \frac{ic}{2i}\frac{\varphi(i/c)}{\varphi'(i/c)}\left[\overline{\varphi\left(\frac{i}{c}\right)}-\frac{1}{\varphi(i/c)}\right]$$

$$= \frac{c}{2}i\sqrt{\frac{1}{c^2}+1}\left(-\frac{2i}{c}\right)=\sec\frac{\theta}{2}>0. \qquad (32)$$

Therefore, we have shown that $G(\zeta) = \Phi(i(\zeta+1)/(\zeta-1))$.

The limit in (32) shows that $\cos(\theta/2)$ is the capacity of γ since $\log|G(\zeta)|$ is Green's function for $\overline{\mathbb{C}}\backslash\gamma$ and the capacity, in such a case, equals $|G'(\infty)|^{-1}$ (see [7, pg. 313]). Thus (31) is (2), and (3) follows immediately.[1]

If γ is not symmetric with respect to $\mathbb{R}$, the problem reduces to the previous case by a simple change of variables (rotation). ∎

Remark 1. If $\text{supp}(\sigma)$ consists of an arc γ plus a finite number of mass points e, and $\sigma' > 0$ a.e. on γ, then (1)–(3) remain valid, where (2), (3) take place uniformly on each compact subset of $\mathbb{C}\backslash(\gamma \cup e)$. In this case, each mass point "attracts" a zero of ϕ_n and the rest "concentrate" on γ. The proof follows arguments similar to the ones above and those employed in the proof of Theorem 1′ of [14].

As usual, given a polynomial $\Theta_n(\zeta)$ of degree n, $\Theta_n^*(\zeta) = \zeta^n\overline{\Theta_n(1/\bar{\zeta})}$ denotes its reversed polynomial.

COROLLARY 1. *Under the assumptions of Theorem* 1, *we have*

$$\lim_{n \to \infty} \frac{\phi_{n+1}^*(\zeta)}{\phi_n^*(\zeta)} = C(\gamma)\,\zeta\overline{G(1/\bar{\zeta})}, \qquad (33)$$

$$\lim_{n \to \infty} \frac{\varphi_{n+1}^*(\zeta)}{\varphi_n^*(\zeta)} = \zeta\overline{G(1/\bar{\zeta})}, \qquad (34)$$

and

$$\lim_{n \to \infty} \phi_{n+1}(0)\frac{\phi_n^*(\zeta)}{\phi_n(\zeta)} = C(\gamma)\,G(\zeta)-\zeta, \qquad (35)$$

uniformly on each compact subset of $\mathbb{C}\backslash\gamma$. *Also,*

$$\lim_{n \to \infty} |\phi_n(0)| = \sqrt{1-C^2(\gamma)}, \qquad \lim_{n \to \infty} \frac{\phi_{n+1}(0)}{\phi_n(0)} = C(\gamma)\,G(0). \qquad (36)$$

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[1] It took us hours of hard computing to double check all formulas and get them right. Being from Cuba we do not have Mathematica at our disposal. This is a joke for good friends.

Proof.   For $\mathbb{C}\backslash(\gamma \cup \{0\})$, formulas (33), (34) are immediate from the definition of reversed polynomial and (2), (3) respectively. Since the zeros of ϕ_n^* are bounded away from the origin, these two formulas extend to $\mathbb{C}\backslash\gamma$ by use of the maximum principle. As for (35) and (36), they follow from the well known relations (see [5, p. 11] or [32, pg. 293])

$$\phi_{n+1}(\zeta) = \zeta\phi_n(\zeta) + \phi_{n+1}(0)\,\phi_n^*(\zeta) \tag{37}$$

and

$$\frac{\alpha_{n-1}^2}{\alpha_n^2} = 1 - |\phi_n(0)|^2,$$

by use of (2) and (1) respectively. The second limit in (36) is (35) at $\zeta = 0$. ∎

Remark 2.   In connection with the second limit in (36), it is not difficult to check that $C(\gamma)\,G(0) = e^{-i\theta_0}$, where θ_0 is the angle one must rotate γ in order to make it symmetric with respect to $\mathbb{R}$, leaving out $z = 1$.

Remark 3.   In the case of the unit circle, an analogue of the Blumenthal–Nevai class of orthogonal polynomials on $\mathbb{R}$ is not known. We suspect that such class is the one formed by those sequences of orthogonal polynomials on Γ whose reflection coefficients satisfy

$$\lim_{n \to \infty} |\phi_n(0)| = a, \qquad \lim_{n \to \infty} \frac{\phi_{n+1}(0)}{\phi_n(0)} = b, \tag{38}$$

where $a \in (0, 1)$.

Under the more restrictive condition

$$\lim_{n \to \infty} \phi_n(0) = \alpha, \qquad 0 < |\alpha| < 1,$$

the support of the orthogonality measure σ has been well studied (see [4], and Theorems 6 and 10 in [6]). In particular, if

$$|\phi_n(0) - \alpha| = 0(r^n), \qquad r < 1,$$

then the measure falls in the category described in Remark 1 (see Theorems 4.1 and 4.2 in [26]); therefore, in this case (2) and (3) hold uniformly on each compact subset of $\mathbb{C}\backslash\text{supp}(\sigma)$. We think that this is true whenever conditions (38) take place.

