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Postnikov factorizations at infinity ☆

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Abstract

We have developed Postnikov sections for Brown–Grossman homotopy groups and for Steenrod homotopy groups in the category of exterior spaces, which is an extension of the proper category. The homotopy fibre of a fibration in the factorization associated with Brown–Grossman groups is an Eilenberg–Mac Lane exterior space for this type of groups and it has two non-trivial consecutive Steenrod homotopy groups. For a space which is first countable at infinity, one of these groups is given by the inverse limit of the homotopy groups of the neighbourhoods at infinity, the other group is isomorphic to the first derived of the inverse limit of this system of groups. In the factorization associated with Steenrod groups the homotopy fibre is an Eilenberg–Mac Lane exterior space for this type of groups and it has two non-trivial consecutive Brown–Grossman homotopy groups. We also obtain a mix factorization containing both kinds of previous factorizations and having homotopy fibres which are Eilenberg–Mac Lane exterior spaces for both kinds of groups.

Given a compact metric space embedded in the Hilbert cube, its open neighbourhoods provide the Hilbert cube the structure of an exterior space and the homotopy fibres of the factorizations above are Eilenberg–Mac Lane exterior spaces with respect to inward (or approaching) Quigley groups.

MSC: 55P57; 55Q52; 55Q70; 55P55; 55U40; 55Q07; 55R65

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0. Introduction

One of the aims of the Algebraic Topology is the study of the spaces and their classification. The usual tools of Algebraic Topology have permitted to obtain many classifications and to analyse some properties. However, there are families of spaces with special singularities that require an adaptation of the standard techniques.

In this paper, we have adapted the standard technique of Postnikov factorizations to the study of non compact spaces and using techniques of shape theory we also have applications to the study of compact metric spaces. For a non compact space it is advisable to consider as neighbourhoods at infinity the complements of closed-compact subsets. The proper category arises when we consider spaces and maps which are continuous at infinity. In order to have a category with limits and colimits it is interesting to extend the proper category to obtain a complete and cocomplete category. The category of exterior spaces satisfies these properties, contains the proper category and has limits and colimits. The study of non compact spaces and more generally exterior spaces has interesting applications, for example, Siebenmann [19] or Brown–Tucker [2] used proper invariants of non compact spaces to obtain some properties and classifications of open manifolds. We can also find applications in the study of compact-metric spaces. Each compact-metric space has an associated fundamental Leftschetz complex, which is a non compact CW-complex. In this way, the proper invariants of non compact CW-complexes became invariants of metric-compact spaces.

To develop the Algebraic Topology at infinity it is useful to consider some analogues of the standard Hurewicz homotopy groups. If instead of n-spheres we use sequences of n-spheres converging to infinity, then we obtain the Brown–Grossman proper homotopy groups, see [1,10]. On the other hand, if we move an n-sphere continuously towards infinity, we get infinity semitubes which represent elements of the Steenrod homotopy groups, see [5] and [4]. For the category of exterior spaces we also have the analogues of the previous groups, see [7] and [8].

Using colimits, we obtain Postnikov sections in the category of exterior spaces for the Brown–Grossman exterior homotopy groups. This gives rise to a tower of sections

$$\cdots \to X_B^{[n+1]} \to X_B^{[n]} \to \cdots \to X_B^{[0]}$$

for groups of Brown–Grossman type, and for the class of cce exterior spaces given at the beginning of Section 4, using towers of spaces and telescopic constructions we obtain another tower

$$\cdots \to X_S^{[n+1]} \to X_S^{[n]} \to \cdots \to X_S^{[0]}$$

for groups of Steenrod type.

For the same class of cce exterior spaces we are able to construct a mix factorization

$$\cdots \to X_S^{[n]} \to X_B^{[n]} \to X_S^{[n-1]} \to \cdots \to X_B^{[1]} \to X_S^{[0]} \to X_B^{[0]}$$

that contains the two previous towers.

An analysis of the homotopy ray fibres of the morphisms of these factorizations provides the following interesting properties.

(i) The homotopy ray fibre of $X_B^{[n+1]} \to X_B^{[n]}$ is an Eilenberg–Mac Lane exterior space $K_B(\pi_{n+1}^B(X), n+1)$ for the exterior homotopy groups of Brown–Grossman type. However, this fibre can have two non trivial consecutive Steenrod groups, one of these groups is given by the inverse limit of the $(n+1)$th homotopy groups of the neighbourhoods at infinity and the other is given by the first derived of the inverse limit of this system of groups.
(ii) The homotopy ray fibre of $X_S^{[n+1]} \to X_S^{[n]}$ is an Eilenberg–Mac Lane exterior space $K_S(\pi_{n+1}^S(X), n+1)$ for the exterior homotopy groups of Steenrod type. This fibre can have two non trivial consecutive Brown–Grossman groups, which we have described using analogues of the functor $\mathcal{P}^\infty$ introduced by Brown [1].
(iii) The homotopy ray fibres of morphisms of the mix factorization are Eilenberg–Mac Lane exterior spaces for both kinds of groups. We also give a description of these groups using analogues of the $\mathcal{P}^\infty$ functor, the inverse limit functor and its first derived functor.

1. Preliminaries

Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of spaces and proper maps will be denoted by **P**. This category and the corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, this category does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera.

In [7] there is a solution for this problem introducing the notion of exterior space. The category of exterior spaces and exterior maps, **E**, is complete and cocomplete and contains **P** as a full subcategory. Furthermore, **E** has a closed simplicial model category structure in the sense of Quillen [18]; hence, it establishes a good framework for the study of proper homotopy theory.

We begin by recalling the notion of exterior space. Roughly speaking, an exterior space is a topological space X with a neighbourhood system at infinity.

Definition 1.1. An exterior space (or exterior topological space) $(X, \varepsilon \subset \tau)$ consists of a topological space (X, τ) together with a non empty collection ε of open subsets satisfying,

(E1) If $E_1, E_2 \in \varepsilon$ then $E_1 \cap E_2 \in \varepsilon$;
(E2) If $E \in \varepsilon$, $U \in \tau$ and $E \subset U$, then $U \in \varepsilon$.

An open E which is in ε is said to be an exterior-open subset or for shorting an e-open subset. The family of e-open subsets ε is called the externology of the exterior space. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be exterior if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.

Given an space (X, τ), we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ and the total exterior space if one takes $\varepsilon = \tau$. In this paper, an important role will be played by the family ε^X_{cc} of the complements of closed-compact subsets of a topological space X, that will be called the cocompact externology. We denote by $\mathbb{N}$ and $\mathbb{R}_+$ the exterior spaces of non negative integers and non negative real numbers having the usual topology and the cocompact externology.

Notice that if **E** denotes the category of exterior spaces and exterior maps, then we have the following full embedding $e : \mathbf{P} \hookrightarrow \mathbf{E}$: It carries an space X to the exterior space X_e which is provided with the topology of X and ε^X_{cc}. A proper map $f : X \to Y$ is carried to the exterior map $f_e : X_e \to Y_e$ given by $f_e = f$.

Remark 1.2. Notice the following two differences between an externology and a topological filter: (i) the empty set is never a member of a topological filter, however it is a member of the total externology, (ii) all members of an externology are open sets and this property need not be satisfied by a topological filter.

Definition 1.3. Let $(X, \varepsilon \subset \tau)$ be an exterior space. An exterior neighbourhood base for $(X, \varepsilon \subset \tau)$ is a collection of subsets of X, β, satisfying that for every e-open subset E there exists $B \in \beta$ such that $B \subset E$ and for every $B' \in \beta$ there exists an e-open subset E' such that $E' \subset B'$.

If an exterior space X has a countable exterior neighbourhood base $\beta = \{X_n\}_{n=0}^{\infty}$ then we say that X is first countable at infinity.

Observe that for these exterior spaces we can suppose without loosing generality that

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots.$$

Notice that the trivial and total exterior spaces and every σ-compact space provided with ε^X_{cc} are first countable at infinity.

Definition 1.4. Let X be an exterior space, Y a topological space. Consider on $X \times Y$ the product topology and the distinguished open subsets E of $X \times Y$ such that for each $y \in Y$ there exists $U_y \in \tau_Y$, $y \in U_y$ and $E_y \in \varepsilon_X$ such that $E_y \times U_y \subset E$. This exterior space will be denoted by $X \bar{\times} Y$.

This construction gives a functor $\mathbf{E} \times \mathbf{Top} \to \mathbf{E}$, where **Top** denotes the category of topological spaces. When Y is a compact space then E is an e-open subset if and only if it is an open subset and there exists $G \in \varepsilon_X$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon_X = \varepsilon^X_{cc}$ then $\varepsilon_{X \bar{\times} Y} = \varepsilon^{X \times Y}_{cc}$.

Let $\mathbf{E}^{\mathbb{N}}$ be the category of exterior spaces under $\mathbb{N}$, where an object is given by an exterior map $\rho : \mathbb{N} \to X$, denoted by (X, ρ), and the morphisms are given by commutative triangles in **E**

$$\begin{array}{ccc} & \mathbb{N} & \\ {}^{\rho_X}\swarrow & & \searrow^{\rho_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

denoted by $f:(X,\rho_X)\to(Y,\rho_Y)$.

Definition 1.5. Let f, g be in $\mathrm{Hom}_{\mathbf{E}^{\mathbb{N}}}((X,\rho_X),(Y,\rho_Y))$, then we say f is e-homotopic to g relative to $\mathbb{N}$, written $f\simeq_e g$, if there is an exterior map $F:X\bar{\times} I\to Y$ such that $F(x,0)=f(x)$, $F(x,1)=g(x)$ and $F(\rho_X(k),t)=\rho_Y(k)$, for all $x\in X$, $k\in\mathbb{N}$ and $t\in I$. The map F is called an exterior homotopy relative to $\mathbb{N}$ from f to g and we sometimes write $F:f\simeq_e g$. The set of exterior homotopy classes relative to $\mathbb{N}$ will be denoted by $[(X,\rho_X),(Y,\rho_Y)]^{\mathbb{N}}$.

In [7], it is proved that $[(\mathbb{N}\bar{\times} S^q,\mathrm{id}_{\mathbb{N}}\bar{\times}*),(X,\rho_X)]^{\mathbb{N}}$ has the structure of a group for $q\geqslant 1$, which is abelian for $q\geqslant 2$; if $q=0$ we get a pointed set.

