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MODELS FOR TORSION HOMOTOPY TYPES*

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ABSTRACT

Given an integer $n > 1$ and any set P of positive integers, one can assign to each topological space X a homotopy universal map $X^{(P,n)} \to X$ where $X^{(P,n)}$ is an $(n-1)$-connected CW-complex whose homotopy groups are P-torsion. We analyze this construction and its properties by means of a suitable closed model category structure on the pointed category of topological spaces.

Introduction

This article aims to link recent work of Blanc [Bl], Chachólski [Ch], Dror Farjoun [DF96], Hirschhorn [Hir] and Nofech [N93] with parallel advances by Elvira–Hernández [E-H] and Extremiana–Hernández–Rivas [E-H-R]. We exploit a closed model category structure [Q67] on the category Top_* of pointed topological spaces, for each $n \geq 2$ and each set of positive integers P, in which the class of weak equivalences is the class of maps $X \to Y$ inducing isomorphisms of homotopy groups with mod m coefficients,

$$\pi_r(X;\mathbb{Z}/m) \cong \pi_r(Y;\mathbb{Z}/m), \qquad \text{for } r \geq n+1 \text{ and } m \in P.$$

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By suitably factoring, in this closed model category, each map of the form $\star \to X$ into a cofibration followed by a trivial fibration, $\star \to X^{(P,n)} \to X$, one obtains a colocalization functor which we call a (P,n)-CW-approximation. It is indeed reminiscent from the usual CW-approximation, where one associates with any space X a CW-complex K together with a map $K \to X$ inducing isomorphisms of homotopy groups. The space $X^{(P,n)}$ is built from torsion Moore spaces of type $M(\mathbb{Z}/m, r)$, with $r \geq n$ and $m \in P$, by means of a countable sequence of push-outs. Approximations of spaces using Moore spaces as building blocks have also been discussed by Blanc in [Bl], where interesting applications have been given.

The closed model category structure used in our article is directly inspired by the one given in [E-H-R] for the case of ordinary homotopy groups. It does not coincide with the structure studied by Hirschhorn [Hir] and Nofech [N95], [N96], although the associated homotopy categories are indeed equivalent.

Of course, it is also possible to factor each map $X \to \star$ into a cofibration followed by a trivial fibration, $X \to X_{(P,n)} \to \star$. This yields a localization functor assigning to each X a space whose homotopy groups are uniquely P-divisible in dimensions $r \geq n+1$ and P-torsion-free in dimension n. (An abelian group A is said to be uniquely P-divisible if multiplication by m is an automorphism of A for every $m \in P$, and an element $a \in A$ is said to be P-torsion if there are integers $m_1, \ldots, m_r$ in P, not necessarily distinct, such that $m_1 \cdots m_r a = 0$.) Those functors are variants of the classical localization of spaces at sets of primes. We shall not insist in their analysis, as they have been previously discussed by Bousfield [B94], [B96], and Casacuberta-Rodríguez [C-R]. However, we emphasize that the study of such localizations in the framework of abstract homotopy theory is more naturally associated with a different closed model category structure, in which a functorial model for the localization of a space X is obtained by suitably factoring the map $X \to \star$ into a trivial cofibration followed by a fibration. This is precisely the point of view adopted by Quillen in his pioneering work on rational homotopy theory [Q69]; it was exploited further by Bousfield [B75] in connection with homological localization, and by several other authors since then.

This paper intends to be largely self-contained, except for standard input from homotopical algebra. Thus we supply alternative, direct proofs of earlier results due to Blanc [Bl] and Dror Farjoun [DF92], and improve some of them. Notably, Theorem 5.2 below shows that the homotopy groups of $X^{(P,n)}$ coincide with those of the homotopy fibre of the localization map $X \to X_{(P,n)}$ in all dimensions except possibly in dimension n; this gives a positive answer to a question raised in [DF92]. Using this fact, we compute $K(A,d)^{(P,n)}$ for any abelian group A and every $d \geq 1$.

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1. Preliminaries

We shall work in the pointed category Top_* of topological spaces. Thus, all maps will preserve basepoints and $[X,Y]$ will denote the set of pointed homotopy classes of maps from X to Y.