Before ending this section, let us apply the results above to obtain asymptotic formulas for the reproducing kernel function $K_n(\zeta, \eta)$ and the

Christoffel function $w_n(\zeta)$ relative to the measure σ. We recall that these functions are defined by the relations

$$K_n(\zeta, \eta) = \sum_{k=0}^{n-1} \varphi_k(\zeta) \, \overline{\varphi_k(\eta)} = \frac{\varphi_n^*(\zeta) \, \overline{\varphi_n^*(\eta)} - \varphi_n(\zeta) \, \overline{\varphi_n(\eta)}}{1 - \zeta\bar{\eta}} \qquad (39)$$

and

$$w_n(\zeta) = K_n^{-1}(\zeta, \zeta). \qquad (40)$$

First, let us study some properties of the function $F(\zeta) = C(\gamma) \, G(\zeta) - \zeta$.

LEMMA 6.   *Under the assumptions of Theorem* 1, *we have*

(i) $|F(\zeta)| \begin{cases} > \sqrt{1 - C^2(\gamma)}, & |\zeta| < 1, \\ = \sqrt{1 - C^2(\gamma)}, & [\,|\zeta| = 1\,]\backslash\gamma, \\ < \sqrt{1 - C^2(\gamma)}, & |\zeta| > 1. \end{cases}$

(ii) $F(\zeta) \, \overline{F(1/\bar{\zeta})} = 1 - C^2(\gamma), \; \zeta \in \overline{\mathbb{C}}\backslash\gamma.$

(iii) $F(\zeta)$ *is one–one in* $\overline{\mathbb{C}}\backslash\gamma.$

(iv)   *For all* $\zeta \in \mathbb{C}\backslash\gamma,$

$$F(\zeta) = \tfrac{1}{2}[\, C(\gamma) \, G(0) - \zeta + \sqrt{(C(\gamma) \, G(0) - \zeta)^2 + 4\zeta(1 - C^2(\gamma)) \, C(\gamma) \, G(0)}\,],$$

where the root is taken so that $F(0) = C(\gamma) \, G(0)$. *In particular, if* γ *is symmetric with respect to* $\mathbb{R}$, $1 \notin \gamma$, *then*

$$F(\zeta) = \frac{1 - \zeta + \sqrt{(1 + \zeta)^2 - 4\zeta C^2(\gamma)}}{2}$$

and

(v) $F_+(\zeta) = -ie^{i\alpha/2}\left[\sqrt{C^2(\gamma) - \cos^2 \frac{\alpha}{2}} + \sin \frac{\alpha}{2}\right], \qquad \zeta = e^{i\alpha},$

$ F_-(\zeta) = ie^{i\alpha/2}\left[\sqrt{C^2(\gamma) - \cos^2 \frac{\alpha}{2}} - \sin \frac{\alpha}{2}\right], \qquad \zeta = i^{i\alpha},$

$\theta < \alpha < 2\pi - \theta$, $e^{\pm i\theta}$ *are the endpoints of* γ, *and* F_+, F_- *denote the limit values of* F *as the variable tends to points in* γ *from the interior and exterior of the unit circle respectively.* (*Note that* $F_+(\zeta) \, \overline{F_-(\zeta)} = 1 - C^2(\gamma), \; \zeta \in \gamma.$)

(vi)   *The nearest and furthest points from the origin of the image of* γ *by* F *have modulus* $1 - C(\gamma)$ *and* $1 + C(\gamma)$ *respectively.*

(vii) *The following relation takes place for all* $(\zeta, \eta) \in (\mathbb{C}\backslash\gamma)^2,$

$$(F(\zeta) \, \overline{F(\eta)} + C^2(\gamma) - 1)(G(\zeta) \, \overline{G(\eta)} - 1) = (1 - C^2(\gamma))(1 - \zeta\bar{\eta}).$$

Proof. The function $F(\zeta)$ is the uniform limit on compact subsets of $\overline{\mathbb{C}}\backslash\gamma$ of the sequence $\{\phi_{n+1}(0)(\phi_n^*(\zeta)/\phi_n(\zeta))\}$ (see (35)). Given a compact set $K \subset \overline{\mathbb{C}}\backslash\gamma$, the functions in the sequence are never zero on K, for all sufficiently large n. Therefore, their limit F must either be identically equal to zero on $\overline{\mathbb{C}}\backslash\gamma$ or never equal to zero on that region. Since $|F(0)| = 1$ (see (36)), then $F(\zeta) \neq 0$, $\zeta \in \overline{\mathbb{C}}\backslash\gamma$.

For each fixed n

$$\left| \phi_{n+1}(0) \frac{\phi_n^*(\zeta)}{\phi_n(\zeta)} \right| \begin{cases} > |\phi_{n+1}(0)|, & |\zeta| < 1, \\ = |\phi_{n+1}(0)|, & |\zeta| = 1, \\ < |\phi_{n+1}(0)|, & |\zeta| > 1. \end{cases}$$

This is true because $\phi_n^*(\zeta)/\phi_n(\zeta)$ is the reverse of a Blaschke product. Using (35) and (36), we obtain the relations (i) with $\geqslant$ and $\leqslant$ in the first and third parts. But equality is not possible by the minimum and maximum principle applied to F in $[|\zeta| < 1]$ and $[|\zeta| > 1]$ respectively, because F cannot be identically equal to a constant (that would imply that G is an affine transformation which is not possible since it transforms $\overline{\mathbb{C}}\backslash\gamma$ conformally onto the complement of the unit disk).

