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Zero Asymptotics of Laurent Orthogonal Polynomials

MANUEL BELLO HERNÁNDEZ AND ANDREI MARTÍNEZ FINKELSHTEIN*

Facultad de Matemática y Computación, Universidad de la Habana, Havana, Cuba

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Let $\{h_n(z)\}$ be the sequence of polynomials, satisfying

$$\int_0^{+\infty} h_m(x)\, h_n(x)\, x^{-\lambda_n}\, d\rho(x) = \delta_{mn}, \qquad 0 \leqslant m \leqslant n,$$

where $\lambda_n \in [0, 2n]$, $n \in \mathbf{N}$. For a wide class of weights $d\rho(x)$ and under the assumption $\lim_{n\to\infty} \lambda_n/(2n) = \theta \in [0, 1]$, two descriptions of the zero asymptotics of $\{h_n(z)\}$ are obtained. Furthermore, their analogues for polynomials orthogonal on $[-1, 1]$ with respect to varying weights are considered. These results continue the study begun in [3]. © 1996 Academic Press, Inc.

1. INTRODUCTION: STATEMENT OF RESULTS

Orthogonal polynomials with respect to "varying weights" (weights depending on the degree of the polynomial) have been studied intensively in the last ten years in connection with rational approximation of analytic functions. Namely, given a Stieltjes function, that is, the Cauchy transform of a measure (or distribution) ρ on the real line with bounded or unbounded support $S(\rho)$

$$\hat{\rho}(z) = \int \frac{d\rho(t)}{t - z}, \tag{1}$$

and starting from its asymptotic expansion at the endpoints of the convex hull of $S(\rho)$, we can construct the so-called two-point Padé approximants:

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rational functions whose denominators in this case will be orthogonal with respect to ρ modified by a factor that depends on the degree of the polynomial. Hence, convergence and pole distribution are clearly connected with the behaviour of these orthogonal polynomials.

In [3] a class of distributions supported on the positive semiaxis was studied. Namely, let ρ be a positive Borel measure on $\mathbf{R}_+ = [0, +\infty)$. Consider a sequence of integers $\{\lambda_n\}$, $n \geqslant 0$, and set

$$\Lambda_1 = \sup_n \lambda_n, \qquad \Lambda_2 = \sup_n (2n - \lambda_n).$$

Assume that

$$\int_0^{+\infty} x^\nu \, d\rho(x) < +\infty, \quad -\Lambda_1 \leqslant \nu \leqslant \Lambda_2, \quad \nu \in \mathbf{Z}. \tag{2}$$

This condition guarantees that all forthcoming integrals exist. If (2) holds, we say that the pair $(\rho, \{\lambda_n\})$ is *admissible.*

By $h_n(z) = \kappa_n z^n + \cdots$, $\kappa_n > 0$, we denote the nth orthonormal polynomial with respect to the measure $x^{-\lambda_n} \, d\rho(x)$, $x > 0$; hence,

$$\int_0^{+\infty} h_n(x) \, h_m(x) \, x^{-\lambda_n} \, d\rho(x) = \delta_{n,m}, \qquad 0 \leqslant m \leqslant n. \tag{3}$$

Then for the Stieltjes function (1), once $n \in \mathbf{N}$ and $0 \leqslant \lambda_n \leqslant 2n$ are fixed, there exists a polynomial P_n, $\deg P_n \leqslant n-1$, satisfying

$$(h_n \hat{\rho} - P_n)(z) = O(z^{-n+\lambda_n-1}), \qquad z \to -\infty,$$

$$(h_n \hat{\rho} - P_n)(z) = O(z^{\lambda_n}), \qquad z \to 0^-,$$

and the rational function $R_n := P_n / h_n$ is the *two-point Padé approximant* (of type $[n/n]$), interpolating $\hat{\rho}$ at 0 and ∞. Furthermore, using standard arguments the Hermite formula for the remainder can be worked out:

$$\hat{\rho}(z) - R_n(z) = \frac{1}{2\pi i} \frac{z^{\lambda_n}}{h_n^2(z)} \int_0^{+\infty} \frac{h_n^2(x)}{x - z} \frac{d\rho(x)}{x^{\lambda_n}}, \qquad z \notin \mathbf{R}_+.$$

Hence, the rate of convergence of Padé approximants in $\mathbf{C}/\mathbf{R}_+$ heavily depends upon the asymptotic behavior of the sequence

$$H_n(z) = h_n(z) \, z^{-\lambda_n/2} \tag{4}$$

as $n \to \infty$. When $\lambda_n \neq 0$, the expression appearing in (4) is commonly called a *Laurent polynomial*. We study the case when the interpolation conditions are proportionally distributed at 0 and ∞,

$$\lim_n \frac{\lambda_n}{2n} = \theta \in [0, 1], \tag{5}$$

in this way continuing the research initiated in [3] (see also [4]).

For $\gamma > 0$, $s > 0$, we introduce the class $\mathbf{F}_s(\gamma)$ of functions τ, continuous on $(0, +\infty)$, such that

$$\lim_{x \to 0^+} (sx)^{\gamma}\, \tau(x) = \lim_{x \to +\infty} (sx)^{-\gamma}\, \tau(x) = 1. \tag{6}$$

We will limit ourselves to measures of the form

$$d\rho(x) = x^{\alpha} \exp(-\tau(x))\, dx, \qquad x > 0, \tag{7}$$

where $\tau \in \mathbf{F}_s(\gamma)$, with $\alpha \in \mathbf{R}$, $s > 0$ and $\gamma > 1/2$ fixed. These kind of measures form admissible pairs $(\rho, \{\lambda_n\})$ for any sequence of reals $\{\lambda_n\}$ even when $\gamma > 0$, but for our purposes we require the determinacy of the moment problem, and then the condition $\gamma > 1/2$ is sufficient.