Given any space M, a space X is called M-cellular [DF96], or an M-CW-complex [Bl], if X belongs to the smallest class of spaces which contains M and is closed under pointed homotopy colimits and homotopy equivalences. A map $f\colon X \to Y$ is said to be an M-equivalence if the induced map of based mapping spaces

$$\mathrm{map}_*(M,X) \to \mathrm{map}_*(M,Y)$$

is a weak homotopy equivalence. As explained in [DF96, 2.B], for every space X there exists an M-equivalence $\mathrm{CW}_M(X) \to X$ where $\mathrm{CW}_M(X)$ is an M-CW-complex; see also [Ch]. This map is called an M-CW-approximation to X. On the other hand, a space X is said to be M-null if the space $\mathrm{map}_*(M,X)$ is weakly contractible. For every space X there is a homotopy universal map $X \to P_M X$ into an M-null space; see [B94], [Ch], [DF96, § 1]. This is called an M-nullification of X.

We shall analyze further these concepts in an important special case. For any positive integer m and $n \geq 2$, let $M(\mathbb{Z}/m,n)$ denote the homotopy cofibre of the standard self-map of S^n of degree m, which is an $(n+1)$-dimensional CW-complex such that $H_n(M(\mathbb{Z}/m,n)) \cong \mathbb{Z}/m$ and $\tilde{H}_r(M(\mathbb{Z}/m,n)) = 0$ for $r \neq n$. We shall adhere to Neisendorfer's notation [Ne] for homotopy groups with coefficients, by writing

$$\pi_r(X;\mathbb{Z}/m) = [M(\mathbb{Z}/m,r-1),X],$$

which is a group if $r \geq 3$. It follows that, if $n \geq 2$, then a map $f\colon X \to Y$ is an $M(\mathbb{Z}/m,n)$-equivalence if and only if the induced homomorphisms

$$f_*\colon \pi_r(X;\mathbb{Z}/m) \to \pi_r(Y;\mathbb{Z}/m)$$

are isomorphisms for $r \geq n+1$. A space X is $M(\mathbb{Z}/m, n)$-null if $\pi_r(X; \mathbb{Z}/m) = 0$ for $r \geq n+1$; this amounts to saying that multiplication by m is a monomorphism in $\pi_n(X)$ and an automorphism of $\pi_r(X)$ for $r \geq n+1$, since the following sequence is exact [Ne, § 1]:

$$(1.1) \qquad \cdots \to \pi_r(X) \stackrel{m}{\to} \pi_r(X) \to \pi_r(X; \mathbb{Z}/m) \to \pi_{r-1}(X) \stackrel{m}{\to} \pi_{r-1}(X) \to \cdots .$$

The machinery developed by Quillen in [Q67] and [Q69] provides a suitable framework to discuss CW-approximations and nullifications, yielding explicit models which are functorial in Top_*. Recall that a closed model category $\mathcal{C}$ is a category endowed with three distinguished families of maps called cofibrations, fibrations and weak equivalences, satisfying certain axioms. For details, properties and further terminology we refer the reader to [Q67] and [Q69]. See also the recent survey by Dwyer and Spalinski [D-S].

A map which is a weak equivalence and a fibration will be called a trivial fibration, and a map which is a weak equivalence and a cofibration will be called a trivial cofibration. Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y, \end{array}$$

the map $i\colon A \to B$ is said to have the left lifting property (LLP) with respect to $p\colon X \to Y$ if a map $B \to X$ exists making both triangles commute. In this situation, one also says that p has the right lifting property (RLP) with respect to i.

2. A generalization

If X and Y are arbitrary pointed spaces, we denote by $X \rtimes Y$ the half-smash product $X \wedge Y^+$, where Y^+ denotes the union of Y with a disjoint basepoint. Thus $X \rtimes I$ is the ordinary pointed cylinder.

In [E-H-R], the following closed model category structures were considered on the category Top_* of pointed topological spaces, for each $n \geq 1$. A map $f\colon X \to Y$ is said to be an n-fibration if f has the RLP with respect to the family of inclusions

$$(D^{n+r} \rtimes \{0\}) \cup (S^{n+r-1} \rtimes I) \to D^{n+r} \rtimes I, \qquad \text{for } r \geq 0;$$

a map f is a weak n-equivalence if the induced homomorphisms $\pi_r(X) \to \pi_r(Y)$ are isomorphisms for $r \geq n$; f is an n-cofibration if it has the LLP with respect to all trivial n-fibrations. As explained in [E-H-R], the corresponding homotopy category is equivalent to the ordinary homotopy category of $(n-1)$-connected CW-complexes.

These closed model category structures can be generalized in the following way. Let $M = \Sigma M'$ be any space which is the pointed suspension of a CW-complex M'. Consider the following families of maps in the category Top_* of pointed topological spaces.