Definition 1.6. Let (X,ρ) be an object of $\mathbf{E}^{\mathbb{N}}$. For $q\geqslant 0$, the qth exterior homotopy group functor of (X,ρ) is given by $\pi_q^B(X,\rho)=[(\mathbb{N}\bar{\times} S^q,\mathrm{id}_{\mathbb{N}}\bar{\times}*),(X,\rho)]^{\mathbb{N}}$. It is also said that $\pi_q^B(X,\rho)$ is the qth Brown–Grossman exterior homotopy group of (X,ρ). As in standard homotopy, for a given exterior pair (X,A) and a base sequence of the form $\sigma:\mathbb{N}\to A$, we also have the relative Brown–Grossman exterior homotopy groups $\pi_q^B(X,A,\rho)$.

In an analogous way, one can consider the category $\mathbf{E}^{\mathbb{R}_+}$ of exterior spaces under $\mathbb{R}_+$. In this case, the set of exterior homotopy classes relative to $\mathbb{R}_+$ will be denoted by $[(X,\sigma_X),(Y,\sigma_Y)]^{\mathbb{R}_+}$. Similarly, see [8], for $q\geqslant 1$ $[(\mathbb{R}_+\bar{\times} S^q,\mathrm{id}_{\mathbb{R}_+}\bar{\times}*),(X,\sigma_X)]^{\mathbb{R}_+}$ admits the structure of a group, which is abelian if $q\geqslant 2$; and for $q=0$ a pointed set is obtained.

Definition 1.7. Let (X,σ) be an object of $\mathbf{E}^{\mathbb{R}_+}$. For every $q\geqslant 0$, the qth cylindric homotopy group functor of (X,σ) is given by

$$\pi_q^S(X,\sigma)=\left[(\mathbb{R}_+\bar{\times} S^q,\mathrm{id}_{\mathbb{R}_+}\bar{\times}*),(X,\sigma)\right]^{\mathbb{R}_+}.$$

It is also said that $\pi_q^S(X,\sigma)$ is the qth Steenrod exterior homotopy group of (X,σ). As above we also have the corresponding relative groups $\pi_q^S(X,A,\sigma)$ for $\sigma:\mathbb{R}_+\to A$.

In [7] the following classes of maps were considered.

Definition 1.8. Let $f:X\to Y$ be an exterior map.

(i) f is a weak exterior equivalence, called in this paper a weak B-equivalence, in either of the following cases:
   (a) if $\mathrm{Hom}_{\mathbf{E}}(\mathbb{N},X)=\emptyset$ then $\mathrm{Hom}_{\mathbf{E}}(\mathbb{N},Y)=\emptyset$,
   (b) if $\mathrm{Hom}_{\mathbf{E}}(\mathbb{N},X)\neq\emptyset$ then $\pi_q^B(f):\pi_q^B(X,\rho)\to\pi_q^B(Y,f\rho)$ is an isomorphism for all $\rho\in\mathrm{Hom}_{\mathbf{E}}(\mathbb{N},X)$, $q\geqslant 0$.

   Sometimes, we short the fact that a map satisfies either (a) or (b) by writing that $\pi_q^B(f):\pi_q^B(X)\to\pi_q^B(Y)$ is an isomorphism. We are thinking that $\pi_q^B(X)$ is the family of $\pi_q^B(X,\rho)$ for all $\rho\in\mathrm{Hom}_{\mathbf{E}}(\mathbb{N},X)$.

(ii) f is an exterior fibration, or B-fibration, if it has the RLP with respect to $\partial_0 : \mathbb{N} \bar{\times} D^q \to \mathbb{N} \bar{\times} (D^q \times I)$ for all $q \geqslant 0$, where $\partial_0(n, x) = (n, x, 0)$.
A map which is both a B-fibration and a weak B-equivalence is said to be a trivial B-fibration.

(iii) f is an exterior cofibration, or B-cofibration, if it has the LLP with respect to any trivial B-fibration.
A map which is both a B-cofibration and a weak B-equivalence is said to be a trivial B-cofibration.

We can develop homotopy theory for the category of exterior spaces using the following result given in [7].

Theorem 1.9. *The category of exterior spaces, $\mathbf{E}$, together with the classes of B-fibrations, B-cofibrations and weak B-equivalences has a closed simplicial model category structure.*

We denote by $\mathbf{Ho}(\mathbf{E})$ the category obtained from $\mathbf{E}$ by inverting the weak B-equivalences. If a exterior space satisfies that $\mathrm{Hom}_{\mathbf{E}}(\mathbb{N}, X) \cong \emptyset$, then we have that $\emptyset \to X$ is a weak B-equivalence, in this case it is said that X is trivial in $\mathbf{Ho}(\mathbf{E})$.

On the other hand, in [8], it is showed that $\mathbf{E}$ has a closed simplicial model category structure with the following classes of maps.

Definition 1.10. Let $f : X \to Y$ be an exterior map.

(i) f is a weak cylindric equivalence, called in this paper a weak S-equivalence, in either of the following cases:
  (a) if $\mathrm{Hom}_{\mathbf{E}}(\mathbb{R}_+, X) = \emptyset$ then $\mathrm{Hom}_{\mathbf{E}}(\mathbb{R}_+, y) = \emptyset$,
  (b) if $\mathrm{Hom}_{\mathbf{E}}(\mathbb{R}_+, X) \neq \emptyset$ then $\pi_q^S(f) : \pi_q^S(X, \sigma) \to \pi_q^S(Y, f\sigma)$ is an isomorphism for all $\sigma \in \mathrm{Hom}_{\mathbf{E}}(\mathbb{R}_+, X)$, $q \geqslant 0$.
As in the case of Brown–Grossman groups, we short these conditions by writing that $\pi_q^S(f) : \pi_q^S(X) \to \pi_q^S(Y)$ is an isomorphism.

(ii) f is a cylindric fibration, or S-fibration, if it has the RLP with respect to

$$\partial_0 : \mathbb{R}_R \bar{\times} D^q \to \mathbb{R}_+ \bar{\times} (D^q \times I) \quad \text{for all } q \geqslant 0,$$

where $\partial_0(r, x) = (r, x, 0)$.
A map which is both an S-fibration and a weak S-equivalence is said to be a trivial S-fibration.

(iii) f is a cylindric cofibration, or S-cofibration, if it has the LLP with respect to any trivial S-fibration.
A map which is both an S-cofibration and a weak S-equivalence is said to be a trivial S-cofibration.

We summarize in the following result, see [8], the existence of this “cylindric structure”.

Theorem 1.11. *The category* **E** *of exterior spaces together with the classes of S-cofibrations, S-fibrations and weak S-equivalences has a closed simplicial model category structure.*

For an exterior base ray $\sigma : \mathbb{R}_+ \to X$ denote by $\sigma|_{\mathbb{N}} = \sigma$ in, where in $: \mathbb{N} \to \mathbb{R}_+$ is the inclusion map. Suppose that we have an exterior map $f : X \to Y$ and take in **E** the pullback

$$\begin{array}{ccc} F & \xrightarrow{v} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{R}_+ & \xrightarrow[f\sigma]{} & Y \end{array}$$

then, the exterior space F is said to be the ray fibre of f with respect to the exterior base ray $f\sigma$. Notice that F is an exterior space over and under $\mathbb{R}_+$ and the pair of maps $\mathrm{id}_{\mathbb{R}_+} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\sigma : \mathbb{R}_+ \to X$ induce the base ray $\tau = (\mathrm{id}_{\mathbb{R}_+}, \sigma) : \mathbb{R}_+ \to F$. As in standard homotopy, we also have long exact sequences associated to a B-fibration. We refer the reader to [13] for a proof of the two following results.

Theorem 1.12. *Let $f : X \to Y$ be an exterior map and let $\sigma : \mathbb{R}_+ \to X$ be an exterior base ray, then*

(i) *if f is a B-fibration, then f is a S-fibration,*
(ii) *if f is a B-fibration with ray fibre F, then the following sequences are exact*

$$\begin{aligned} \cdots \to \pi^B_{q+1}(Y, f\sigma|_{\mathbb{N}}) \to \pi^B_q(F, \tau|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(Y, f\sigma|_{\mathbb{N}}) \\ \to \cdots \to \pi^S_{q+1}(Y, f\sigma) \to \pi^S_q(F, \tau) \to \pi^S_q(X, \sigma) \to \pi^S_q(Y, f\sigma) \to \cdots \end{aligned}$$

where the base sequences and rays are determined by σ.

The Brown–Grossman exterior homotopy groups and the Steenrod exterior homotopy groups are related by a long exact sequence that is a version for exterior spaces of the exact sequence given by Quigley [17] in shape theory or by Porter [14] in proper homotopy theory.

Theorem 1.13. *Let X be an exterior space and let $\sigma : \mathbb{R}_+ \to X$ be an exterior base ray, then there is an exact sequence*

$$\begin{aligned} \cdots \to \pi^B_{q+1}(Y, \sigma|_{\mathbb{N}}) \to \pi^S_q(X, \sigma) \to \pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}}) \\ \to \cdots \to \pi^B_1(X, \sigma|_{\mathbb{N}}) \to \pi^S_0(X, \sigma) \to \pi^B_0(X, \sigma|_{\mathbb{N}}) \to \pi^B_0(X, \sigma|_{\mathbb{N}}). \end{aligned}$$

Moreover, this exact sequence is natural with respect to exterior rayed spaces (X, σ).

Remark 1.14. If ϕ is an element of $\pi^B_q(X, \sigma|_{\mathbb{N}})$, then for $q \geqslant 1$ this element is applied by the map $\pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}})$ into the element $\phi^{-1} \cdot \mathrm{sh}_\sigma\, \phi$, where ϕ^{-1} is the inverse of ϕ in the group $\pi^B_q(X, \sigma|_{\mathbb{N}})$ and $\mathrm{sh}_\sigma\, \phi$ is given by the shift induced by the base ray σ. The exactness in the last part of the long sequence $\pi^S_0(X, \sigma) \to \pi^B_0(X, \sigma|_{\mathbb{N}}) \to \pi^B_0(X, \sigma|_{\mathbb{N}})$

means that the image of the map $\pi_0^S(X,\sigma) \to \pi_0^B(X,\sigma|_{\mathbb{N}})$ is the igualator of the maps id, $\mathrm{sh}_\sigma : \pi_0^B(X,\sigma|_{\mathbb{N}}) \to \pi_0^B(X,\sigma|_{\mathbb{N}})$, where if $S^0 = \{-1, 1\}$ and the base point of S^0 is -1 if an element is represented by $\psi : \mathbb{N} \bar{\times} S^0 \to X$ then $\mathrm{sh}_\sigma[\psi]$ is represented by the map ψ' given by $\psi'(k,1) = \psi(k+1,1)$, $\psi'(k,-1) = \sigma(k)$, for $k \in \mathbb{N}$.