For $d\rho$ given in (7), we denote by $h_{n,m}(\tau; z)$ the orthonormal polynomial of degree m with respect to the measure $x^{-\lambda_n}\, d\rho(x)$, $x \in \mathbf{R}_+$:

$$h_{n,m}(\tau; z) = \kappa_{n,m}(\tau)\, z^m + \cdots, \qquad \kappa_{n,m}(\tau) > 0.$$

We omit the explicit reference to τ when it cannot lead to confusion, using notation introduced in (3).

In [3] the following result regarding the asymptotics of such polynomials was obtained:

THEOREM A. *Let ρ and $\{\lambda_n\}$ be as stated above. Then, for the orthonormal polynomials $h_{n,n}(\tau; z)$*

$$\lim_n \frac{\log |h_{n,n}(\tau; z)\, z^{-\lambda_n/2}|}{(2n)^{1-1/(2\gamma)}} = D(\gamma)\{(1-\theta)^{1-1/(2\gamma)} \operatorname{Im}[(sz)^{1/2}] + \theta^{1-1/(2\gamma)} \operatorname{Im}[(sz)^{-1/2}]\}, \quad z \in \mathbf{C} \backslash \mathbf{R}_+, \tag{8}$$

where

$$D(\gamma) = \frac{2\gamma}{2\gamma - 1} \left[\frac{\Gamma(\gamma + 1/2)}{\pi^{1/2}\Gamma(\gamma)} \right]^{1/(2\gamma)},$$

$\Gamma(z)$ is Euler's gamma function and we take the branch of the root so that $(-1)^{1/2}=i$. In (8) convergence holds uniformly on each compact subset of $\mathbf{C}\backslash\mathbf{R}_+$.

Here we describe the (weak) zero asymptotics of $\{h_{n,n}(\tau; z)\}$ in two different ways. To each polynomial $P(z)=A(z-z_1)\cdots(z-z_n)$ we associate a discrete measure

$$\nu(P)=\sum_{i=1}^{n}\delta_{z_i},$$

where δ_{z_i} denotes the unit measure whose support is the point z_i. An important contribution in this field is due to E. A. Rakhmanov [6] (see also [1] for a general approach). For $\lambda_n\equiv 0$ and for Freud-type weights $d\rho(x)=g(x)\,dx$, $x>0$, satisfying

$$\lim_{x\to+\infty} x^{-\gamma}\log g(x)=-r, \qquad r>0, \quad \gamma>1/2,$$

he proved the existence of the "contracted" zero asymptotics (for simplicity, we take $r=1$): if $Q_n(x)$ satisfies

$$\deg Q_n=n, \qquad \int_0^{+\infty} x^{\nu}Q_n(x)\,g(x)\,dx=0, \quad \nu=0, ..., n-1,$$

$$c_n=\left(2\pi^{1/2}\frac{\Gamma(\gamma)}{\Gamma(\gamma+1/2)}\,n\right)^{1/\gamma} \tag{9}$$

and $Q_n^*(x)=Q_n(c_n x)$, then

$$\frac{1}{n}\,\nu(Q_n^*)\to U, \tag{10}$$

where U is the so-called unit Nevai–Ullmann distribution on $[0, 1]$

$$dU(x)=\frac{\gamma}{\pi}\int_x^1\frac{t^{\gamma-1}\,dt}{\sqrt{x(t-x)}}\,dx, \qquad x\in[0, 1]. \tag{11}$$

In (10) (and in the sequel) the symbol $\to$ referring to measures denotes weak-* convergence.

In the case we are dealing with, the situation is rather different. It is not difficult to prove (see (43) below) that

$$\frac{1}{n}\,\nu(h_{n,n})\to\theta\delta_0.$$

Roughly speaking, this means that approximately θn zeros of $h_{n,n}$ "concentrate" at $z=0$, while $(1-\theta)n$ "escape" to infinity. Hence, any contraction (or dilation) on the real axis gives rise to mass points at $z=0$ or $z=\infty$.

In order to circumvent this undesirable effect, we analyze the rescaled asymptotics of the "large" and "small" zeros of $h_{n,n}$ separately. If

$$h_{n,n}(\tau; z)=\kappa_{n,n}(\tau)\prod_{i=1}^{n}(z-z_{i,n}), \qquad 0<z_{1,n}<\cdots<z_{n,n}<\infty \tag{12}$$

define

$$A_n(z)=\prod_{z_{i,n}\geqslant 1}(z-z_{i,n}), \qquad B_n(z)=\prod_{z_{i,n}<1}\left(z-\frac{1}{z_{i,n}}\right). \tag{13}$$

THEOREM 1. *Under the assumptions of Theorem A the following zero distributions take place*:

(i) *if* $0<\theta<1$, *and*

$$A_n^*(z)=A_n(c_n(1-\theta)^{\gamma}z/s), \quad B_n^*(z)=B_n(c_n\theta^{\gamma}z/s), \tag{14}$$

then

$$\nu(A_n^*)\to(1-\theta)\,U, \qquad \nu(B_n^*)\to\theta U. \tag{15}$$

(ii) *if* $\theta=0$, *for* $P_n^*(z)=h_{n,n}(c_n z/s)$,

$$\nu(P_n^*)\to U. \tag{16}$$

Analogously, if $\theta=1$ *and* $Q_n^*(z)=z^n h_{n,n}(1/(c_n zs))$, *then*

$$\nu(Q_n^*)\to U. \tag{17}$$

Here U *is the Nevai–Ullmann distribution on* $[0,1]$ *and* $\{c_n\}$ *is defined in* (9).