Formula (ii) is an immediate consequence of the second relation in (i) and the symmetry principle of analytic functions.

In order to prove the rest of the statements, let us find a compact analytic expression for F. To this end, we shall use (37) and its symmetric form

$$\phi_{n+1}^*(\zeta) = \phi_n^*(\zeta) + \zeta \overline{\phi_{n+1}(0)}\, \phi_n(\zeta). \tag{41}$$

Dividing (41) by (37) and multiplying either sides by $\phi_{n+2}(0)$, we obtain

$$\phi_{n+2}(0) \frac{\phi_{n+1}^*(\zeta)}{\phi_{n+1}(\zeta)} = \frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} \frac{\phi_{n+1}(0)((\phi_n^*(\zeta))/(\phi_n(\zeta))) + \zeta\, |\phi_{n+1}(0)|^2}{\zeta + \phi_{n+1}(0)((\phi_n^*(\zeta))/(\phi_n(\zeta)))}.$$

Taking limits, using (35) and (36), we have

$$F(\zeta) = \frac{F(\zeta) + \zeta(1 - C^2(\gamma))}{\zeta + F(\zeta)}\, C(\gamma)\, G(0)$$

(note that $\zeta + F(\zeta) = C(\gamma)\, G(\zeta) \neq 0$). This last relation is equivalent to

$$F^2(\zeta) - C(\gamma)\, G(0)\, F(\zeta) = \zeta[(1 - C^2(\gamma))\, C(\gamma)\, G(0) - F(\zeta)]. \tag{42}$$

Let us prove that $(1 - C^2(\gamma))\, C(\gamma)\, G(0) - F(\zeta) \neq 0$ for $\zeta \in \overline{\mathbb{C}}\backslash\gamma$. In fact, if $(1 - C^2(\gamma))\, C(\gamma)\, G(0) - F(\zeta_1) = 0$ for some $\zeta_1 \in \overline{\mathbb{C}}\backslash\gamma$, then

$$F^2(\zeta_1) = C(\gamma)\, G(0)\, F(\zeta_1),$$

or what is the same

$$(1 - C^2(\gamma))^2 \, C^2(\gamma) \, G^2(0) = (1 - C^2(\gamma)) \, C^2(\gamma) \, G^2(0).$$

This is impossible because $0 < C(\gamma) < 1$ and $G(0) \neq 0$.

Therefore, from (42) it readily follows that

$$\frac{F^2(\zeta) - C(\gamma) \, G(0) \, F(\zeta)}{(1 - C^2(\gamma)) \, C(\gamma) \, G(0) - F(\zeta)} = \zeta. \tag{43}$$

Assume that $F(\zeta_1) = F(\zeta_2)$, $\zeta_1, \zeta_2 \in \overline{\mathbb{C}} \backslash \gamma$; then, from (43), we obtain that $\zeta_1 = \zeta_2$. In other words, F is one to one on $\overline{\mathbb{C}} \backslash \gamma$ as stated in (iii).

Formula (42) may be rewritten as

$$F^2(\zeta) - (C(\gamma) \, G(0) - \zeta) \, F(\zeta) - \zeta(1 - C^2(\gamma)) \, C(\gamma) \, G(0) = 0.$$

Solving this quadratic equation for F, we obtain

$$F(\zeta) = \tfrac{1}{2} [\, C(\gamma) \, G(0) - \zeta + \sqrt{(C(\gamma) \, G(0) - \zeta)^2 + 4\zeta(1 - C^2(\gamma)) \, C(\gamma) \, G(0)} \,] \tag{44}$$

(the root is taken so that $F(0) = C(\gamma) \, G(0)$). Relation (44) is (iv), and in the symmetric case, with $1 \notin \gamma$, it reduces to the second formula in (iv) since then $C(\gamma) \, G(0) = 1$.

By the principle of correspondence of boundaries under conformal representations, we know that, for each $\zeta \in \gamma$, there exist $F_+(\zeta)$ and $F_-(\zeta)$. From (ii) these limit values must be symmetric with respect to the circle of center 0 and radius $\sqrt{1 - C^2(\gamma)}$. From (i) it follows that $F_+(\zeta)$ is the one of greater module and $F_-(\zeta)$ the one of smaller absolute value.