Denote by *Gps* the category of groups and by *towGps* the category of tower of groups, see [4]. In 1975, E.M. Brown [1] gave a definition of the proper fundamental group ${}^B\pi_1^\infty(X)$ of a σ-compact space X with a base ray. He also defined a functor $\mathcal{P}^\infty : \mathit{towGps} \to \mathit{Gps}$, that gives the relation between the tower of fundamental groups, $\{\pi_1(X_i)\}$, of a tower of neighbourhoods of X at infinity and Brown's proper fundamental group.

In the case of exterior homotopy theory, if we restrict ourselves to exterior spaces having a countable exterior neighbourhood base $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, and we suppose that there is an exterior base ray $\sigma : \mathbb{R}_+ \to X$ such that $\sigma(i) \in X_i$, then we have that the base σ induces the following tower of groups

$$\cdots \to \pi_1\bigl(X_{i+1}, \sigma(i+1)\bigr) \to \pi_1\bigl(X_i, \sigma(i)\bigr) \to \cdots \to \pi_1\bigl(X_0, \sigma(0)\bigr)$$

where the element of $\pi_1(X_{i+1}, \sigma(i+1)$ represented by $\alpha : I \to X_{i+1}$, $\alpha(0) = \sigma(i+1) = \alpha(1)$, is applied to the element represented by $\sigma|_{[i,i+1]} \cdot \alpha \cdot (\sigma|_{[i,i+1]})^{-1}$. This base ray induces similar sequences for the homotopy group functors π_q. If $q = 0$ we have a tower of pointed sets and for $q > 1$ a tower of abelian groups. The relation between this tower of groups and the Brown–Grossman exterior homotopy groups is given by a global version $\mathcal{P}$ of Brown's functor (we refer the reader to [12] and [11] for the exact formulation of this global and other versions of this functor).

For exterior spaces, similarly to the results given in the mentioned references, we obtain the following result.

Theorem 1.15. *Let X be an exterior space with a countable exterior neighbourhood base of the form $X = X_0 \supset X_1 \supset X_2 \supset \cdots$. Suppose that we have an exterior base ray $\sigma : \mathbb{R}_+ \to X$ such that $\sigma(i) \in X_i$. Then*

$$\pi_q^B(X,\sigma) \cong \mathcal{P}\bigl\{\pi_q\bigl(X_i, \sigma(i)\bigr)\bigr\}.$$

In this paper we shall also use the inverse limit functor and its first derived in the case of tower of groups.

Let $\cdots \to G_2 \xrightarrow{p_1} G_1 \xrightarrow{p_0} G_0$ be a tower of groups. Consider the map $d : \prod_{i=0}^\infty G_i \to \prod_{i=0}^\infty G_i$ given by

$$d(g_0, g_1, g_2, \ldots) = \bigl(g_0^{-1} p_0(g_1), g_1^{-1} p_1(g_2), g_2^{-1} p_2(g_3), \ldots\bigr).$$

Then the inverse limit is given by $\mathrm{Lim}\{G_i, p_i\} = \mathrm{Ker}\, d$. We have the right action $\prod_{i=0}^\infty G_i \times \prod_{i=0}^\infty G_i \to \prod_{i=0}^\infty G_i$ given by

$$\begin{aligned} x \cdot g &= (x_0, x_1, x_2, \ldots) \cdot (g_0, g_1, g_2, \ldots) \\ &= \bigl(g_0^{-1} x_0 p_0(g_1), g_1^{-1} x_1 p_1(g_2), g_2^{-1} x_2 p_2(g_3), \ldots\bigr). \end{aligned}$$

The pointed set of orbits of this action is denoted by $\mathrm{Lim}^1\{G_i, p_i\}$ and it is called the first derived of the Lim functor. We shall also use the shorten notation, $\mathrm{Lim}\{G_i\}$ and $\mathrm{Lim}^1\{G_i\}$. For more properties of this functors we refer the reader to [3] and [4].

Theorem 1.16. *Let* $\cdots \to G_2 \overset{p_1}{\to} G_1 \overset{p_0}{\to} G_0$ *be a tower of groups.*

(i) *If each* $p_i : G_{i+1} \to G_i$ *is an epimorphism then* $\mathrm{Lim}^1\{G_i\} = *$,
(ii) *the* Lim *and* Lim^1 *functors of a short exact sequence*

$$0 \to \{G_i\} \to \{H_i\} \to \{K_i\} \to 0,$$

are connected by the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Lim}\{G_i\} \to \mathrm{Lim}\{H_i\} \to \mathrm{Lim}\{K_i\} \\ &\to \mathrm{Lim}^1\{G_i\} \to \mathrm{Lim}^1\{H_i\} \to \mathrm{Lim}^1\{K_i\} \to 0. \end{aligned}$$

The exact sequence given in Theorem 1.13 and the Lim and Lim^1 functors are related as follows.

Theorem 1.17. *Let X be a first countable at infinity exterior space having an exterior neighbourhood base $X = X_0 \supset X_1 \supset \cdots \supset X_i \supset \cdots$ and let $\sigma : \mathbb{R}_+ \to X$ be an exterior base ray such that $\sigma(i) \in X_i$, then in the exact sequence*

$$\cdots \to \pi^B_{q+1}(X, \sigma|_{\mathbb{N}}) \to \pi^S_q(X, \sigma) \to \pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}}) \to \cdots$$

we also have that $\mathrm{Ker}(\pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}})) \cong \mathrm{Lim}\{\pi_q(X_i, \sigma(i))\}$ *and for* $q > 0$, $\mathrm{Coker}(\pi^B_q(X, \sigma|_{\mathbb{N}}) \to \pi^B_q(X, \sigma|_{\mathbb{N}})) \cong \mathrm{Lim}^1\{\pi_q(X_i, \sigma(i))\}$.

Remark 1.18. For $q = 0$, $\mathrm{Ker}(\pi^B_0(X, \sigma|_{\mathbb{N}}) \to \pi^B_0(X, \sigma|_{\mathbb{N}}))$ is the fibre of a map between pointed sets and $\mathrm{Lim}\{\pi_0(X_i, \sigma(i))\}$ in the inverse limit of a tower on pointed sets. For $q = 1$, $\mathrm{Coker}(\pi^B_1(X, \sigma|_{\mathbb{N}}) \to \pi^B_1(X, \sigma|_{\mathbb{N}}))$ is interpreted as the pointed set of the orbits of the right action $\pi^B_1(X, \sigma|_{\mathbb{N}}) \times \pi^B_1(X, \sigma|_{\mathbb{N}}) \to \pi^B_1(X, \sigma|_{\mathbb{N}})$ given by $x \cdot g = g^{-1}x(\mathrm{sh}_\sigma\, g)$, where sh_σ denotes the shift operator induced by σ, see [11].

2. Postnikov factorization for Brown–Grossman groups

In this section we construct, for an exterior space X, a canonical factorization

$$\cdots \to P^B_{n+1}(X) \to P^B_n(X) \to \cdots \to P^B_0(X)$$

where each exterior space $P^B_n(X)$ is a section of Postnikov type for the exterior Brown–Grossman homotopy groups π^B_q.

For each $n \geqslant 0$, consider the set of all exterior maps $\{u_\lambda : \mathbb{N} \bar{\times} S^{n+1} \to X\}_{\lambda \in \Lambda}$ and denote by $Q_n^B(X)$ the exterior space obtained by the following push-out in the category **E**

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda}(\mathbb{N}\bar{\times} S^{n+1})_\lambda & \xrightarrow{\coprod_{\lambda\in\Lambda} u_\lambda} & X \\ \downarrow & & \downarrow l_n \\ \coprod_{\lambda\in\Lambda}(\mathbb{N}\bar{\times} D^{n+2})_\lambda & \xrightarrow{\coprod_{\lambda\in\Lambda} v_\lambda} & Q_n^B(X) \end{array}$$

Since the relative homotopy groups $\pi_q^B(Q_n^B(X), X)$ are trivial for each $q \leqslant n+1$ and every base sequence $\mathbb{N} \to X$, the induced homomorphism $\pi_q^B(l_n)$ is an isomorphism for $q \leqslant n$ and an epimorphism for $q = n+1$. Then, an element of $\pi_{n+i}^B(Q_n^B(X))$ can be represented by an exterior map of the form

$$\mathbb{N} \bar{\times} S^{n+1} \xrightarrow{u_\lambda} X \xrightarrow{l_n} Q_n^B(X), \quad \text{for some } \lambda \in \Lambda,$$

which has an exterior extension v_λ. Therefore $\pi_{n+1}^B(Q_n^B(X))$ is a trivial group for any base sequence. We remark that in the case that the set Λ is empty we have that $Q_n^B(X) = X$.

By iterating the functorial construction above, we obtain a sequence of inclusions

$$X \to Q_n^B(X) \to Q_{n+1}^B Q_n^B(X) \to \cdots \to Q_k^B Q_{k-1}^B \cdots Q_n^B(X) \to \cdots.$$

Now consider the exterior space

$$P_n^B(X) = \operatorname*{colim}_{k \geqslant n} Q_k^B \cdots Q_n^B(X)$$

and denote by $\eta_n^X : X \to P_n^B(X)$ the natural inclusion.

Note that, for every $q \geqslant 0$, $\mathbb{N} \bar{\times} S^q$ has the cocompact externology. Since the inclusions $\mathbb{N} \bar{\times} S^m \to \mathbb{N} \bar{\times} D^{m+1}$ are closed and e-closed, it follows that each inclusion of the above sequence is closed and e-closed. On the other hand, $\{p\}$ is closed and e-closed in $Q_k^B \cdots Q_n^B(X)$ for all $p \in Q_k^B \cdots Q_n^B(X) \setminus X$. Taking into account that $P_n^B(X)$ has the colim topology and the colim externology, using an argument analogous to those given in Section 4 of [7], we can conclude that for every exterior map $\mathbb{N} \bar{\times} S^q \to P_n^B(X)$ there is an integer k, $k \geqslant \max\{n, q+1\}$, such that this map factors through $Q_k^B \cdots Q_n^B(X)$.