As was shown in [9], another appropriate description of the zero behavior is given by the weighted asymptotics. Now we associate with $h_{n,n}(\tau,z)$ the discrete measure

$$\alpha_n=\sum_{i=1}^{n}\frac{z_{i,n}}{1+z_{i,n}^2}\,\delta_{z_{i,n}}. \tag{18}$$

In this way, we "diminish" the weight of the smaller and larger zeros, avoiding the mass point effect they produce at the end points. In fact, the following result holds:

THEOREM 2. *Under the assumptions of Theorem A,*

$$\frac{1}{(2n)^{1-1/(2\gamma)}}\,\alpha_n \to \beta, \qquad n\to\infty,$$

where β is an absolutely continuous measure on $\mathbf{R}_+$ given by

$$d\beta(x)=\frac{D(\gamma)}{2\pi(1+x^2)}\,((1-\theta)^{1-1/(2\gamma)}\,(sx)^{1/2}+\theta^{1-1/(2\gamma)}(sx)^{-1/2})\,dx, \qquad x>0.$$

As a consequence, for every bounded and continuous function f on $[0,+\infty)$

$$\lim_{n\to\infty}\frac{1}{(2n)^{1-1/(2\gamma)}}\sum_{i=1}^{n}\frac{z_{i,n}f(z_{i,n})}{1+z_{i,n}^2}=\int_0^{+\infty} f(x)\,d\beta(x).$$

Now we consider briefly the case of measures with bounded support (the so-called Markov case). Suppose μ is a finite positive Borel measure on $\Delta=[-1,1]$ of the form

$$d\mu(x)=C(1-x)^{\alpha}\,(1+x)^{\beta}\exp(-\tau(x))$$

with α, $\beta\in\mathbf{R}$ and τ continuous on $(-1,1)$ and such that

$$\lim_{x\to-1^+} s^{\gamma}(1+x)^{\gamma}\,\tau(x)=\lim_{x\to1^-} s^{-\gamma}(1-x)^{\gamma}\,\tau(x)=2^{\gamma},$$

where $\gamma>1/2$, $s>0$. Then, for $0\leqslant\lambda_n\leqslant 2n$, we define the polynomials $l_n(x)$, $\deg l_n=n$, such that

$$\int_{\Delta} l_n(x)\,l_m(x)\,\frac{d\mu(x)}{(1-x)^{2n-\lambda_n}\,(1+x)^{\lambda_n}}=\delta_{mn}, \qquad 0\leqslant m\leqslant n, \quad n\in\mathbf{N}. \tag{19}$$

Again, for

$$\hat{\mu}(z)=\int_{\Delta}\frac{d\mu(t)}{t-z}$$

and $n\in\mathbf{N}$ we can construct p_n, $\deg p_n\leqslant n-1$, satisfying

$$(l_n\hat{\mu}-p_n)(z)=O((1-z)^{2n-\lambda_n}), \qquad z\to1^+,$$
$$(l_n\hat{\mu}-p_n)(z)=O((1+z)^{\lambda_n}), \qquad z\to-1^-,$$

then $r_n=p_n/l_n$ is the two-point Padé approximant of type $[n/n]$ interpolating $\hat{\mu}$ at -1 and $+1$. The Hermite formula for the remainder $(\hat{\mu}-r_n)(z)$ permits to deduce the rate of convergence of r_n to $\hat{\mu}$ in

$\mathbf{C}\backslash[-1, 1]$, knowing the asymptotic properties of $\{l_n(z)\}$. In particular, under assumption (5) a formula of type (8) can be obtained.

Although both results stated in Theorems 1 and 2 may be translated into the Markov case by an appropriate change of variable, for the sake of brevity we restrict ourselves to the weighted zero distribution of polynomials l_n.

Define ρ on $[0, +\infty)$ such that

$$d\mu(x) = \frac{1}{2}(1-x)\, d\rho\left(\frac{1+x}{1-x}\right), \qquad x \in (-1, 1).$$

Then ρ satisfies (6)–(7) and

$$h_n(x) = K_n \left(\frac{x+1}{2}\right)^n l_n\left(\frac{x-1}{x+1}\right)$$

satisfy (3). The constants K_n can be explicitly computed, but play no role in the zero distribution. Hence, with Theorem 2 at hand, we arrive at

THEOREM 3. *Let* $l_n(x) = k_n \prod_{i=1}^n (x - x_{i,n})$ *satisfy* (19) *and*

$$\alpha_n^* = \sum_{i=1}^n \frac{1 - x_{i,n}^2}{1 + x_{i,n}^2}\, \delta_{x_{i,n}}.$$

Then, if (5) *holds,*

$$\frac{1}{(2n)^{1-1/(2\gamma)}}\, \alpha_n^* \to \beta^*, \qquad n \to \infty,$$

where β^* *is an absolutely continuous measure on* $[-1, 1]$ *given by its density*

$$d\beta^*(x) = \frac{D(\gamma)}{4\pi(1+x^2)} \left\{(1-\theta)^{1-1/(2\gamma)} \left(s\,\frac{1+x}{1-x}\right)^{1/2} + \theta^{1-1/(2\gamma)} \left(s\,\frac{1+x}{1-x}\right)^{-1/2}\right\} dx, \qquad x \in (-1, 1).$$

As a consequence, for every bounded and continuous function f *on* $[-1, 1]$

$$\lim_{n\to\infty} \frac{1}{(2n)^{1-1/(2\gamma)}} \sum_{i=1}^n \frac{1 - x_{i,n}^2}{1 + x_{i,n}^2} f(x_{i,n}) = \int_{-1}^1 f(x)\, d\beta^*(x).$$

The structure of the paper is as follows. For completeness we state in Section 2 the results that we need appearing in [3] and [4]. In Section 3, we obtain a bound for the greatest zero $z_{n,n}$ of $h_{n,n}(\tau, z)$, which is used in the proof of Theorem 1 in Section 4. Theorem 2 is proved in Section 5.