Definition 2.1: Let $f\colon X \to Y$ be a map. We say that

(i) f is a weak M-equivalence if the induced homomorphisms

$$f_*\colon [\Sigma^r M, X] \to [\Sigma^r M, Y]$$

are isomorphisms for $r \geq 0$;

(ii) f is an M-fibration if it has the RLP with respect to the family of maps

$$(C\Sigma^r M' \rtimes \{0\}) \cup (\Sigma^r M' \rtimes I) \to C\Sigma^r M' \rtimes I \qquad \text{for } r \geq 0,$$

where C denotes the pointed cone functor;

(iii) f is an M-cofibration if it has the LLP with respect to every trivial fibration.

Since each map in (ii) is both a CW-inclusion and a homotopy equivalence, every Serre fibre map is an M-fibration. However, in contrast with [Hir] or [N95], an M-fibration need not be a Serre fibre map (for instance, every map between non-connected spaces with the same basepoint component is an M-fibration). As usual, a space X will be called M-fibrant if the map $X \to \star$ is an M-fibration (hence, all spaces are M-fibrant) and X will be called M-cofibrant if the map $\star \to X$ is an M-cofibration.

PROPOSITION 2.2: *A map $f\colon X \to Y$ is a trivial M-fibration if and only if it has the right lifting property with respect to the family $\mathcal{C}$ of inclusions*

$$\star \to M, \qquad \Sigma^r M \to C\Sigma^r M, \qquad r \geq 0.$$

Proof: Note that, if a map $f\colon X \to Y$ has the RLP with respect to $\Sigma^r M \to C\Sigma^r M$, then in particular every diagram of the following form (where the upper arrow is the constant map) admits a lifting

$$\begin{array}{ccc} \Sigma^r M & \xrightarrow{\ *\ } & X \\ \downarrow & & \downarrow f \\ C\Sigma^r M & \longrightarrow & Y. \end{array}$$

Therefore, f has the RLP with respect to $\star \to \Sigma^{r+1}M$ as well. As a consequence, if a map f has the RLP with respect to the maps in $\mathcal{C}$, then the induced homomorphisms $[\Sigma^r M, X] \to [\Sigma^r M, Y]$ are isomorphisms for all r, so that f is a weak M-equivalence. In order to check that f is an M-fibration, we use the fact that by glueing together two copies of $(C\Sigma^r M' \rtimes \{0\}) \cup (\Sigma^r M' \rtimes I)$ one obtains a space which is homeomorphic to $\Sigma^{r+1}M'$, while $C\Sigma^{r+1}M'$ is homeomorphic to the space obtained by glueing together two copies of $C\Sigma^r M' \rtimes I$ in the same way.

Conversely, let $f: X \to Y$ be a trivial M-fibration. Suppose given a commutative diagram of the form

$$\begin{array}{ccc} \Sigma^r M & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ C\Sigma^r M & \xrightarrow[v]{} & Y \end{array}$$

with $r \geq 0$. Then we may argue as follows; cf. [E-H, 2.4]. Since f is a weak M-equivalence, there is a map $w: C\Sigma^r M \to X$ such that $wi = u$ and $fw \simeq v$. Let $H: C\Sigma^r M \rtimes I \to Y$ be a homotopy with $H\partial_0 = fw$ and $H\partial_1 = v$, where ∂_0, ∂_1 denote the face maps. Using the fact that f is an M-fibration, we can find a homotopy $F: C\Sigma^r M \rtimes I \to X$ such that $fF = H$, extending both w and the constant map $(x,t) \mapsto u(x)$ for $x \in \Sigma^r M$ and $t \in I$. Then $w' = F\partial_1$ satisfies $fw' = v$ and $w'i = u$, as desired. A similar argument shows that f has the RLP with respect to the map $* \to M$, hence completing the proof. ∎

THEOREM 2.3: *For every space M which is the suspension of a CW-complex, the category of pointed topological spaces together with the above families of weak M-equivalences, M-fibrations and M-cofibrations has the structure of a closed model category.*

We denote by Top_*^M this closed model category structure on the category Top_*, and thus by $\mathrm{Ho}(\mathrm{Top}_*^M)$ the category obtained from Top_*^M by formally inverting the family of weak M-equivalences. For pointed spaces X and Y, the set of morphisms from X to Y in the category $\mathrm{Ho}(\mathrm{Top}_*^M)$ will be denoted by $[X,Y]^M$.

The routine verification of the Quillen axioms CM1 to CM5 in order to prove Theorem 2.3 proceeds as in [D-S, § 8], [E-H-R, § 2], or [Q67, II.3]; compare with the approaches of Hirschhorn [Hir] and Nofech [N95]. In order to construct the factorizations stated in axiom CM5, we resort to Quillen's "small object argument" (see [Q67, II.3.3] or [D-S, 7.12]), using the maps given in Proposition 2.2 above. Hence, the resulting factorizations are functorial.