Since the inclusion induces isomorphisms $\pi_i^B(X) \cong \pi_i^B Q_k^B \cdots Q_n^B(X)$ for $i \leqslant n$ and the groups $\pi_i^B Q_k^B \cdots Q_n^B(X)$ are trivial for $n+1 \leqslant i \leqslant k+1$, we obtain that $\eta_n^X : X \to P_n^B(X)$ induces isomorphisms $\pi_q^B(\eta_n^X)$ for $q \leqslant n$ and the groups $\pi_q^B(P_n^B(X))$ are trivial for $q \geqslant n+1$.

Note that the construction $P_n^B(\cdot)$ is functorial and the pair $(P_n^B(X), X)$ is a relative $\mathbb{N}$-complex, where we are using the notion of $\mathbb{N}$-complex introduced in [9].

On the other hand, the inclusions

$$\left\{Q_k^B \cdots Q_{n+1}^B(X) \to Q_k^B \cdots Q_n^B(X)\right\}_{k \geqslant n+1}$$

induce an exterior map $f_{n+1} : P_{n+1}^B(X) \to P_n^B(X)$ such that $f_{n+1}\eta_{n+1}^X = \eta_n^X$.

Then, we obtain the following canonical tower of exterior spaces

$$\cdots \to P_{n+1}^B(X) \xrightarrow{f_{n+1}} P_n^B(X) \to \cdots \to P_0^B(X)$$

and canonical inclusions $\eta_n^X : X \to P_n^B(X)$ with the following properties:

(1) the pair $(P_n^B(X), X)$ is a relative $\mathbb{N}$-complex,
(2) $f_{n+1}\eta_{n+1}^X = \eta_n^X$,
(3) $\eta_q^B(\eta_n^X)$ is an isomorphism for all $q \leqslant n$,
(4) $\pi_q^B(P_n^B(X))$ is trivial for all $q \geqslant n+1$.

We summarize the properties of the construction above in the following result.

Theorem 2.1. *Let X be an exterior space, then there is a commutative diagram*

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & \swarrow^{\eta_{n+1}^X} & \downarrow{\scriptstyle \eta_n^X} & \searrow^{\eta_{n-1}^X} & & & \\ \cdots \longrightarrow & P_{n+1}^B(X) & \xrightarrow[f_{n+1}]{} & & P_n^B(X) & & \xrightarrow[f_n]{} & P_{n-1}^B(X) & \longrightarrow \cdots \end{array}$$

such that

(i) $\pi_q^B(\eta_n^X)$ *is an isomorphism for all* $q \leqslant n$,
(ii) $\pi_q^B(P_n^B(X))$ *is trivial for all* $q \geqslant n+1$.

Consider $\mathbf{Ho}(\mathbf{E})$ the localized category obtained by inverting the weak B-equivalences of the closed model B-structure of $\mathbf{E}$. Then as for the standard Postnikov sections we obtain the following universal property.

Proposition 2.2. *If $f : X \to Y$ is an exterior map and $\pi_q^B(Y)$ are trivial for all $q \geqslant n+1$, there is a unique map $\tilde{f} : P_n^B(X) \to Y$ in $\mathbf{Ho}(\mathbf{E})$ such that $\tilde{f}\eta_n^X = f$.*

Remark 2.3. Let X be an exterior space and suppose that

$$\cdots \to X_{n+1} \xrightarrow{g_{n+1}} X_n \to \cdots \to X_{n+1} \xrightarrow{g_1} X_0$$

is a tower of exterior spaces verifying that for every $n \geqslant 0$ there is an exterior map $\mu_n : X \to X_n$ such that $g_{n+1}\mu_{n+1} = \mu_n$, the induced homomorphism $\pi_q^B(\mu_n)$ is an isomorphism for all $q \leqslant n$ and the homotopy groups $\pi_q^B(X_n)$ are trivial for all $q \geqslant n+1$.

Then there are isomorphisms $h_n : P_n^B(X) \to X_n$, $n \geqslant 0$, in $\mathbf{Ho}(\mathbf{E})$, such that the following diagrams

$$\begin{array}{ccc} P_{n+1}^B(X) & \xrightarrow{f_{n+1}} & P_n^B(X) \\ \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} \\ X_{n+1} & \xrightarrow{g_{n+1}} & X_n \end{array} \qquad \begin{array}{ccc} P_n^B(X) & & \\ \downarrow{\scriptstyle h_n} & \nwarrow^{\eta_n^X} & \\ X_n & \xleftarrow{\mu_n} & X \end{array}$$

are commutative.

Remark 2.4. Taking into account that every exterior map $f : A \to B$ can be factored as $A \xrightarrow{p} Z \xrightarrow{g} B$ where p is a trivial B-cofibration and g is a B-fibration, the canonical factorization $\{P_n^B(X), f_n, \eta_n^X\}$ is isomorphic in $\mathbf{Ho}(\mathbf{E})$ to a factorization $\{P'_n(X), f'_n, \eta'_n\}$ where each f'_n is a B-fibration.

Remark 2.5. Given a topological space X consider the exterior space X_{tr} which has the trivial externology $\varepsilon = \{X\}$ and the exterior space X_{to} which has the total externology. If $X^{[n]}$ denotes the Postnikov section in standard homotopy theory, see [15,16], then the standard sections $(X^{[n]})_{\mathrm{tr}}$ are weak B-equivalent to the new sections $P_n^B(X_{\mathrm{tr}})$. Nevertheless, the sections $P_n^B(X_{\mathrm{to}})$ are trivial in $\mathbf{Ho}(\mathbf{E})$.

3. B-factorizations of first countable at infinity exterior spaces

In this section, we observe that if an exterior space is first countable at infinity we can construct, up to weak B-equivalence, sections which are also first countable at infinity.

Lemma 3.1. *Let X, Y exterior spaces with exterior neighbourhood bases $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, $Y = Y_0 \supset Y_1 \supset Y_2 \supset \cdots$. Suppose that $f : X \to Y$ is an exterior map such that $f(X_i) \subset Y_i$, $i \geqslant 0$, and for all $x \in X_i$, the induced maps $\pi_q(X_i, x) \to \pi_q(Y_i, f(x))$ are isomorphisms for a given integer $q \geqslant 0$. Then for any base sequence $\rho : \mathbb{N} \to X$, $\pi_q^B(X, \rho) \to \pi_q^B(Y, f\rho)$ is an isomorphism.*

Proof. Let $p : \mathbb{N} \to X$ be a base sequence. Suppose that $\beta : \mathbb{N} \bar{\times} S^q \to Y$ is an exterior map such that $\beta(\,, *) = f\rho(n)$, $n \geqslant 0$. For each $i \in \mathbb{N}$ there is $\phi(i) \geqslant i$ such that $\beta(\{n\} \bar{\times} S^q) \subset Y_i$ and $\rho(n) \in X_i$ for all $n \geqslant \phi(i)$. The map $\phi : \mathbb{N} \to \mathbb{N}$ can be chosen satisfying that $\phi(0) = 0$ and if $i < j$ then $\phi(i) < \phi(j)$. Given k such that $\phi(i+1) > k \geqslant \phi(i)$, since the restriction map, $f_i = f|_{X_i} : X_i \to Y_i$, verifies that $\pi_q(X_i, x) \to \pi_q(Y_i, f(x))$ is an isomorphims for every $x \in X_i$, there is $\alpha_k : (S^q, *) \to (X_i, \rho(k))$ such that $f_{i_*}[\alpha_k] = [\beta|_{\{k\} \bar{\times} S^q}]$ in $\pi_q(Y_i, f\rho(k))$. If we consider the exterior map $\alpha : \mathbb{N} \bar{\times} S^q \to X$ given by $\alpha|_{\{k\} \bar{\times} S^q} = \alpha_k$, then we have that $f_*[\alpha] = [\beta]$. This implies that $f_* : \pi_q^B(X, \rho) \to \pi_q^B(Y, f\rho)$ is epimorphic.

Using a similar argument we can prove that $f_* : \pi_q^B(X, \rho) \to \pi_q^B(Y, f\rho)$ is also an injective map. □

Given a tower of topological spaces $\{\cdots \to X_2 \stackrel{X_1^2}{\to} X_1 \stackrel{X_0^1}{\to} X_0\}$, the telescope of $\{X_i\}$ is constructed as the following quotient space

$$\mathrm{Tel}\{X_i\} = \left(X_0 \times \{0\} \cup \coprod_1^\infty X_i \times [i-1, i] \right) \Big/ \sim$$

where $(X_i^{i+1}(x), i) \sim (x, i)$, $x \in X_{i+1}$, $i \geqslant 0$. $\mathrm{Tel}\{X_i\}$ has a unique externology such that the family

$$\left\{ E_n = \left(X_n \times \{n\} \cup \coprod_{n+1}^\infty X_i \times [i-1, i] \right) \Big/ \sim \;\middle|\; n \geqslant 0 \right\}$$

is a countable exterior neighbourhood base. Moreover, given a level map of towers $\{f_i : X_i \to Y_i\}$, we get an induced exterior map $\mathrm{Tel}\{f_i\} : \mathrm{Tel}\{X_i\} \to \mathrm{Tel}(Y_i\}$.

Note that the map of towers, $\{p_n : E_n \to X_n\}$ given by $p_n[x,t] = X_n^i x$, $x \in X_i$, $i \geqslant n$, where X_n^i denotes the corresponding boundary composition, satisfies that each p_n is a weak equivalence. Then, if X is an exterior space with a countable exterior neighbourhood base $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, by Lemma 3.1, we have that the exterior map $p : \mathrm{Tel}\{X_i\} \to X$ given by $p[x,t] = x$, $x \in X$, is a weak B-equivalence. Therefore, we obtain the following result.