Finally, we should point out that the mentioned "mass point effect" appears only when $\theta \in (0, 1)$. Asymptotic formulas for $\theta \in \mathbf{R} \setminus [0, 1]$ were worked out in [4]; in this case, the Laurent polynomials H_n, defined in (4), exhibit a contracted nth root asymptotics, from which the corresponding contracted zero asymptotics is easily obtained.

2. Background

Here we overview some facts, proved in [3] and [4], which we will use below. We also keep the notation introduced in Section 1. Note first of all, that if $\tau(x) \in \mathbf{F}_s(\gamma)$ and $g(x) = \tau(x/s)$, then

$$h_{n,n}(\tau; x) = s^{(\alpha+1-\lambda_n)/2} h_{n,n}(g; sx), \qquad \kappa_{n,n}(\tau) = s^{(2n+\alpha+1-\lambda_n)/2} \kappa_{n,n}(g),$$

so that $s \neq 1$ is trivially reduced to $s = 1$. Hence, in what follows we take $s = 1$. The parameter α plays no role in the proof, except that $(\rho, \{\lambda_n\})$ be admissible, so for simplicity we assume that $\alpha = 0$. For $n \in \mathbf{N}$, $\mathbf{P}_n$ denotes the class of all polynomials of degree $\leqslant n$. Given a positive Borel measure σ on $\mathbf{R}_+$ the notation $S(\sigma)$ stands for its support and V_σ for its logarithmic potential:

$$V_\sigma(x) = -\int \log |z - t| \, d\sigma(t).$$

Moreover, in the sequel we say that any property holds *quasi-everywhere* (briefly, q.e.) in $\Omega \subset \mathbf{C}$ if it is satisfied for all $z \in \Omega \setminus e$, where e is a Borel subset of zero logarithmic capacity.

For $n \in \mathbf{N}$ let M_n be the class of all positive Borel measures σ on $\mathbf{R}_+$ that satisfy $\|\sigma\| = \int d\sigma = n$.

As it was already shown in [3], a key role in the proof of Theorem 1 is played by a sequence of equilibrium measures $\mu_n(\tau)$ in the presence of the external fields $\tau(x) + \lambda_n \log x$. These measures can be defined by the relations

$$\begin{aligned} &\mu_n(\tau) \in M_n, \\ &2V_{\mu_n}(x) + \tau(x) + \lambda_n \log x = \omega_n(\tau), \qquad && x \in S(\mu_n(\tau)), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \geqslant \omega_n(\tau), && x \in \mathbf{R}_+. \end{aligned}$$

In particular, for $\varphi(x) = x^\gamma + x^{-\gamma} \in \mathbf{F}_1(\gamma)$ and $A > 0$, we write

$$\mu_{n,A} = \mu_n(A\varphi), \qquad \omega_{n,A} = \omega_n(A\varphi),$$

which are respectively the equilibrium measure and the equilibrium constant of the problem

$$\begin{aligned} 2V_{\mu_{n,A}}(x)+\varphi_{n,A}(x) &= \omega_{n,A}, \qquad x\in S(\mu_{n,A}(\tau)),\\ &\geqslant \omega_{n,A}, \qquad x\in \mathbf{R}_+, \end{aligned} \tag{20}$$

where $\varphi_{n,A}(x)=A\varphi(x)+\lambda_n \log x$. In [3] it was proved that $S(\mu_{n,A})=[r_{n,A}, R_{n,A}]$ with $0<r_{n,A}<R_{n,A}<+\infty$ satisfying the equations

$$\frac{1}{\pi}\int_{r_{n,A}}^{R_{n,A}} \frac{\varphi_n'(x)\,dx}{\sqrt{(R_{n,A}-x)(x-r_{n,A})}}=0,$$

$$\frac{1}{\pi}\int_{r_{n,A}}^{R_{n,A}} \frac{x\varphi_n'(x)\,dx}{\sqrt{(R_{n,A}-x)(x-r_{n,A})}}=2n.$$

The following asymptotic formulas were also obtained:

$$R_{n,A}=\left[\frac{2n(1-\theta)}{A\gamma B(\gamma)}\right]^{1/\gamma}+o(n^{1/\gamma}), \qquad r_{n,A}^{-1}=\left[\frac{2n\theta}{A\gamma B(\gamma)}\right]^{1/\gamma}+o(n^{1/\gamma}), \tag{21}$$

with

$$B(\gamma):=\pi^{-1}\int_0^1 x^{\gamma}\{x(1-x)\}^{-1/2}\,dx=\pi^{-1/2}\Gamma(\gamma+1/2)/\Gamma(\gamma+1).$$

Furthermore, the equilibrium constant satisfies

$$\omega_{n,A}=(2n-\lambda_n)\log\frac{4e^{1/\gamma}}{R_{n,A}}+o(n). \tag{22}$$

Along with the equilibrium problem (20), an essential role is played by the asymptotics of the Christoffel functions

$$\mathscr{K}_n(\tau;z)=\sup_{P\in\mathbf{P}_n}|P(z)|^2\left\{\int_0^{+\infty}|P(x)|^2\,\frac{\exp(-\tau(x))}{x^{\lambda_n}}\,dx\right\}^{-1}, \qquad z\in\mathbf{C}.$$

It is known (see, e.g., [5]) that

$$\mathscr{K}_n(\tau;z)=\sum_{m=0}^{n}|h_{n,m}(\tau;z)|^2.$$

For any subinterval Δ of the real line, $g_\Delta(z,\infty)$ denotes the Green function of $\hat{\mathbf{C}}\setminus\Delta$ with pole at ∞, and $\Phi_\Delta(z)$ is a conformal mapping of $\hat{\mathbf{C}}\setminus\Delta$ onto the exterior of the unit disk, such that $g_\Delta(z,\infty)=\log|\Phi_\Delta(z)|$.