From this and formula (iv), it follows that ($\zeta = e^{i\alpha}$, $\theta < \alpha < 2\pi - \theta$, $|\cos(\alpha/2)| \leqslant C(\gamma)$)

$$\begin{aligned}
F_\pm(\zeta) &= \frac{1}{2} \, [\, 1 + \zeta + \sqrt{(1 + \zeta)^2 - 4\zeta C^2(\gamma)} \,] - \xi \\
&= e^{i\alpha/2} \left[\frac{e^{i\alpha/2} + e^{-i\alpha/2}}{2} + \sqrt{\frac{1}{2} \, (e^{i\alpha/2} + e^{-i\alpha/2})^2 - C^2(\gamma)} - e^{i\alpha/2} \right] \\
&= e^{i\alpha/2} \left[\sqrt{\cos^2 \frac{\alpha}{2} - C^2(\gamma)} - i \sin \frac{\alpha}{2} \right] \\
&= i e^{i\alpha/2} \left[\mp \sqrt{C^2(\gamma) - \cos^2 \frac{\alpha}{2}} - \sin \frac{\alpha}{2} \right].
\end{aligned}$$

Thus we have proved (v).

In the symmetric case with $1 \notin \gamma$, the statement in (vi) is an immediate consequence of formula (v). Moreover, $F_-(-1) = 1 - C(\gamma) \leqslant |F_-(\zeta)| \leqslant |F_+(\zeta)| \leqslant F_+(-1) = 1 + C(\gamma)$. The general case reduces to the symmetric one by a rotation in the variable which does not affect the absolute value of the function (compare both formulas in (iv) and take into consideration that $|C(\gamma) G(0)| = 1$ according to (36)).

Finally, let us prove (vii). Since $C(\gamma) G(z) = F(\zeta) + \zeta$, it is equivalent to prove that

$$(F(\zeta) \overline{F(\eta)} + C^2(\gamma) - 1)((F(\zeta) + \zeta)(\overline{F(\eta)} + \bar{\eta}) - C^2(\gamma))$$
$$= C^2(\gamma)(1 - C^2(\gamma))(1 - \zeta\bar{\eta}),$$

for all $(\zeta, \eta) \in (\mathbb{C}\backslash\gamma)^2$.

From (42), we obtain

$$F^2(\zeta) + \zeta F(\zeta) = C(\gamma) G(0)(F(\zeta) + (1 - C^2(\gamma)) \zeta).$$

Using this relation and that $|C(\gamma) G(0)| = 1$, it follows that

$$(F(\zeta) \overline{F(\eta)} + C^2(\gamma) - 1)((F(\zeta) + \zeta)(\overline{F(\eta)} + \bar{\eta}) - C^2(\gamma))$$
$$= (F^2(\zeta) + \zeta F(\zeta))(\overline{F^2(\eta)} + \overline{\eta F(\eta)}) - C^2(\gamma) F(\zeta) \overline{F(\eta)}$$
$$+ (C^2(\gamma) - 1)(F(\zeta) + \zeta)(\overline{F(\eta)} + \bar{\eta}) + C^2(\gamma)(1 - C^2(\gamma))$$
$$= |C(\gamma) G(0)|^2 (F(\zeta) + (1 - C^2(\gamma)) \zeta)(\overline{F(\eta)} + (1 - C^2(\gamma)) \bar{\eta})$$
$$- C^2(\gamma) F(\zeta) \overline{F(\eta)} + (C^2(\gamma) - 1) F(\zeta) \overline{F(\eta)} + (C^2(\gamma) - 1) \bar{\eta} F(\zeta)$$
$$+ (C^2(\gamma) - 1) \zeta \overline{F(\eta)} + (C^2(\gamma) - 1) \zeta\bar{\eta} + C^2(\gamma)(1 - C^2(\gamma))$$
$$= C^2(\gamma)(1 - C^2(\gamma))(1 - \zeta\bar{\eta}),$$

which is what we needed to prove. ∎

Remark 4.   From (vii) in Lemma 6, it follows that in $(\mathbb{C}\backslash\gamma)^2$

$$F(\zeta) \overline{F(\eta)} - 1 + C^2(\gamma) = 0 \Leftrightarrow \eta = 1/\bar{\zeta},$$

because in this set $G(\zeta) \overline{G(\eta)} - 1$ is never zero.

We are ready for the proof of

COROLLARY 2.   *Under the assumptions of Theorem* 1, *we have*

$$\lim_{n \to \infty} \frac{K_n(\zeta, \eta)}{\varphi_n(\zeta) \overline{\varphi_n(\eta)}} = \frac{1}{G(\zeta) \overline{G(\eta)} - 1}, \tag{45}$$

uniformly on each compact subset of $(\overline{\mathbb{C}}\backslash\gamma)^2$. *In particular,*

$$\lim_{n\to\infty}\frac{1}{w_n(\zeta)\,|\varphi_n(\zeta)|^2}=\frac{1}{|G(\zeta)|^2-1}, \tag{46}$$

uniformly on each compact subset of $\overline{\mathbb{C}}\backslash\gamma$. *Since the right-hand sides of* (45) *and* (46) *are different from zero on* $(\mathbb{C}\backslash\gamma)^2$ *and* $\mathbb{C}\backslash\gamma$ *respectively, these formulas may be reversed with uniform limit on compact subsets of* $(\mathbb{C}\backslash\gamma)^2$ *and* $\mathbb{C}\backslash\gamma$ *respectively.*