Proposition 3.2. *Let X be a first countable at infinity exterior space. For every $n \geqslant 0$, the nth Postnikov section $P_n^B(X)$ is weak B-equivalent to an exterior space $X_B^{[n]}$ which is first countable at infinity.*

Proof. Take a countable exterior neighbourhood base $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, and the induced natural maps $\eta_i : X_i \to X_i^{[n]}$, where $X_i^{[n]}$ denotes an nth Postnikov section of X_i in standard homotopy theory. The zig-zag mappings $X \xleftarrow{p} \mathrm{Tel}\{X_i\} \to \mathrm{Tel}\{X_i^{[n]}\}$ give a natural map $\eta : X \to \mathrm{Tel}\{X_i^{[n]}\}$ in the localized category $\mathbf{Ho}(\mathbf{E})$. By Lemma 3.1 and Remark 2.3 we have that $P_n^B(X)$ is weak B-equivalent to $X_B^{[n]} = \mathrm{Tel}\{X_i^{[n]}\}$, which is first countable at infinity. □

4. Postnikov factorization for Steenrod groups

In order to obtain Postnikov sections for Steenrod groups we remark that it is not possible to develop a construction similar to the one given in Section 2 for Brown–Grossman groups. Nevertheless, for an important class of exterior spaces, that we have called cce exterior spaces, the authors have found a different argument that permits to construct these sections for Steenrod groups.

Definition 4.1. An exterior space X is said to be a cce exterior space if X is first countable at infinity and it has a countable exterior neighbourhood base $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ such that

(a) X_i is 0-connected for all $i \geqslant 0$,
(b) for any base point $x \in X_{i+1}$ the induced map $\pi_1(X_{i+1}, x) \to \pi_1(X_i, x)$ is an epimorphism.

The cce exterior spaces have the following properties:

(i) If X is a cce exterior space, then X has only one Freudenthal end, see [6].
(ii) There is an exterior base ray $\sigma : \mathbb{R}_+ \to X$ such that $\sigma(i) \in X_i$, and for every base ray σ satisfying this condition the induced tower of groups

$$\cdots \to \pi_1\big(X_{i+1}, \sigma(i+1)\big) \to \pi_1\big(X_i, \sigma(i)\big) \to \cdots \to \pi_1\big(X_0, \sigma(0)\big)$$

verifies that all boundary homomorphisms $\pi_1(X_{i+1}, \sigma(i+1) \to \pi_1(X_i, \sigma(i))$, $i \geqslant 0$, are surjective.

(iii) A cce exterior space verifies that for any base sequence $\rho : \mathbb{N} \to X$, $\pi_0^B(X, \rho)$ is trivial, and for any base ray β, $\pi_0^S(X, \beta)$ is trivial.

Let X be a cce exterior space, a base ray σ induces the inverse tower of groups

$$\cdots \to \pi_n(X_2) \to \pi_n(X_1) \to \pi_n(X_0)$$

and we can consider the group $\check{\pi}_n(X, \sigma) = \mathrm{Lim}\{\pi_n(X_i)\}$. As in this case we have that $\pi_0^B(X, \sigma)$ is trivial, it follows that two base rays σ, σ' are exteriorly homotopic and then $\check{\pi}_n(X, \sigma)$ is isomorphic to $\check{\pi}_n(X, \sigma')$.

In this section, when the spaces are 0-connected, sometimes, we omit the base point in the notation $\pi_q(X, x)$ and for cce exterior spaces in $\pi_q^B(X, \rho)$, $\pi_q^S(X, \beta)$, $\check{\pi}_q(X, x)$ we omit the base sequence or base ray.

Proposition 4.2. *Let X be a cce exterior space. Then for $n \geqslant 0$ the map $\eta : X \to X_B^{[n]}$ satisfies*

(i) *if $n \geqslant 1$, the induced map $\pi_q^S(X) \to \pi_q^S(X_B^{[n]})$ is an isomorphism for $q \leqslant n-1$,*
(ii) *the induced map $\pi_n^S(X) \to \pi_n^S(X_B^{[n]})$ is an epimorphism and $\pi_n^S(X_B^{[n]}) \cong \check{\pi}_n(X)$,*
(iii) *for $q > n$, $\pi_q^S(X_B^{[n]})$ is trivial.*

Proof. By Theorem 1.13, we can consider the following commutative diagram induced by $\eta : X \to X_B^{[n]}$ between horizontal long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \to & \pi_{q+1}^B(X) & \to & \pi_q^S(X) & \to & \pi_q^B(X) & \to & \pi_q^B(X) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \pi_{q+1}^B(X_B^{[n]}) & \to & \pi_q^S(X_B^{[n]}) & \to & \pi_1^B(X_B^{[n]}) & \to & \pi_q^B(X_B^{[n]}) & \to & \cdots
\end{array}$$

By Proposition 3.2 and Theorem 2.1, it follows that, if $q > n$, $\pi_q^S(X_B^{[n]}) \cong 0$; for the case $q = n$, by Theorem 1.17 one has that

$$\begin{aligned}
\pi_n^S\big(X_B^{[n]}\big) &\cong \mathrm{Ker}\big(\pi_n^B\big(X_B^{[n]}\big) \to \pi_n^B\big(X_B^{[n]}\big)\big) \cong \mathrm{Ker}\big(\pi_n^B(X) \to \pi_n^B(X)\big) \\
&\cong \mathrm{Lim}\big\{\pi_n(X_i)\big\} \cong \check{\pi}_n(X).
\end{aligned}$$

Finally, if $q < n$, by the five lemma, we have that $\pi_q^S(X) \to \pi_q^S(X_B^{[n]})$ is an isomorphism. $\square$

We can consider the following construction in standard homotopy theory. Let Y be a 0-connected pointed space, and suppose that, for $n \geqslant 1$, N is a subgroup of $\pi_n(Y)$ invariant by the action of the fundamental group $\pi_1(Y)$. If A is a set of generators of the $\pi_1(Y)$-module N, we can define the space $Y[N]$ as the pushout

$$\begin{array}{ccc} \bigvee_{a\in A} S^n_a & \xrightarrow{\sum_{a\in A} a} & Y \\ \downarrow & & \downarrow \\ \bigvee_{a\in A} D^{n+1}_a & \longrightarrow & Y[N] \end{array}$$

Then it is not difficult to check that the following sequence is exact

$$0 \to N \to \pi_n(Y) \to \pi_n\big(Y[N]\big) \to 0.$$

The natural map $\mu : Y \to Y[N]$ has the following property: the induced homomorphism $\pi_q(\mu)$ is an isomorphism if $q < n$, and, given any map $f : Y \to Z$ such that $\pi_n(f)(N) = 0$, where Z is a pointed space, there exists a map $\bar{f} : Y[N] \to Z$ such that $\bar{f}_\mu = f$, and this map is unique up to homotopy if we have the additional condition $\pi_{n+1}(Z) \cong 0$.

Proposition 4.3. *Let Y be a cce exterior space. Then, there exists a map $\zeta : Y \to Y^{(n)}$ in* **Ho**(**E**)*, where $Y^{(n)}$ is an exterior space first countable at infinity, such that*

(i) *For $q < n$, $\pi^B_q(\zeta)$ is an isomorphism,*
(ii) *if $q = n$, $\pi^B_n(\zeta)$ is an epimorphism and $\pi^B_n(Y^{(n)}) \cong \mathcal{P}\{\pi_n(Y_i)/I_n(Y_i)\}$, where $I_n(Y_i)$ denotes the image of the canonical map $\check{\pi}_n(Y) \to \pi_n(Y_i)$ for a countable exterior neighbourhood base $\{Y_i\}$ of Y,*
(iii) *the group $\check{\pi}_n(Y^{(n)})$ is trivial.*

Proof. The space Y has a countable exterior neighbourhood base, $Y = Y_0 \supset Y_1 \supset Y_2 \supset \cdots$, such that the boundary morphisms of the fundamental tower, $\cdots \to \pi_1(Y_{i+1}) \to \pi_1(Y_i) \to \cdots \to \pi_1(Y_0)$, are surjective. So, we have that $I_n(Y_i)$ is invariant by the action of $\pi_1(Y_i)$ and therefore we can consider the construction developed above to obtain a map $\mu_i : Y_i \to Y_i[I_n(Y_i)]$ and we also have induced maps $Y_{i+1}[I_n(Y_{i+1})] \to Y_i[I_n(Y_i)]$. The zig-zag mappings $Y \xleftarrow{q} \mathrm{Tel}\{Y_i\} \to \mathrm{Tel}\{Y_i[I_n(Y_i)]\}$ gives a natural map $\zeta : Y \to \mathrm{Tel}\{Y_i[I_n(Y_i)]\}$ in the localized category **Ho**(**E**). By the properties of the telescopic construction we have that $Y^{(n)} = \mathrm{Tel}\{Y_i[I_n(Y_i)]\}$ is an exterior space first countable at infinity, and, as a consequence of Lemma 3.1 and Theorem 1.15, one has that $Y^{(n)}$ satisfies properties (i) and (ii). From the short exact sequence $0 \to \{I_n(Y_i)\} \to \{\pi_n(Y_i)\} \to \{\pi_n(Y_i)/I_n(Y_i)\} \to 0$, by Theorem 1.16 we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Lim}\big\{I_n(Y_i)\big\} \to \mathrm{Lim}\big\{\pi_n(Y_i)\big\} \to \mathrm{Lim}\big\{\pi_n(Y_i)/I_n(Y_i)\big\} \\ &\to \mathrm{Lim}^1\big\{I_n(Y_i)\big\} \to \mathrm{Lim}^1\big\{\pi_n(Y_i)\big\} \to \mathrm{Lim}^1\big\{\pi_n(Y_i)/I_n(Y_i)\big\} \to 0. \end{aligned}$$

Taking into account that each $I_n(Y_{i+1}) \to I_n(Y_i)$ is an epimorphism, we have that $\mathrm{Lim}^1\{I_n(Y_i)\} \cong 0$; this implies that $\check{\pi}(Y^{(n)}) \cong \mathrm{Lim}\{\pi_n(Y_i)/I_n(Y_i)\} \cong 0$. □

Theorem 4.4. *Let Y be a cce exterior space. Then there exists a map $\eta\zeta : Y \to Y^{[n]}_S$, where $Y^{[n]}_S = (Y^{(n+1)})^{[n+1]}_B$ is an exterior space first countable at infinity and such that*

(i) *For* $q \leqslant n$, $\pi_q^S(\eta\zeta)$ *is an isomorphism,*
(ii) *if* $q > n$, $\pi_q^S(Y_S^{[n]}) \cong 0$.