Notice that, in the process of constructing such factorizations, it suffices to take the colimit of a countable sequence whenever the space M is compact. Otherwise it will normally require transfinite sequences, as in [B75], [Hir], or [J]. However, if the space M is a (possibly infinite) wedge $\bigvee_{\alpha\in\Lambda} M_\alpha$ where each M_α is compact, then one can still avoid the use of transfinite sequences by replacing the family $\mathcal{C}$ in Proposition 2.2 by the family consisting of $\star \to M_\alpha$ and $\Sigma^r M_\alpha \to C\Sigma^r M_\alpha$ for $r \geq 0$ and all $\alpha \in \Lambda$; further details are given in the next section.

3. Localization and colocalization

If one considers the M-cofibrant space X^M constructed by factoring a map $\star \to X$ into an M-cofibration followed by a trivial M-fibration,

$$\star \to X^M \to X,$$

by means of the "small object argument", what one has is a functor $(-)^M$: $\mathrm{Top}_* \to \mathrm{Top}_*$ together with a natural transformation ε: $(-)^M \to \mathrm{Id}$. This is in fact a model for an M-CW-approximation in the sense of [DF96]. On the other hand, by factoring each map $X \to \star$ into an M-cofibration followed by a trivial M-fibration,

$$X \to X_M \to \star,$$

one obtains a functor $(-)_M$: $\mathrm{Top}_* \to \mathrm{Top}_*$ together with a natural transformation η: $\mathrm{Id} \to (-)_M$, yielding a model for M-nullification. The canonical maps $X^M \to X$ and $X \to X_M$ will be called colocalization and localization, respectively. In this section we describe some basic properties of colocalization.

Since M-cofibrations are ordinary cofibrations and Serre fibre maps are M-fibrations, it follows from standard arguments (see e.g. Theorem 9.7 in [D-S]) that for all spaces X and Y there is a natural bijection

$$[X,Y]^M \cong [X^M, Y], \tag{3.1}$$

that is, the functor $(-)^M$ is left adjoint to the "identity" functor from $\mathrm{Ho}(\mathrm{Top}_*)$ to $\mathrm{Ho}(\mathrm{Top}_*^M)$.

If we suppose in addition that X is M-cofibrant, then, since all spaces are M-fibrant, the set $[X,Y]^M$ is in one-to-one correspondence with the set of homotopy classes maps from X to Y in Top_*^M; see [Q67, 1.16]. Now, arguing as in [D-S, 4.15] and [D-S, 9.10], we infer from (3.1) that if X is M-cofibrant and Y is any space then there is a natural bijection $[X,Y]^M \cong [X,Y]$. Since weak M-equivalences are isomorphisms in $\mathrm{Ho}(\mathrm{Top}_*^M)$, we have the following.

THEOREM 3.1: *If $f\colon Y \to Z$ is a weak M-equivalence, then f induces a bijection $[X,Y] \cong [X,Z]$ for every M-cofibrant space X.*

As an immediate consequence, one obtains a broad generalization of the classical Whitehead theorem; see also [DF96, 2.E].

THEOREM 3.2: *If X and Y are M-cofibrant spaces, then a map $f\colon X \to Y$ is a homotopy equivalence if and only if it is a weak M-equivalence.*

COROLLARY 3.3: *For every space Y, the colocalization map $Y^M \to Y$ has the following universal properties:*

(1) *It is homotopy initial among weak M-equivalences $f\colon X \to Y$.*
(2) *It is homotopy terminal among maps $f\colon X \to Y$ where X is M-cofibrant.*

COROLLARY 3.4: *If X is M-cofibrant, then the colocalization map $X^M \to X$ is a homotopy equivalence.*

COROLLARY 3.5: *The adjoint pair*

$$\mathrm{Ho}(\mathrm{Top}_*^M) \underset{\mathrm{Id}}{\overset{(-)^M}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_*)$$

sets up an equivalence of categories between $\mathrm{Ho}(\mathrm{Top}_*^M)$ *and the full subcategory of* $\mathrm{Ho}(\mathrm{Top}_*)$ *whose objects are the M-cofibrant spaces.*

The M-cofibrant spaces are precisely the retracts of M-CW-complexes, since for every cofibrant X the map $\star \to X$ has the LLP with respect to $X^M \to X$. A more explicit description of M-cofibrant spaces is given in the next section in the special case where M is a wedge of torsion Moore spaces.