Proof. We can rewrite (39) as

$$\frac{K_n(\zeta,\eta)}{\varphi_n(\zeta)\,\overline{\varphi_n(\eta)}}=\frac{1}{|\phi_{n+1}(0)|^2}$$

$$\times\frac{\phi_{n+1}(0)((\phi_n^*(\zeta))/\phi_n(\zeta))\,\overline{\phi_{n+1}(0)(\phi_n^*(\eta)/\phi_n(\eta))}-|\phi_{n+1}(0)|^2}{1-\zeta\bar\eta}. \tag{47}$$

Using (47), (35), and the first part of (36), we obtain

$$\lim_{n\to\infty}\frac{K_n(\zeta,\eta)}{\varphi_n(\zeta)\,\overline{\varphi_n(\eta)}}=\frac{1}{1-C^2(\gamma)}\frac{(C(\gamma)\,G(\zeta)-\zeta)(C(\gamma)\,\overline{G(\eta)}-\bar\eta)+C^2(\gamma)-1}{1-\zeta\bar\eta},$$

uniformly on each compact subset of

$$(\overline{\mathbb{C}}\backslash\gamma)^2\backslash\{(\zeta,1/\bar\zeta):\zeta\in\overline{\mathbb{C}}\backslash\gamma\}.$$

This formula has the problem that its right-hand side is not determined for $\eta=1/\bar\zeta$. But, according to (vii) in Lemma 6 (see also Remark 4), the right-hand side equals $1/(G(\zeta)\,G(\bar\eta)-1)$ on $(\overline{\mathbb{C}}\backslash\gamma)^2\backslash\{(\zeta,1/\bar\zeta):\zeta\in\overline{\mathbb{C}}\backslash\gamma\}$. Thus, we have proved (45) on compact subsets of $(\overline{\mathbb{C}}\backslash\gamma)^2\backslash\{(\zeta,1/\bar\zeta):\zeta\in\overline{\mathbb{C}}\backslash\gamma\}$. In order to conclude the proof of (45), it is sufficient to show that on a neighborhood of each point in $\{(\zeta,1/\bar\zeta):\zeta\in\overline{\mathbb{C}}\backslash\gamma\}$, (45) takes place.

Fix $\zeta_0\in\mathbb{C}\backslash(\gamma\cup\{0\})$. Take $r>0$ sufficiently small so that the disk $B_1=\{\zeta:|\zeta-\zeta_0|\leqslant r\}$ is contained in $\mathbb{C}\backslash(\gamma\cup\{0\})$. From symmetry with respect to the unit circle, we have that the disk $B_2=\{1/\bar\zeta:\zeta\in B_1\}$ is also contained in $\mathbb{C}\backslash(\gamma\cup\{0\})$. Therefore, $B_1\times B_2$ is a compact neighborhood of $(\zeta_0,1/\overline{\zeta_0})$ contained in $(\mathbb{C}\backslash\gamma)^2$. Let us prove that on $B_1\times B_2$ there is uniform convergence. Let $r_1>r$ be such that $\{\zeta:|\zeta-\zeta_0|\leqslant r_1\}$ is contained in $\mathbb{C}\backslash(\gamma\cup\{0\})$. Set $\rho=\{1/\bar\zeta:|\zeta-\zeta_0|=r_1\}$. Obviously, $\rho\subset\mathbb{C}\backslash\gamma$ and it is a simple curve which surrounds the set B_2. From Cauchy's Theorem, we

have that for all $(\zeta, \eta) \in B_1 \times B_2$ (and all sufficiently large n so that the zeros of φ_n lie in the exterior of B_1 and ρ)

$$\frac{\overline{K_n(\zeta, \eta)}}{\overline{\varphi_n(\zeta)}\,\varphi_n(\eta)} - \frac{1}{\overline{G(\zeta)}\,G(\eta) - 1} = F_n(\zeta, \eta) - F(\zeta, \eta)$$

$$= \frac{1}{2\pi i} \int_\rho \frac{F_n(\zeta, z) - F(\zeta, z)}{z - \eta}\,dz.$$

The set $B_1 \times \rho$ is a compact subset of $(\mathbb{C}\backslash\gamma)^2 \backslash \{(\zeta, 1/\bar{\zeta}) : \zeta \in \mathbb{C}\backslash\gamma\}$. Therefore, from what was proved above, it follows that $F_n(\zeta, z)$ converges uniformly to $F(\zeta, z)$ on $B_1 \times \rho$. Using the integral expression, we immediately obtain that

$$\lim_{n \to \infty} \frac{\overline{K_n(\zeta, \eta)}}{\overline{\varphi_n(\zeta)}\,\varphi(\eta)} = \frac{1}{\overline{G(\zeta)}\,G(\eta) - 1},$$

uniformly on $B_1 \times B_2$, as we needed to prove. For the points $(0, \infty)$ and $(\infty, 0)$, the proof may be carried out following the same line of reasoning. For $(\infty, 0)$, take $r > 1$ and $B_1 = \{\zeta : |\zeta| \geqslant r\}$, then proceed analogously with $r > r_1 > 1$. For the point $(0, \infty)$ we start out by defining the sets from the second variable and integrate with respect to the first. With this we conclude the proof. ∎

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As in Section 2, we concentrate on the case when γ is symmetric with respect to $\mathbb{R}$ and $1 \notin \gamma$. Let h be a measurable function defined on γ such that $h\,d\sigma$ is a finite positive Borel measure on γ. In the following, $\phi_n(h; \zeta)$ denotes the nth monic orthogonal polynomial with respect to $h\,d\sigma$ and $L_{n,n}(h; \tau)$ the nth monic orthogonal polynomial with respect to $h((t+i)/(t-i))\,d\mu_n(t)$. Analogously, we denote the orthonormal polynomials and the leading coefficients, relative to these measures. For the orthogonal polynomials and leading coefficients relative to $d\sigma$ and $d\mu_n(t)$, we maintain the previous notation. Recall that $d\mu(t) = d\sigma((t+i)/(t-i))$ and $d\mu_n(t) = d\mu(t)/(1 + t^2)^n$.