Proof. Using the exact sequences of the commutative diagram induced by $\eta\zeta$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_{q+1}^B(Y) & \longrightarrow & \pi_q^S(Y) & \longrightarrow & \pi_q^B(Y) & \longrightarrow & \pi_q^B(Y) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \pi_{q+1}^B(Y_S^{[n]}) & \longrightarrow & \pi_q^S(Y_S^{[n]}) & \longrightarrow & \pi_q^B(Y_S^{[n]}) & \longrightarrow & \pi_q^B(Y_S^{[n]}) & \longrightarrow \cdots \end{array}$$

one has that

(a) By Propositions 4.3 and 3.2 and by the five lemma, we have isomorphisms $\pi_q^S(Y) \to \pi_q^S(Y_S^{[n]})$ for $q \leqslant n-1$.
(b) For $q = n$, if we denote by

$$\begin{aligned} &\mathrm{Coker}_{n+1}(Y) = \mathrm{Coker}\big(\pi_{n+1}^B(Y) \to \pi_{n+1}^B(Y)\big), \\ &\mathrm{Coker}_{n+1}\big(Y_S^{[n]}\big) = \mathrm{Coker}\big(\pi_{n+1}^B\big(Y_S^{[n]}\big) \to \pi_{n+1}^B\big(Y_S^{[n]}\big)\big), \\ &\mathrm{Ker}_n(Y) = \mathrm{Ker}\big(\pi_n^B(Y) \to \pi_n^B(Y)\big) \quad \text{and} \\ &\mathrm{Ker}_n\big(Y_S^{[n]}\big) = \mathrm{Ker}\big(\pi_n^B\big(Y_S^{[n]}\big) \to \pi_n^B\big(Y_S^{[n]}\big)\big), \end{aligned}$$

we can consider the induced commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Coker}_{n+1}(Y) & \longrightarrow & \pi_n^S(Y) & \longrightarrow & \mathrm{Ker}_n(Y) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathrm{Coker}_{n+1}(Y_S^{[n]}) & \longrightarrow & \pi_n^S(Y_S^{[n]}) & \longrightarrow & \mathrm{Ker}_n(Y_S^{[n]}) & \longrightarrow 0 \end{array}$$

Then, since $Y_0^{[n]} = (Y^{(n+1)})_B^{[n+1]}$, from Proposition 4.3, the map $\mathrm{Ker}_n(Y) \to \mathrm{Ker}_n(Y_S^{[n]})$ is an isomorphism. On the other hand, $\mathrm{Coker}_{n+1}(Y) \cong \mathrm{Lim}^1\{\pi_{n+1}(Y_i)\}$ and

$$\begin{aligned} \mathrm{Coker}_{n+1}\big(Y_S^{[n]}\big) &\cong \mathrm{Coker}_{n+1}\big(Y^{(n+1)}\big) \cong \mathrm{Lim}^1\big\{\pi_{n+1}\big(Y_i\big[I_{n+1}(Y_i)\big]\big)\big\} \\ &\cong \mathrm{Lim}^1\big\{\pi_{n+1}(Y_i)/I_{n+1}(Y_i)\big\}. \end{aligned}$$

Now, by Theorem 1.16 we can use the exact sequence $\mathrm{Lim} - \mathrm{Lim}^1$ induced by the short exact sequence $0 \to \{I_{n+1}(Y_i)\} \to \{\pi_{n+1}(Y_i)\} \to \{\pi_{n+1}(Y_i)/I_{n+1}(Y_i)\} \to 0$, to obtain that the induced map $\mathrm{Lim}^1\{\pi_{n+1}(Y_i)\} \to \mathrm{Lim}^1\{\pi_{n+1}(Y_i)/I_{n+1}(Y_i)\}$ is an isomorphism. Therefore the map $\pi_n^S(Y) \to \pi_n^S(Y_S^{[n]})$ is also an isomorphism.
(c) As $Y_S^{[n]} = (Y^{(n+1)})_B^{[n+1]}$, it follows from Proposition 4.2 that $\pi_{n+1}^S(Y_S^{[n]}) \cong \check{\pi}_{n+1}(Y^{(n+1)})$ and by Proposition 4.3 we have that $\check{\pi}_{n+1}(Y^{(n+1)}) \cong 0$.
(d) From the long exact sequence of the space $Y_S^{[n]}$ given in Theorem 1.13, we have that $\pi_q^S(Y_S^{[n]}) \cong 0$ for $q > n+1$. □

Theorem 4.5. *Let Y be a cce exterior space. Then the map $\eta\zeta : Y \to Y_S^{[n]}$ satisfies*

(i) *For $q \leqslant n$, $\pi_q^B(\eta\zeta)$ is an isomorphism,*
(ii) *for $q = n+1$, $\pi_{n+1}^B(\eta\zeta)$ is an epimorphism and*

$$\pi_{n+1}^B\big(Y_S^{[n]}\big) \cong \mathcal{P}\big\{\pi_{n+1}(Y_i)/I_{n+1}(Y_i)\big\},$$

(iii) *if $q > n+1$, $\pi_q^B(Y_S^{[n]}) \cong 0$.*

Proof. Since $Y_S^{[n]} = (Y^{(n+1)})_B^{[n+1]}$ it suffices to apply Propositions 3.2 and 4.3 to obtain the desired properties. □

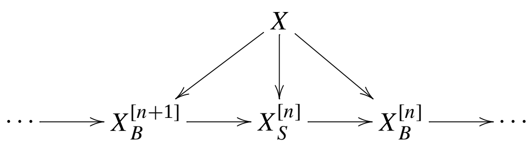
Remark 4.6. It is interesting to note that for a given 0-connected topological space X the standard Postnikov section $(X^{[n]})_{\mathrm{tr}}$ is weak B-equivalent to $(X_{\mathrm{tr}})_S^{[n]}$. The sections $(X_{\mathrm{to}})_S^{[n]}$ are trivial in $\mathbf{Ho}(\mathbf{E})$.

5. Mix factorization and homotopy fibres

In this section, we work with cce exterior spaces X. Suppose that $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ is a countable exterior neighbourhood base of 0-connected spaces such that the induced tower of groups $\cdots \to \pi_1(X_{i+1}) \to \pi_1(X_i) \to \cdots \to \pi_1(X_0)$ has surjective boundary homomorphisms.

We note that the map $X \to X^{(n+1)}$ induces a map $X_B^{[n+1]} \to (X^{(n+1)})_B^{[n+1]} = X_S^{[n]}$. Now, applying the construction $(\cdot)_B^{[n]}$ to the space $X_S^{[n]}$, we get the induced map $X_S^{[n]} \to (X_S^{[n]})_B^{[n]}$. By Theorem 4.5, it follows that $(X_S^{[n]})_B^{[n]}$ is weak B-equivalent to $X_B^{[n]}$. Therefore, we obtain the following result.

Theorem 5.1. *Let X be a cce exterior space, then there is a commutative diagram*



such that

(i) *the subdiagram associated with $\cdots \to X_B^{[n+1]} \to X_B^{[n]} \to \cdots$ is a B-factorization of X,*
(ii) *the subdiagram associated with $\cdots \to X_S^{[n+1]} \to X_S^{[n]} \to \cdots$ is an S-factorization of X,*
(iii) *the maps $X_B^{[n+1]} \to X_S^{[n]}$, $X_S^{[n]} \to X_B^{[n]}$ are B-fibrations (hence, S-fibrations) and the exterior spaces $X_B^{[n]}$, $X_S^{[n]}$ are weak B-equivalent (hence, weak S-equivalent) to exterior spaces which are first countable at infinity.*

Next, we study the exterior homotopy groups of homotopy ray fibres, given in Section 1, of the maps of the mix factorization.

Theorem 5.2. *For $n \geqslant 0$, let F_B^{n+1} be the homotopy ray fibre of the B-fibration $X_B^{[n+1]} \to X_B^{[n]}$. Then, this fibre is an $(n+1)$-dimensional Eilenberg–Mac Lane exterior space with respect to Brown–Grossman exterior homotopy groups, determined by the group $\pi_{n+1}^B(X)$. We will denote a space of this type by $F_B^{n+1} = K_B(\pi_{n+1}^B(X), n+1)$.*

Proof. It follows from the long exact sequence of Brown–Grossman exterior homotopy groups of the B-fibration $F_B^{n+1} \to X_B^{[n+1]} \to X_B^{[n]}$, see Theorem 1.12. □

Theorem 5.3. *For $n \geqslant 0$, let F_B^{n+1} be the homotopy ray fibre of the B-fibration $X_B^{[n+1]} \to X_B^{[n]}$. Then, this fibre has only two possible non trivial consecutive exterior Steenrod homotopy groups*:

(i) $\pi_{n+1}^S(F_B^{n+1}) \cong \mathrm{Lim}\{\pi_{n+1}(X_i)\} = \check{\pi}_{n+1}(X)$,
(ii) $\pi_n^S(F_B^{n+1}) \cong \mathrm{Lim}^1\{\pi_{n+1}(X_i)\}$.

Proof. By Theorem 1.13, we can consider the long exact sequence

$$\cdots \to \pi_{q+1}^B\big(F_B^{n+1}\big) \to \pi_q^S\big(F_B^{n+1}\big) \to \pi_q^B\big(F_B^{n+1}\big) \to \pi_q^B\big(F_B^{n+1}\big) \to \cdots.$$

By Theorem 5.2 we have that $F_B^{n+1} = K_B(\pi_{n+1}^B(X), n+1)$, and by Theorem 1.17 it follows that

$$\pi_{n+1}^S\big(F_B^{n+1}\big) \cong \mathrm{Ker}\big(\pi_{n+1}^B(X) \to \pi_{n+1}^B(X)\big) \cong \mathrm{Lim}\big\{\pi_{n+1}(X_i)\big\},$$
$$\pi_n^S\big(F_B^{n+1}\big) \cong \mathrm{Coker}\big(\pi_{n+1}^B(X) \to \pi_{n+1}^B(X)\big) \cong \mathrm{Lim}^1\big\{\pi_{n+1}(X_i)\big\}.$$

From the long exact sequence above and the fact that F_B^{n+1} is an Eilenberg–Mac Lane exterior space, we also get that for $q \neq n, n+1$, $\pi_q^S(F_B^{n+1}) \cong 0$. □

Theorem 5.4. *For $n \geqslant 1$, let F_S^n be the homotopy ray fibre of the B-fibration $X_S^{[n]} \to X_S^{[n-1]}$. Then, this fibre is an n-dimensional Eilenberg–Mac Lane exterior space with respect to Steenrod exterior homotopy groups, determined by the group $\pi_n^S(X)$. A space of this type will be denoted by $F_S^n = K_S(\pi_n^S(X), n)$.*

Proof. It follows from the long exact sequence of Steenrod exterior homotopy groups of the fibration $F_S^n \to X_S^{[n]} \to X_S^{[n-1]}$, see Theorem 1.12. □

Theorem 5.5. *For $n \geqslant 1$, let F_S^n be the homotopy ray fibre of the B-fibration $X_S^{[n]} \to X_S^{[n-1]}$. Then this fibre has only two possible non trivial consecutive Brown–Grossman exterior homotopy groups*:

(i) $\pi_{n+1}^B(F_S^n) \cong \mathcal{P}\{\pi_{n+1}(X_i)/I_{n+1}(X_i)\}$,

(ii) $\pi_n^B(F_S^n(F_S^n) \cong \mathcal{P}\{I_n(X_i)\}$.