Finally, $\operatorname{dist}(z, K)$ stands for the distance from z to the set K. The following relations were proved in [3] (see also [4]):

LEMMA 1. (i) *For* $\Delta_{n,A} = [r_{n,A}, R_{n,A}]$ *and* $z \in \mathbf{C} \setminus \Delta_{n,A}$,

$$\pi \operatorname{dist}(z, \Delta_{n,A})\, \mathscr{K}_n(A\varphi; z)\, |\Phi_{\Delta_{n,A}}(z)|^{-1} \leqslant \exp\{\omega_{n,A} - 2V_{\mu_{n,A}}(z)\}. \tag{23}$$

(ii) *There exists a constant* $C > 0$ *such that for* $z \in \mathbf{C} \setminus \Delta_{n,A}$, $n \in \mathbf{N}$,

$$\mathscr{K}_n(A\varphi; z) \geqslant Cn^{-5} \left(1 + \frac{R_{n,A} - r_{n,A}}{\operatorname{dist}(z, \Delta_{n,A})}\right)^{-2} \exp\{\omega_{n,A} - 2V_{\mu_{n,A}}(z)\}. \tag{24}$$

(iii) *Furthermore,*

$$f_n(z)\, \mathscr{K}_n(\tau; z) \leqslant |h^2_{n,n}(\tau, z)| \leqslant \mathscr{K}_n(\tau; z), \qquad z \in \mathbf{C} \setminus \mathbf{R}_+, \tag{25}$$

with

$$f_n(z) = \begin{cases} 1 + n^6 (\operatorname{Im} z)^{-2}, & z \in \mathbf{C} \setminus \mathbf{R}, \\ 1 + n^6\, |z|^{-1}, & z < 0. \end{cases}$$

3. Bounds for the Zeros

We maintain the notation $0 < z_{1,n} < \cdots < z_{n,n} < \infty$ for the zeros of $h_{n,n}(\tau, z)$ introduced in (12).

LEMMA 2. *There exists* $k > 1$ *independent from* n, *such that*

$$z_{n,n} \leqslant k R_{n,1}, \qquad z_{1,n} \geqslant r_{n,1}/k.$$

Proof. From the Gauss quadrature formula, applied to

$$p_{n-1}(z) = \frac{h_{n,n}(\tau, z)}{z - z_{n,n}},$$

it follows that

$$\begin{aligned} z_{n,n} &= \frac{\int_0^{+\infty} x p^2_{n-1}(x)\, x^{-\lambda_n}\, d\rho(x)}{\int_0^{+\infty} p^2_{n-1}(x)\, x^{-\lambda_n}\, d\rho(x)} \\ &\leqslant s_n + \int_{s_n}^{+\infty} x p^2_{n-1}(x) \left(\int_0^{+\infty} p^2_{n-1}(t)\, t^{-\lambda_n}\, d\rho(t)\right)^{-1} x^{-\lambda_n}\, d\rho(x) \\ &\leqslant s_n + \int_{s_n}^{+\infty} x \mathscr{K}_n(\tau; x)\, x^{-\lambda_n}\, d\rho(x), \end{aligned} \tag{26}$$

for any $s_n \geqslant 0$.

Fix $\varepsilon > 0$. Then there exist constants A_1 and A_2 such that

$$(1-\varepsilon)\,\varphi(x) + A_1 \leqslant \tau(x) \leqslant (1+\varepsilon)\,\varphi(x) + A_2, \qquad x \in (0, +\infty), \tag{27}$$

and consequently,

$$\exp(A_1)\,\mathcal{K}_n((1-\varepsilon)\,\varphi; z) \leqslant \mathcal{K}_n(\tau; z) \leqslant \exp(A_2)\,\mathcal{K}_n((1+\varepsilon)\,\varphi; z), \qquad z \in \mathbf{C}. \tag{28}$$

For the sake of brevity, in what follows we omit in the notation the explicit reference to the constant $A > 0$ whenever $A = 1$, and substitute the subindex A by the corresponding superindex "+" or "−", depending on whether $A = 1+\varepsilon$ or $A = 1-\varepsilon$. For example,

$$r_n = r_{n,1}, \qquad r_n^+ = r_{n,1+\varepsilon}, \qquad r_n^- = r_{n,1-\varepsilon}.$$

Let $s_n = kR_n^+$, with $k > 1$. Then from (27), (28) and Lemma 1(i), it follows that

$$\begin{aligned}
I_n &= \int_{s_n}^{+\infty} x\mathcal{K}_n(\tau; x)\,\frac{\exp(-\tau(x))}{x^{\lambda_n}}\,dx\\
&\leqslant \frac{C_1}{(k-1)\,R_n^+}\int_{s_n}^{+\infty} x\,|\Phi_{\Delta_n^+}(x)|\\
&\quad\times \exp\{\omega_n^+ - 2V_{\mu_n^+}(x) - (1-\varepsilon)\,\varphi(x)\}\,x^{-\lambda_n}\,dx\\
&\leqslant \frac{C_2\exp(\omega_n^+)}{(k-1)(R_n^+)^2}\int_{s_n}^{+\infty} x^{2n-\lambda_n+4}\exp\{-(1-\varepsilon)\,x^{\gamma}\}\,\frac{dx}{x^2}.
\end{aligned}$$