LEMMA 7. *Under the assumptions of Lemma 3 with respect to σ, and $h > 0$ a.e. on γ, we have*

$$\lim_{n \to \infty} \frac{\phi_n\,(h; (\tau+i)/(\tau-i))\,l_{n,n}(\tau)\,l_{n,n}(h; i)}{\phi_n((\tau+i)/(\tau-i))\,l_{n,n}(h, \tau)\,l_{n,n}(i)} = 1, \tag{48}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus [-c, c]$, *and*

$$\lim_{n \to \infty} \frac{\alpha_n(h)}{\alpha_n} \frac{|l_{n,n}(i)|}{|l_{n,n}(h; i)|} = 1. \tag{49}$$

Proof. Note that not only $d\sigma$ but also $h \, d\sigma$ satisfies the conditions of Lemma 3. Therefore, (48) and (49) are immediate consequences of (14) and (16) applied to the sequences $\{\phi_n(h, \zeta)\}$, $\{\phi_n(\zeta)\}$, $\{\alpha_n(h)\}$, and $\{\alpha_n\}$. ∎

We must investigate the asymptotic behavior of $\{l_{n,n}(h; \tau)/l_{n,n}(\tau)\}$. We will translate the problem to the whole unit circle, where the corresponding results are at hand. To this end, we will connect the orthogonal polynomials we have on $[-c, c]$ with orthogonal polynomials on Γ. We do this for $\{h_{n,n}(\tau)\}$, and then for $\{h_{n,n}(h; \tau)\}$ the formulas are obvious.

Define on Γ a measure $\tilde{\sigma}$, symmetric with respect to the real axis, as follows. Let E be a measurable set on the upper half of the unit circle, then

$$\tilde{\sigma}(E) = \mu\{cx : x = \cos \alpha, \text{ with } e^{i\alpha} \in E\}.$$

In other words,

$$d\tilde{\sigma}(u) = d\mu(c \cos \alpha), \qquad u = e^{i\alpha},$$

in the upper half circle. In the lower half, $\tilde{\sigma}$ is defined by symmetry with respect to $\mathbb{R}$.

Note that $(1 + c^2 \cos^2 \alpha)^n$ is a positive trigonometric polynomial on $\alpha \in [0, 2\pi]$. It is well known that there exists an algebraic polynomial whose zeros lie in $[|\xi| < 1]$, such that its module to the square takes the same value at $u = e^{i\alpha}$ as the trigonometric polynomial at α (see Theorem 1.2.2 in [32]). Let us take a look at this algebraic polynomial.

It is easy to verify that

$$\left| \left(u - \frac{1}{\varphi(i/c)} \right) \left(u - \frac{1}{\overline{\varphi(i/c)}} \right) \right|^2 = \left(u - \frac{1}{\varphi(i/c)} \right) \left(\bar{u} - \frac{1}{\varphi(i/c)} \right) \left(u - \frac{1}{\overline{\varphi(i/c)}} \right)$$

$$\times \left(\bar{u} - \frac{1}{\overline{\varphi(i/c)}} \right)$$

$$= \frac{4(1 + c^2 \cos^2 \alpha)}{c^2 |\varphi(i/c)|^2}, \qquad u = e^{i\alpha}.$$

Denote

$$w_{2n}(u) = \left(u - \frac{1}{\varphi(i/c)} \right)^n \left(u - \frac{1}{\overline{\varphi(i/c)}} \right)^n,$$

then

$$|w_{2n}(u)|^2 = \left(\frac{2}{c\,|\varphi(i/c)|}\right)^{2n}(1 + c^2\cos^2\alpha)^n, \qquad u = e^{i\alpha}.$$

Therefore,

$$\begin{aligned}
d\tilde{\sigma}_{2n}(u) &= \frac{d\tilde{\sigma}(u)}{|w_{2n}(u)|^2} \\
&= \left(\frac{c\,|\varphi(i/c)|}{2}\right)^{2n}\frac{d\mu(c\cos\alpha)}{(1 + c^2\cos^2\alpha)^n}, \qquad u = e^{i\alpha}.
\end{aligned}$$

Since $l_{n,n}(cx)$ is the nth orthonormal polynomial with respect to the measure $d\mu(cx)/(1 + c^2x^2)^n$, $x \in [-1, 1]$, then $(2/c\,|\varphi(i/c)|)^n\,l_{n,n}(cx)$ is the nth orthonormal polynomial with respect to $(c\,|\varphi(i/c)|/2)^{2n}\,d\mu(cx)/(1 + c^2x^2)^n$.