Proof. By Theorem 1.12, we can consider the long exact sequence

$$\cdots \to \pi_q^B\big(F_S^n\big) \to \pi_q^B\big(X_S^{[n]}\big) \to \pi_q^B\big(X_S^{[n-1]}\big) \to \pi_{q-1}^B\big(F_S^n\big) \to \cdots.$$

Taking into account that $X_S^{[n]} = (X^{(n+1)})_B^{[n+1]}$ and $X_S^{[n-1]} = (X^{(n)})_B^{[n]}$, by Theorem 4.5 we have that

If $q > n+1$, $\pi_q^B(F_S^n) \cong 0$.

For $q = n+1$, $\pi_{n+1}^B(F_S^n) \cong \pi_{n+1}^B(X_S^{[n]}) \cong \mathcal{P}\{\pi_{n+1}(X_i)/I_{n+1}(X_i)\}$.

If $q = n$, we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{P}\{I_n(X_i)\} & \longrightarrow & \mathcal{P}\{\pi_n(X_i)\} & \longrightarrow & \mathcal{P}\{\pi_n(X_i)/I_n(X_i)\} \\
 & & & \downarrow & & \downarrow \\
0 \longrightarrow & \pi_n^B(F_S^n) & \longrightarrow & \pi_n^B(X_S^{[n]}) & \longrightarrow & \pi_n^B(X_S^{[n-1]})
\end{array}$$

where the vertical arrows are isomorphisms. Then we obtain that $\pi_n^B(F_S^n) \cong \mathcal{P}\{I_n(X_i)\}$. Since $\pi_n^B(X_S^{[n]}) \to \pi_n^B(X_S^{[n-1]})$ is an epimorphism and if $q < n$, $\pi_q^B(X_S^{[n]}) \cong \pi_q^B(X)$, we have that if $q < n$, $\pi_q^B(F_S^n) \cong 0$. □

Theorem 5.6. *For $n \geqslant 0$, let F_{BS}^{n+1} be the homotopy ray fibre of the B-fibration $X_B^{[n+1]} \to X_S^{[n]}$. Then, this fibre is an Eilenberg–Mac Lane exterior space with respect to Brown–Grossman and Steenrod exterior homotopy groups*:

(i) $F_{BS}^{n+1} = K_B(\mathcal{P}\{I_{n+1}(X_i)\}, n+1)$,
(ii) $F_{BS}^{n+1} = K_S(\check{\pi}_{n+1}(X), n+1)$.

Proof. By Theorem 1.12, we can consider the long exact sequence

$$\cdots \to \pi_{q+1}^B\big(X_S^{[n]}\big) \to \pi_q^B\big(F_{BS}^{n+1}\big) \to \pi_q^B\big(X_B^{[n+1]}\big) \to \pi_q^B\big(X_S^{[n]}\big) \to \cdots.$$

If $q > n+1$, we have that $\pi_q^B(F_{BS}^{n+1}) \cong 0$. For $q = n+1$, by Theorem 4.5 we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{P}\{I_{n+1}(X_i)\} & \longrightarrow & \mathcal{P}\{\pi_{n+1}(X_i)\} & \longrightarrow & \mathcal{P}\{\pi_{n+1}(X_i)/I_{n+1}(X_i)\} \\
 & & & \downarrow & & \downarrow \\
0 \longrightarrow & \pi_{n+1}^B(F_{BS}^{n+1}) & \longrightarrow & \pi_{n+1}^B(X_B^{[n+1]}) & \longrightarrow & \pi_n^B(X_S^{[n]})
\end{array}$$

where the vertical arrows are isomorphisms. Then, we obtain that $\pi_{n+1}^B(F_{BS}^{n+1}) \cong \mathcal{P}\{I_{n+1}(X_i)\}$. Since $\pi_{n+1}^B(X_B^{[n+1]}) \to \pi_{n+1}^B(X_S^{[n]})$ is an epimorphism and if $q \leqslant n$, $\pi_q^B(X_B^{[n+1]}) \cong \pi_q^B(X)$, and $\pi_q^B(X_S^{[n]}) \cong \pi_q^B(X)$ we get that if $q \leqslant n$, $\pi_q^B(F_{BS}^{n+1}) \cong 0$.

Using the long exact sequence given in Theorem 1.12,

$$\cdots \to \pi_{q+1}^S\big(X_S^{[n]}\big) \to \pi_q^S\big(F_{BS}^{n+1}\big) \to \pi_q^S\big(X_B^{[n+1]}\big) \to \pi_q^S\big(X_S^{[n]}\big) \to \cdots,$$

by Proposition 4.2 and Theorem 4.4, if $q > n+1$, $\pi_q^S(F_{BS}^{n+1}) = 0$. For $q = n+1$ we obtain the exact sequence

$$0 \to \pi_{n+1}^S\left(F_{BS}^{n+1}\right) \to \pi_{n+1}^S\left(X_B^{[n+1]}\right) \to 0.$$

By Proposition 4.2, $\pi_{n+1}^S(X_B^{[n+1]}) \cong \check{\pi}_{n+1}(X)$. If $q < n+1$, then we have that $\pi_q^S(F_{BS}^{n+1}) \cong 0$. Therefore $F_{BS}^{n+1} = K_S(\check{\pi}_{n+1}(X), n+1)$. □

Theorem 5.7. *For $n \geqslant 0$, let F_{BS}^n be the homotopy ray fibre of the B-fibration $X_S^{[n]} \to X_B^{[n]}$. Then this fibre is an Eilenberg–Mac Lane exterior space with respect to Brown–Grossman and Steenrod exterior homotopy groups*:

(i) $F_{SB}^n = K_B(\mathcal{P}\{\pi_{n+1}(X_i)/I_{n+1}(X_i)\}, n+1)$,
(ii) $F_{SB}^n = K_S(\mathrm{Lim}^1\{\pi_{n+1}(X_i)\}, n)$.

Proof. Consider the long exact sequence

$$\cdots \to \pi_{q+1}^B\left(X_B^{[n]}\right) \to \pi_q^B\left(F_{SB}^n\right) \to \pi_q^B\left(X_S^{[n]}\right) \to \pi_q^B\left(X_B^{[n]}\right) \to \cdots.$$

By Theorem 4.5, if $q > n+1$, we have that $\pi_q^B(F_{SB}^n) \cong 0$; for $q = n+1$, we have the exact sequence

$$0 \to \pi_{n+1}^B\left(F_{SB}^n\right) \to \mathcal{P}\left\{\pi_{n+1}(X_i)/I_{n+1}(X_i)\right\} \to 0$$

and, if $q \leqslant n$, $\pi_q^B(F_{SB}^n) \cong 0$.

Using the long exact sequence

$$\cdots \to \pi_{q+1}^S\left(X_B^{[n]}\right) \to \pi_q^S\left(F_{SB}^n\right) \to \pi_q^S\left(X_S^{[n]}\right) \to \pi_q^S\left(X_B^{[n]}\right) \to \cdots.$$

By Theorem 4.4 and Proposition 4.2, if $q > n$, $\pi_q^S(F_{SB}^n) \cong 0$. For $q = n$ we obtain the exact sequence

$$0 \to \pi_n^S\left(F_{BS}^n\right) \to \pi_n^S(X) \to \check{\pi}_n(X) \to 0.$$

By Theorem 1.17, from the long exact sequence involving the homotopy groups π_q^S and π_q^B of the space X, we obtain the short exact sequence

$$0 \to \mathrm{Lim}^1\left\{\pi_{n+1}(X_i)\right\} \to \pi_n^S(X) \to \check{\pi}_n(X) \to 0$$

and we have $\pi_n^S(F_{SB}^n) \cong \mathrm{Lim}^1\{\pi_{n+1}(X_i)\}$. If $q < n$, then we have that $\pi_q^S(F_{SB}^n) \cong 0$. Therefore $F_{SB}^n = K_S(\mathrm{Lim}^1\{\pi_{n+1}(X_i)\}, n)$. □

Remark 5.8. In this paper, we have not developed the analogous of standard Postnikov's cohomology invariants for exterior spaces. The authors think that the sequential cohomologies introduced in [9] are the right theories to reconstruct an exterior space from its sections and cohomology invariants.

6. Examples and open questions

In this section, we give some examples of Eilenberg–Mac Lane exterior spaces and we explain some connections between compact metric spaces and exterior spaces. We also give some open questions that the authors are trying to solve in order to develop Postnikov factorizations in shape categories.

6.1. Eilenberg–Mac Lane exterior spaces associated with a tower of groups

A given tower of groups $\{\cdots \to G_2 \to G_1 \to G_0\}$ induces the tower of Eilenberg–Mac Lane spaces $\{\cdots \to K(G_2,1) \to K(G_1,1) \to K(G_0,1)\}$ and we can consider the exterior space $\mathrm{Tel}\{K(G_i,1)\}$, see Section 3, which has a canonical base ray and base sequence induced by the base points of $K(G_i,1)$, $i \geqslant 0$. Notice that $\mathrm{Tel}\{K(G_i,1)\}$ is an Eilenberg–Mac Lane exterior space

$$\mathrm{Tel}\big\{K(G_i,1)\big\} = K_B\big(\mathcal{P}\{G_i\},1\big).$$

The non trivial Steenrod homotopy exterior groups are given by $\pi_0^S(\mathrm{Tel}\{K(G_i,1)\}) \cong \mathrm{Lim}^1\{G_i\}$ and $\pi_1^S(\mathrm{Tel}\{K(G_i,1)\}) \cong \mathrm{Lim}\{G_i\}$.