Here and in the sequel, we denote by $C_1, C_2, ...$ positive constants that do not depend on n. Note that the function

$$g_n(x) = x^{2n-\lambda_n+4}\exp\{-(1-\varepsilon)\,x^{\gamma}\}$$

is decreasing for $x > ((2n-\lambda_n+4)/((1-\varepsilon)\gamma))^{1/\gamma}$. Taking into account (21), it follows that there exists a $k_0 > 1$ such that for $k > k_0$, $kR_n^+ > ((2n-\lambda_n+4)/((1-\varepsilon)\gamma))^{1/\gamma}$, so that by (22)

$$I_n \leqslant kR_n^+\,\frac{C_2k^2}{k-1}\exp\left[(2n-\lambda_n)\left(\log 4ke^{1/\gamma} - \frac{1-\varepsilon}{1+\varepsilon}\,\frac{k^{\gamma}}{\gamma B(\gamma)}\right) + o(n)\right]. \tag{29}$$

In order to establish the first inequality in Lemma 2, it remains to use (21) and (26). The second inequality is directly obtained making the substitution $x \mapsto 1/x$.

4. Contracted Zero Asymptotics

This section is devoted to the proof of Theorem 1. For the sequence $\{\mu_{n,A}\}$ of equilibrium measures (20) define the "contracted" measures

$$d\mu^*_{n,A}(x)=d\mu_{n,A}(R_{n,A}x),\quad x>0,\qquad S(\mu^*_{\mu,A})=[r_{n,A}/R_{n,A},1]=\Delta^*_{n,A}.$$

Then,

$$\begin{aligned}\frac{1}{n}V_{\mu^*_{n,A}}(x)+A\frac{R^\gamma_{n,A}}{2n}x^\gamma+\frac{A}{R^\gamma_{n,A}2n}x^{-\gamma}+\frac{\lambda_n}{2n}\log x\\ =\frac{1}{2n}\{\omega_{n,A}+(2n-\lambda_n)\log R_{n,A}\},\qquad x\in\Delta^*_{n,A},\\ \geqslant\frac{1}{2n}\{\omega_{n,A}+(2n-\lambda_n)\log R_{n,A}\},\qquad x>0.\end{aligned}\tag{30}$$

Since the supports of the unit measures $(1/n)\mu^*_{n,A}$ are uniformly bounded, they form a weakly compact sequence. Hence, we can fix a $\Lambda\subseteq\mathbf{N}$ such that

$$\frac{1}{n}\mu^*_{n,A}\to\mu^*_A,\qquad n\in\Lambda,\tag{31}$$

where μ^*_A is a unit measure, supported on a subset of $[0,1]$. From (5), (21) and (22) by the lower envelope principle for potentials (see [2, Theorem 3.8], [7]), it follows that

$$\begin{aligned}V_{\mu^*_A}(x)+\frac{1-\theta}{\gamma B(\gamma)}x^\gamma+\theta\log x=(1-\theta)\log\{4e^{1/\gamma}\},\qquad \text{q.e. on } (0,1]\\ \geqslant(1-\theta)\log\{4e^{1/\gamma}\},\qquad \text{q.e. on } (1,+\infty).\end{aligned}\tag{32}$$

Suppose that $0\leqslant\theta<1$. Then

$$\zeta_A=\frac{1}{1-\theta}(\mu^*_A-\theta\delta_0)$$

is a charge (signed measure) with $S(\zeta_A)\subseteq[0,1]$, satisfying

$$\begin{aligned}V_{\zeta_A}(x)+\frac{1}{\gamma B(\gamma)}x^\gamma=\log\{4e^{1/\gamma}\},\qquad \text{q.e. on } [0,1]\\ \geqslant\log\{4e^{1/\gamma}\},\qquad x>1.\end{aligned}\tag{33}$$

Note that the Nevai–Ullmann distribution on $[0, 1]$ also satisfies the relation (33) (see [6], [8] or [9]). To establish that $\zeta_A = U$, we use the following strong uniqueness result:

LEMMA 3 [2, Chap. IV]. *Suppose we have two signed measures ν_1 and ν_2, such that for the set I of irregular points of $S(\nu_1) \cup S(\nu_2)$,*

$$\nu_1|_I \equiv \nu_2|_I \equiv 0,$$

(where $\nu|_I$ denotes the restriction of ν to I).
Then if

$$V_{\nu_1} = V_{\nu_2} \text{ q.e. on } S(\nu_1) \cup S(\nu_2),$$

it follows that $\nu_1 = \nu_2$.

Now, for $\nu_1 = \zeta_A$ and $\nu_2 = U$ we have $S(\nu_1) \cup S(\nu_2) = [0, 1]$, so that $I = \varnothing$. Hence,

$$\zeta_A = U, \tag{34}$$

and since $\Lambda \subseteq \mathbf{N}$ in (31) was arbitrary,

$$\frac{1}{n} \mu^*_{n, A} \to (1-\theta)\, U + \theta \delta_0, \qquad n \to \infty. \tag{35}$$

From (32) and the unicity of the solution of the corresponding equilibrium problem it follows that (35) holds for $\theta = 1$ also.