Let $\tilde{\varphi}_{2n,2n}$ be the $(2n)$th orthonormal polynomial, with positive leading coefficient, with respect to $(1/2\pi)\,d\tilde{\sigma}_{2n}$. As usual, $\tilde{\varphi}^*_{2n,2n}$ denotes its reversed polynomial and $\tilde{\phi}_{2n,2n}$ the corresponding monic orthogonal polynomial. Since n is fixed, the following well-known formula takes place (see [32, Theorem 11.5] and [5, Theorem 9.1])

$$\left(\frac{2}{c\,|\varphi(i/c)|}\right)^n l_{n,n}(c\chi) = \frac{\tilde{\varphi}_{2n,2n}(\xi) + \tilde{\varphi}^*_{2n,2n}(\xi)}{\xi^n\sqrt{2\pi(1 + \tilde{\phi}_{2n,2n}(0))}}, \qquad \chi = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right). \tag{50}$$

Analogously,

$$\left(\frac{2}{c\,|\varphi(i/c)|}\right)^n l_{n,n}(h; c\chi) = \frac{\tilde{\varphi}_{2n,2n}(h;\xi) + \tilde{\varphi}^*_{2n,2n}(h;\xi)}{\xi^n\sqrt{2\pi(1 + \tilde{\phi}_{2n,2n}(h;0))}}, \qquad \chi = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right), \tag{51}$$

where $\tilde{\phi}_{2n,2n}(h;\xi)$ and $\tilde{\varphi}_{2n,2n}(h;\xi)$ denote the monic and orthonormal polynomials of degree $2n$ with respect to the measure

$$\frac{1}{2\pi}\frac{d\tilde{\sigma}_h(u)}{|w_{2n}(u)|^2}, \qquad d\tilde{\sigma}_h(u) = h\left(\frac{c\cos\alpha + i}{c\cos\alpha - i}\right)d\tilde{\sigma}(u), \qquad u = e^{i\alpha}.$$

LEMMA 8. *Under the assumptions of Lemma 3 with respect to σ, and $h > 0$ a.e. on γ, we have*

$$\lim_{n\to\infty}\frac{l_{n,n}(h; c\chi)}{l_{n,n}(c\chi)}\frac{\tilde{\varphi}_{2n,2n}(\xi)}{\tilde{\varphi}_{2n,2n}(h;\xi)} = 1, \qquad \chi = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right), \tag{52}$$

uniformly on each compact subset of $[\,|\xi| > 1]$.

Proof. From (50) and (51), it follows that

$$\frac{l_{n,\,n}(h;c\chi)}{l_{n,\,n}(c\chi)}\frac{\tilde{\varphi}_{2n,\,2n}(\xi)}{\tilde{\varphi}_{2n,\,2n}(h;\xi)}=\frac{1+((\tilde{\varphi}^*_{2n,\,2n}(h;\xi))/(\tilde{\varphi}_{2n,\,2n}(h;\xi)))}{1+((\tilde{\varphi}^*_{2n,\,2n}(\xi))/(\tilde{\varphi}_{2n,\,2n}(\xi)))}\frac{\sqrt{1+\tilde{\phi}_{2n,\,2n}(0)}}{\sqrt{1+\tilde{\phi}_{2n,\,2n}(h;0)}}. \tag{53}$$

Since $\tilde{\sigma}'(u)>0$ a.e. on Γ and $h((c\cos\alpha+i)/(c\cos\alpha-i))>0$ a.e. on Γ ($e^{i\alpha}=u$), we have from Lemmas 4 and 5 in [13] that

$$\lim_{n\to\infty}\tilde{\phi}_{2n,\,2n}(h,0)=\lim_{n\to\infty}\tilde{\phi}_{2n,\,2n}(0)=0 \tag{54}$$

and

$$\lim_{n\to\infty}\frac{\tilde{\varphi}^*_{2n,\,2n}(h;\xi)}{\tilde{\varphi}_{2n,\,2n}(h;\xi)}=\lim_{n\to\infty}\frac{\tilde{\varphi}^*_{2n,\,2n}(\xi)}{\tilde{\varphi}_{2n,\,2n}(\xi)}=0, \tag{55}$$

uniformly on each compact subset of $[|\xi|>1]$. From (53)–(55), we obtain (52). ∎

We referred to [13] because the zeros of $\{w_{2n}(\xi)\}$ are bounded away from the unit circle (in fact, they are fixed at two points within the unit circle), which is the kind of varying weights considered there. In [14] and [17], you will find generalizations of Lemmas 4 and 5 from [13] which also lead to (54), (55).

We have reduced the problem to the study of the asymptotic behavior of the sequence $\{\varphi_{2n,\,2n}(h,\xi)/\varphi_{2n,\,2n}(\xi)\}$. This question is considered in Theorem 1 of [15]. Unfortunately, there, the corresponding result is proved for the particular case when $w_{2n}(\xi)=(\xi-1)^{2n}$, which was the case needed for the application to relative asymptotics with respect to fixed measures supported on unbounded sets. The proof in the present situation follows using the same arguments as those in [15]. One good reason why this is so is that here we are much better off. No singularities are placed on the support of the measure; therefore, Carleman-type conditions for the uniqueness of analytic functions are not needed. You will see that no special use is made of the form of $\{w_{2n}\}$ as long as Lemma 1 of [15] is satisfied. For a proof of the statements of that Lemma in the present setting, we refer the reader to Lemmas 1, 4, and 5 of [13].