In the case that all bonding maps $G_{i+1} \to G_i$, $i \geqslant 0$, are surjective, $\mathrm{Tel}\{K(G_i,1)\}$ is a cce exterior space and in this case

$$\mathrm{Tel}\big\{K(G_i,1)\big\} = K_S\big(\mathrm{Lim}\{G_i\},1\big)$$

is also an Eilenberg–Mac Lane exterior space for the Steenrod exterior homotopy groups.

Similarly, for $n \geqslant 2$, a given tower of abelian groups $\{\cdots \to H_2 \to H_1 \to H_0\}$ induces the tower of Eilenberg–Mac Lane spaces $\{\cdots \to K(H_2,n) \to K(H_1,n) \to K(H_0,n)\}$ and the cce exterior space $\mathrm{Tel}\{K(H_i,n)\}$.

By taking the mix factorization of this exterior space, we have the following sections

$$\cdots \to \big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_S^{[n]} \to \big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_B^{[n]} \to \big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_S^{[n-1]} \to \cdots$$

with the non trivial homotopy ray fibres $F_{SB}^{n-1} = F_S^{n-1}$, $F_{BS}^n = F_S^n$ and F_B^n which are Eilenberg–Mac Lane exterior spaces.

$$\begin{aligned} F_{SB}^{n-1} = F_S^{n-1} &= \big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_S^{[n-1]} = K_S\big(\mathrm{Lim}^1\{H_i\},n-1\big) \\ &= K_B\big(\mathcal{P}\big\{H_i/I(H_i)\big\},n\big), \\ F_{BS}^n = F_S^n &= K_S\big(\mathrm{Lim}\{H_i\},n\big) = K_B\big(\mathcal{P}\big\{I(H_i)\big\},n\big) \end{aligned}$$

where $I(H_i) = \mathrm{image}(\mathrm{Lim}\{H_i\} \to H_i)$. For $q \geqslant n$ the B-sections and S-sections satisfy that

$$\big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_B^{[q]} = \big(\mathrm{Tel}\big\{K(H_i,n)\big\}\big)_S^{[q]} = \mathrm{Tel}\big\{K(H_i,n)\big\} = K_B\big(\mathcal{P}\{H_i\},n\big) = F_B^n$$

are Eilenberg–Mac Lane exterior spaces for Brown–Grossman groups, but they have two consecutive non trivial Steenrod groups.

We also remark that the B-sections are trivial for $q \leqslant n-1$ and the S-sections are trivial for $q \leqslant n-2$.

6.2. Exterior factorizations associated with a compact metrisable space

It is well known that a compact metrisable space X is homeomorphic to a closed subset of the Hilbert cube Q. Therefore to study this class of spaces we can consider closed subspaces of the Hilbert cube Q. Given a compact metric space $X \subset Q$, we associate with X the exterior space Q_X which is the topological space Q provided with the externology of all open neighbourhoods of X in Q. (We remark that for a compact metric space $Y \subset Q$ if X is homeomorphic to Y, the exterior space Q_X has the same exterior homotopy type that Q_Y.) For a given base point $x_0 \in X$, the following base ray $x_0^S : \mathbb{R}_+ \to Q_X$ and base sequence $x_0^B : \mathbb{N} \to Q_X$ are induced by $x_0^S(r) = x_0$, $r \in \mathbb{R}_+$ and $x_0^B(k) = x_0$, $k \in \mathbb{N}$.

It is very interesting to note that the Brown–Grossman exterior homotopy group $\pi_q^B(Q_X, x_0^B)$ is isomorphic to the qth Quigley inward group of (X, x_0), see [7] and [17], and the Steenrod exterior homotopy group $\pi_q^S(Q_X, x_0^S)$ is isomorphic to the qth Quigley approaching group of (X, x_0).

For the exterior space Q_X, we have the exterior "B-sections" $P_n^B(Q_X)$ that give the "B-factorization" of Q_X in the category of exterior spaces. Since the exterior space Q_X is first countable at infinity, by Proposition 3.2 we have that $P_n^B(Q_X)$ is weak B-equivalent to $(Q_X)_B^{[n]}$ which is first countable at infinity.

In a natural way the following question arises:

Question 1. Given a compact metric space $X \subset Q$, under which conditions for each $n \geqslant 0$ there is a compact metric space $X_I^{[n]}$ $(\subset Q,\ I = \text{Inward})$ such that $Q_{X_I^{[n]}}$ is weak B-equivalent to $(Q_X)_B^{[n]}$.

Given a compact metric space, we remark that X is connected if and only if Q_X has a countable exterior neighbourhood base $Q_X = Q_0 \supset Q_1 \supset Q_2 \supset \cdots$, $(\bigcap_0^\infty Q_i = X)$ such that for $i \geqslant 0$, Q_i is a 0-connected space.

It is interesting to observe that if we also have that for $i \geqslant 0$, $\pi_1(Q_{i+1}, x_0) \to \pi_1(Q_i, x_0)$ is an epimorphism; that is, Q_X is a cce exterior space, then the pointed continuum (X, x_0) is a pointed 1-movable space.

For a given compact metric space $X \subset Q$ such that Q_X is a cce exterior space, we have constructed in Section 4, the "S-sections" $(Q_X)_S^{[n]}$ and the following natural question arises:

Question 2. Given a continuum $X \subset Q$ such that Q_X is a cee exterior space, under which conditions for each $n \geqslant 0$ there is a continuum $X_A^{[n]}$ $(\subset Q,\ A = \text{Approaching})$ such that $Q_{X_A^{[n]}}$ is weak B-equivalent to $(Q_X)_S^{[n]}$.

The answer to these interesting questions and its extension for more general settings give the construction in the shape (or strong shape) category of Postnikov decompositions for Quigley groups.

Example 1. The Hawaiian earring (Fig. 1) H is a continuum embedded into the plane $\mathbb{R}^2$ which is formed by a sequence of circles $C_1, C_2, C_3, \ldots$ that are all tangent to each other at the same point and the sequence of radii converges to zero. Since $\mathbb{R}^2$ is homeomorphic to $(0,1) \times (0,1)$, which is homeomorphic to $(0,1) \times (0,1) \times \{1/2\} \times \cdots \subset Q$, we can suppose that H is also a closed subspace of the Hilbert cube. If we consider on the space

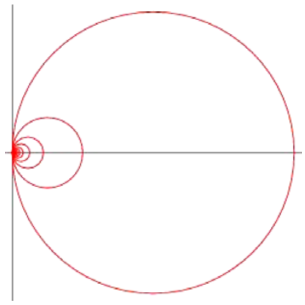


Fig. 1.

$\mathbb{R}^2$ the externology of open neighbourhoods of H in $\mathbb{R}^2$, the induced exterior space $\mathbb{R}^2_H$ and Q_H have the same exterior homotopy type. Associated with each circle C_i, we can take a generator x_i, $i \geqslant 1$. Now, let $F[x_1,\ldots,x_n]$ be the free group generated by the finite set $\{x_1,\ldots,x_n\}$. It is not hard to take a sequence of open neighbourhoods $\mathbb{R}^2 = E_0 \supset E_1 \supset E_2 \supset \cdots$ such that $\bigcap_0^\infty E_i = H$ and for $n \geqslant 1$ E_n has the homotopy type of $C_1 \cup \cdots \cup C_n$. The inclusion $E_{n+1} \to E_n$ induces the map $b:\pi_1(E_{n+1}) \to \pi_1(E_n)$, where $\pi_1(E_{n+1}) \cong F[x_1,\ldots,x_n,x_{n+1}]$, $\pi_1(E_n) \cong F[x_1,\ldots,x_n]$ and b carries x_1 to $x_1,\ldots,x_n$ to x_n and x_{n+1} to 1. Since E_n is 0-connected for $n \geqslant 0$, we have that $\mathbb{R}^2_H$ is a cee exterior space.

Note that the mix factorization of the exterior space $\mathbb{R}^2_H$ has only the non trivial homotopy ray fibres $F^1_B = F^1_S = F^1_{BS} = \mathbb{R}^2_H$ and we obtain that for $q \geqslant 1$, $(\mathbb{R}^2_H)^{[q]}_B = \mathbb{R}^2_H$ which is an Eilenberg–Mac Lane exterior space for Brown–Grossman groups

$$\mathbb{R}^2_H = K_B\big(\mathcal{P}\big\{F[x_1,\ldots,x_n]\big\},1\big)$$

and for $q \geqslant 1$, $(\mathbb{R}^2_H)^{[q]}_S = \mathbb{R}^2_H$ which is also an Eilenberg–Mac Lane exterior space for Steenrod groups

$$\mathbb{R}^2_H = K_S\big(\mathrm{Lim}\big\{F[x_1,\ldots,x_n]\big\},1\big).$$

Example 2. Suppose that we have a tower of groups $\{\cdots \to G_2 \to G_1 \to G_0\}$ such that for each $i \geqslant 0$ the map $G_{i+1} \to G_i$ is surjective and each $K(G_i,1)$ has the homotopy type of a finite CW-complex. Therefore we can take a tower of Eilenberg–Mac Lane spaces $\{\cdots \to K(G_2,1) \to K(G_1,1) \to K(G_0,1)\}$ of finite CW-complexes and the compact metrisable space $X = \mathrm{Lim}\{K(G_i,1)\}$ satisfies that Q_X has the exterior homotopy type of the exterior space $\mathrm{Tel}\{K(G_i,1)\}$. In this case, one obtains that

$$Q_X = K_B\big(\mathcal{P}\{G_i\},1\big), \qquad Q_X = K_S\big(\mathrm{Lim}\{G_i\},1\big).$$

We leave to the reader the case of a tower of abelian groups $\{\cdots \to H_2 \to H_1 \to H_0\}$ and $n \geqslant 2$ such that $K(H_i,n)$ has the homotopy type of a finite CW-complex, one can take the compact metrisable space obtained by the inverse limit and to study the associated exterior spaces and factorizations.

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