Fix $\varepsilon > 0$. Lemma 1 along with (28) yields the inequality

$$\begin{aligned} &\frac{Ce^{A_1}}{n^5} f_n(z) \left(1 + \frac{R_n^- - r_n^-}{\operatorname{dist}(z, \Delta_n^-)}\right)^{-2} \exp\{\omega_n^- - 2V_{\mu_n^-}(z)\} \\ &\quad \leqslant |h^2_{n,n}(\tau, z)| \\ &\quad \leqslant \frac{e^{A_2}\, |\Phi_{\Delta_n^+}(z)|}{\pi \operatorname{dist}(z, \Delta_n^+)} \exp\{\omega_n^+ - 2V_{\mu_n^+}(z)\}, \qquad z \in \mathbf{C} \setminus \mathbf{R}_+, \end{aligned}$$

that is,

$$\begin{aligned} &\log(\pi \operatorname{dist}(z, \Delta_n^+)) - A_2 - g_{\Delta_n^+}(z) + 2V_{\mu_n^+}(z) - \omega_n^+ \\ &\quad \leqslant 2V_{\nu(h_{n,n})}(z) - \log \kappa^2_{n,n}(\tau) \\ &\quad \leqslant -\log\left[\frac{C}{n^5} f_n(z) \left(1 + \frac{R_n^- - r_n^-}{\operatorname{dist}(z, \Delta_n^-)}\right)^{-2}\right] - A_1 + 2V_{\mu_n^-}(z) - \omega_n^-, \\ &\qquad z \in \mathbf{C} \setminus \mathbf{R}_+. \end{aligned} \tag{36}$$

Then, for the sequence

$$\nu_n^* = \sum_{i=1}^{n} \delta_{z_{i,n}/R_n}$$

with an appropriate scaling in (36), we obtain

$$\begin{aligned}
&\log(\pi R_n^+ \operatorname{dist}(z, \Delta_{n,1+\varepsilon}^*)) - A_2 - g_{\Delta_{n,1+\varepsilon}^*}(z, \infty) + 2V_{\mu_{n,1+\varepsilon}^*}(z) - \omega_n^+ \\
&\qquad + \frac{n}{\gamma} \log(1+\varepsilon) + \log \kappa_{n,n}^2(\tau) \\
&\qquad \leqslant 2V_{\nu_n^*}(z) \\
&\qquad \leqslant -\log\left[\frac{C}{n^5} f_n(R_n^- z)\left(1 + \frac{1 - r_n^-/R_n^-}{\operatorname{dist}(z, \Delta_{n,1-\varepsilon}^*)}\right)^{-2}\right] - A_1 + 2V_{\mu_n^-}(z) - \omega_n^- \\
&\qquad + \frac{n}{\gamma} \log(1-\varepsilon) + \log \kappa_{n,n}^2(\tau). \qquad (37)
\end{aligned}$$

Moreover, if in (36) $z = ix$, $x > 0$ with $x \to +\infty$, we have

$$\log\{(R_n^+ - r_n^+)/4\} - A_2 - \omega_n^+ \leqslant -\log \kappa_{n,n}^2(\tau) \leqslant -\log Cn^{-5} - A_1 - \omega_n^-. \qquad (38)$$

On the other hand, from Lemma 2 it follows that the supports of $\{\nu_n^*\}$ are uniformly bounded. Hence, we can fix an arbitrary subsequence (that we denote by Λ again), $\Lambda \subseteq \mathbf{N}$, such that

$$\frac{1}{n} \nu_n^* \to \nu^*, \qquad n \in \Lambda.$$

Now, dividing (37) by n and taking into account (29), (35), and (38), we obtain for $n \to \infty$, $n \in \Lambda$,

$$\begin{aligned}
&2V_{(1-\theta)U + \theta\delta_0}(z) + \frac{1-\theta}{\gamma} \log\left(\frac{1-\varepsilon}{1+\varepsilon}\right) + \frac{1}{\gamma}\log(1+\varepsilon) \\
&\qquad \leqslant 2V_{\nu^*}(z) \\
&\qquad \leqslant 2V_{(1-\theta)U + \theta\delta_0}(z) + \frac{1-\theta}{\gamma} \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) + \frac{1}{\gamma}\log(1-\varepsilon), \qquad z \in \mathbf{C} \setminus \mathbf{R}_+.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finally arrive at

$$V_{\nu^*}(z) = V_{(1-\theta)U + \theta\delta_0}(z), \qquad z \in \mathbf{C} \setminus \mathbf{R}_+.$$

The interior of $\mathbf{R}_+$ is empty, so this proves that

$$\nu^* = (1-\theta)\, U + \theta \delta_0. \tag{39}$$

On the other hand, the limit

$$\frac{1}{n}\, \nu(h_{n,n}) \to \theta \delta_0 \tag{40}$$

will be established below (see (43)). From this, for the polynomials $A_n(z)$ and $B_n(z)$, defined in (13), we have

$$\deg A_n(z) = (1-\theta) n + o(n), \qquad \deg B_n(z) = \theta n + o(n).$$

Thus for $P_n(z) = h_{n,n}(z)/A_n(z)$,

$$\frac{1}{n}\, \nu(P_n) \to \theta \delta_0,$$

and then

$$\frac{1}{n}\, \nu(P_n^*) \to \theta \delta_0, \tag{41}$$

where $P_n^*(z) = P_n(R_n z)$.

Taking into account (39)–(41),

$$\frac{1}{n}\, \nu(A_n^*) \to (1-\theta)\, U$$

readily follows. The second limit in (15) is established in a similar way. Moreover, (ii) of Theorem 1 is a particular case when $\theta = 0$ or $\theta = 1$. Theorem 1 is proved.

5. Weighted Zero Asymptotics

The proof of Theorem 2 is based on the formula (8), obtained in [3], and follows the scheme, proposed by Van Assche in [9]. We may take again $s = 1$.