LEMMA 9. *Under the assumptions of Theorem 2 with γ symmetric with respect to $\mathbb{R}$ and $1\notin\gamma$, we have*

$$\lim_{n\to\infty}\frac{\tilde{\varphi}_{2n,\,2n}(h;\xi)}{\tilde{\varphi}_{2n,\,2n}(\xi)}=S_h(\xi), \tag{56}$$

uniformly on each compact subset of $[|\xi| > 1]$, *where*

$$S_h(\xi) = \exp\left\{\frac{1}{4\pi}\int_\Gamma \log h\left(\frac{c\cos\alpha + i}{c\cos\alpha - i}\right)\frac{u+\xi}{u-\xi}|du|\right\}, \qquad u = e^{i\alpha}.$$

Proof. From the assumptions, it is obvious that there exists an algebraic polynomial Q such that

$$Q(u)\left[h\left(\frac{c\cos\alpha + i}{c\cos\alpha - i}\right)\right]^{\pm 1}\in L_\infty(\tilde\sigma), \qquad u = e^{i\alpha}.$$

On the other hand, $h((c\cos\alpha + i)/(c\cos\alpha - i)) > 0$ a.e. on Γ, $\tilde\sigma'(u) > 0$ a.e. on Γ, and the zeros of $\{w_{2n}\}$ are bounded away from Γ. It only remains to follow the scheme of the proof of Theorem 1 in [15]. ∎

Now, we are ready for the

Proof of Theorem 2. First, let us consider the case when γ is symmetric with respect to $\mathbb{R}$ and $1 \notin \gamma$. The assumptions in Theorem 2 imply the conditions of Lemma 3, and $h > 0$ a.e. on γ. Therefore, we can use Lemmas 7–9. From (48), (52), and (56), we obtain

$$\lim_{n\to\infty}\frac{\phi_n(h; (\tau+i)/(\tau-i))}{\phi_n((\tau+i)/(\tau-i))} = \frac{S_h(\varphi(\tau/c))}{S_h(\varphi(i/c))},$$

uniformly on each compact subset of $\bar{\mathbb{C}}\backslash[-c, c]$. This is equivalent to saying that

$$\lim_{n\to\infty}\frac{\phi_n(h; \zeta)}{\phi_n(\zeta)} = \frac{S_h(\varphi(i(\zeta+1)/c(\zeta-1)))}{S_h(\varphi(i/c))}, \tag{57}$$

uniformly on each compact subset of $\bar{\mathbb{C}}\backslash\gamma$. On the other hand, from (49), (52), and (56) it follows that

$$\lim_{n\to\infty}\frac{\alpha_n(h)}{\alpha_n} = \left|S_h\left(\varphi\left(\frac{i}{c}\right)\right)\right|. \tag{58}$$

Therefore, using (57) and (58), we obtain that

$$\lim_{n\to\infty}\frac{\varphi_n(h; \zeta)}{\varphi_n(\zeta)} = \frac{S_h(\varphi(i(\zeta+1)/c(\zeta-1)))\,|S_h(\varphi(i/c))|}{S_h(\varphi(i/c))}. \tag{59}$$

This proves the existence of limit in (4). On the other hand, from the properties of the (exterior) Szegő function for the case of the unit disk it is obvious that the function on the right-hand side of (59) satisfies the conditions (i)–(iii) which characterize $D(h; \zeta)$ (see point 3 in Section 1).

As in Theorem 1, the general case reduces to the symmetric one by rotation. With this we conclude the proof of Theorem 2. ∎

In terms of the asymptotic behavior of the reflection coefficients, results on relative asymptotics of polynomials orthogonal on arcs of the unit circle may be found in [27].

Let $K_n(h; \zeta, \eta)$ and $w_n(h; \zeta)$ denote the reproducing kernel and the Christoffel function relative to the measure $h \, d\sigma$.

COROLLARY 3. *Under the assumptions of Theorem 2, we have*

$$\lim_{n \to \infty} \frac{K_n(h; \zeta, \eta)}{K_n(\zeta, \eta)} = D(h; \zeta) \, \overline{D(h; \eta)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash \gamma)^2$. *On the other hand,*

$$\lim_{n \to \infty} \frac{w_n(h; \zeta)}{w_n(\zeta)} = |D(h; \zeta)|^{-2},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \backslash \gamma$.

Proof.   These results follow immediately form Corollary 2 and Theorem 2.
∎

Remark 5.   In Widom's paper [34] and more recently in [10] there are very general results regarding strong asymptotics for polynomials orthogonal with respect to measures, supported on curves, which satisfy Szegö's condition. We point out that the method we exposed in the present paper may be used also to obtain such asymptotics when the curve is an arc of the unit circle. You will find in [16] the results needed relative to strong asymptotics of polynomials orthogonal with respect to varying measures on the unit circle.

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