Here we use the notation

$$\nu_n = \nu(h_{n,n}).$$

Moreover, $\mathscr{V}_n(z)$ stands for a multivalued and analytic function in $\mathbf{C}\setminus S(v_n)$ whose real part coincides with $V_{v_n}(z)$:

$$\operatorname{Re}\mathscr{V}_n(z)=V_{v_n}(z),\qquad \frac{d}{dz}\mathscr{V}_n(z)=\int\frac{dv_n(t)}{t-z},\qquad z\in\mathbf{C}\setminus S(v_n).$$

Due to the uniform convergence in (8), we can take derivatives on both sides of this formula to obtain

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}}\left(\int\frac{dv_n(t)}{t-z}+\frac{\lambda_n}{2z}\right)$$
$$=D(\gamma)\left\{\frac{i}{2}(1-\theta)^{1-1/(2\gamma)}z^{-1/2}+\frac{i}{2}\theta^{1-1/(2\gamma)}z^{-3/2}\right\},\qquad z\in\mathbf{C}\setminus\mathbf{R}_+.\quad(42)$$

Then

$$\lim_n \frac{1}{n}\left(\int\frac{dv_n(t)}{t-z}+\frac{\lambda_n}{2z}\right)=0,$$

so that

$$\frac{1}{n}\int\frac{dv_n(t)}{t-z}\to\theta\int\frac{d\delta_0(t)}{t-z}$$

uniformly on any compact subset of $z\in\mathbf{C}\setminus\mathbf{R}_+$. Since $\{v_n/n\}$ are unit measures, they form a weakly compact sequence on $\bar{\mathbf{R}}_+$. The Stieltjes–Perron inversion formula, applied to any of its limit points, shows that necessarily

$$\frac{1}{n}v(h_{n,n})\to\theta\delta_0,\qquad n\to\infty\qquad(43)$$

((43) is also a direct consequence of the Grommer and Hamburger continuity theorem, see for example [9]).

On the other hand, taking in (42) $z=i$ and $z=-i$ respectively, we obtain

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}}\left(\int\frac{dv_n(t)}{t-i}+\frac{\lambda_n}{2i}\right)$$
$$=\frac{D(\gamma)}{2^{1/2}(1+i)}\left\{i(1-\theta)^{1-1/(2\gamma)}+\theta^{1-1/(2\gamma)}\right\}\qquad(44)$$

and

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}} \left(\int \frac{dv_n(t)}{t+i} - \frac{\lambda_n}{2i} \right)$$
$$= \frac{D(\gamma)}{2^{1/2}(i-1)} \{ i(1-\theta)^{1-1/(2\gamma)} - \theta^{1-1/(2\gamma)} \}. \tag{45}$$

Adding (44) and (45), it follows that

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}} \int \frac{t\, dv_n(t)}{1+t^2} = \frac{D(\gamma)}{2^{3/2}} \{ (1-\theta)^{1-1/(2\gamma)} + \theta^{1-1/(2\gamma)} \}. \tag{46}$$

In the same way, subtracting (45) from (44) we get

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}} \left(\int \frac{dv_n(t)}{1+t^2} - \frac{\lambda_n}{2} \right) = \frac{D(\gamma)}{2^{3/2}} \{ (1-\theta)^{1-1/(2\gamma)} - \theta^{1-1/(2\gamma)} \}. \tag{47}$$

Finally, using the identity

$$\frac{t}{(1+t^2)(t-z)} = \frac{1}{1+z^2} \left\{ \frac{z}{t-z} - \frac{zt}{1+t^2} + \frac{1}{1+t^2} \right\}$$

and formulas (44)–(47), it gives

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}} \int \frac{t}{1+t^2} \frac{dv_n(t)}{t-z}$$
$$= \frac{D(\gamma)}{2} \left\{ (1-\theta)^{1-1/(2\gamma)} \frac{iz^{1/2} - 2^{-1/2}z + 2^{-1/2}}{1+z^2} \right.$$
$$\left. + \theta^{1-1/(2\gamma)} \frac{iz^{-1/2} - 2^{-1/2}z - 2^{-1/2}}{1+z^2} \right\}. \tag{48}$$

Integrating along an appropriate path in the complex domain, the following identities for $z \in \mathbf{C} \setminus \mathbf{R}_+$ are easily obtained:

$$\frac{iz^{1/2} - 2^{-1/2}z + 2^{-1/2}}{1+z^2} = \frac{1}{\pi} \int_0^{+\infty} \frac{t^{1/2}}{1+t^2} \frac{dt}{t-z},$$

$$\frac{iz^{-1/2} - 2^{-1/2}z - 2^{-1/2}}{1+z^2} = \frac{1}{\pi} \int_0^{+\infty} \frac{t^{-1/2}}{1+t^2} \frac{dt}{t-z}.$$

Then,

$$\lim_n \frac{1}{(2n)^{1-1/(2\gamma)}} \int \frac{t}{1+t^2} \frac{d\nu_n(t)}{t-z}$$

$$= \frac{D(\gamma)}{2\pi} \int_0^{+\infty} \{(1-\theta)^{1-1/(2\gamma)}\, t^{1/2} + \theta^{1-1/(2\gamma)} t^{-1/2}\}$$

$$\times \frac{dt}{(1+t^2)(t-z)}, \qquad z \in \mathbf{C} \setminus \mathbf{R}_+.$$

Since (46) shows that the sequence of measures

$$\frac{1}{(2n)^{1-1/(2\gamma)}}\, \alpha_n$$

with α_n defined in (18), is uniformly bounded, a scheme of reasoning analogous to the one used to establish (43) yields the asymptotics

$$\frac{1}{(2n)^{1-1/(2\gamma)}}\, \alpha_n \to \frac{D(\gamma)}{2\pi(1+x^2)} \{(1-\theta)^{1-1/(2\gamma)}\, x^{1/2} + \theta^{1-1/(2\gamma)} x^{-1/2}\}\, dx,$$

for $x > 0$. This concludes the proof of Theorem 2.

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