Let F be the homotopy fibre of the localization map $X \to X_M$. Since $X_M \to \star$ is a weak M-equivalence, the map $F \to X$ is a weak M-equivalence as well. Hence, $F^M \to X^M$ is a weak M-equivalence and we infer the following result, which will be used for calculations in Section 5.

THEOREM 3.6: *Let X be any space and let F be the homotopy fibre of the localization map $X \to X_M$. Then $F^M \simeq X^M$.*

If the space M is an infinite wedge $\bigvee_{\alpha\in\Lambda} M_\alpha$, where each M_α is compact, but M itself is not compact, then the construction of X^M described above will require the use of transfinite sequences in general. However, we can obtain a model for X^M whose construction stops at the first infinite ordinal by proceeding as follows.

Notice that a map $f\colon X \to Y$ is a trivial M-fibration if and only if it has the RLP with respect to the family $\mathcal{C}'$ of inclusions $\star \to M_\alpha$ and $\Sigma^r M_\alpha \to C\Sigma^r M_\alpha$

with $r \geq 0$ and $\alpha \in \Lambda$; cf. Proposition 2.2. Hence, for each space X, we can construct a suitable model for X^M by means of the "small object argument" using the family $\mathcal{C}'$ instead of the family $\mathcal{C}$ displayed in Proposition 2.2. For convenience, we next recall the details of the process used to decompose a given map $f: A \to X$ into an M-cofibration followed by a trivial M-fibration.

Firstly, we consider all maps of the form $g: M_\alpha \to X$, with $\alpha \in \Lambda$, and use them to construct a space $X^0 = A \vee (\bigvee_{g,\alpha} M_\alpha)$ equipped with a map $p^0: X^0 \to X$ which coincides with f on A and with g on the wedge summand labelled with g, for each g. This map $p^0: X^0 \to X$ has the RLP with respect to $\star \to M_\alpha$ for all $\alpha \in \Lambda$. Next, we construct inductively a sequence

$$X^0 \xrightarrow{j^1} X^1 \xrightarrow{j^2} X^2 \longrightarrow \cdots$$

together with maps $p^r: X^r \to X$ such that $p^r j^r = p^{r-1}$. Assuming that the map p^{r-1} has been constructed, we take all commutative diagrams D of the form

$$\begin{array}{ccc} \Sigma^r M_\alpha & \xrightarrow{u_D} & X^{r-1} \\ \downarrow & & \downarrow{\scriptstyle p^{r-1}} \\ C\Sigma^r M_\alpha & \xrightarrow[v_D]{} & X \end{array} \tag{3.2}$$

with $r \geq 0$ and $\alpha \in \Lambda$, and define $j^r: X^{r-1} \to X^r$ by the push-out

$$\begin{array}{ccc} \bigvee_D \Sigma^r M_\alpha & \longrightarrow & X^{r-1} \\ \downarrow & & \downarrow{\scriptstyle j^r} \\ \bigvee_D C\Sigma^r M_\alpha & \longrightarrow & X^r. \end{array} \tag{3.3}$$

The map $p^r: X^r \to X$ is the sum of p^{r-1} and all the maps v_D in diagram (3.2). Passage to the direct limit yields a trivial M-fibration $p: X^\infty \to X$ and the desired factorization of f as

$$A \to X^\infty \to X,$$

where X^∞ is M-cofibrant. In particular, if we choose A to be a point, then $X^\infty \simeq X^M$, by Theorem 3.2.

This construction can be modified in order to obtain substantially smaller (although possibly non-functorial) models for X^M. For instance, it suffices to pick one representative within each pointed homotopy class of maps at each step of the process. Thus, if $f: A \to X$ is a map of CW-complexes and we use cellular maps in the construction above, then we obtain a factorization $A \to \bar{X} \to X$, where $A \to \bar{X}$ is an M-cofibration, $\bar{X} \to X$ is a weak M-equivalence (which need not be an M-fibration) and $\bar{X}$ is a CW-complex. If X is M-cofibrant then X itself is homotopy equivalent to X^M. If all maps $M_\alpha \to X$ are nullhomotopic, then X^M is homotopy equivalent to a point.

4. The case of torsion Moore spaces

In the rest of the paper we specialize to the case where M is a wedge of certain compact, torsion Moore spaces. Thus let P be any set of positive integers, not necessarily prime, and $n \geq 2$ a fixed integer. Let $M = \bigvee_{m \in P} M(\mathbb{Z}/m, n)$. We shall use the notation $\mathrm{Top}_*^{(P,n)}$ for the associated closed model category structure, and refer to the corresponding families of maps as weak (P,n)-equivalences, (P,n)-fibrations and (P,n)-cofibrations, respectively. Likewise, we denote the localization $(-)_M$ by $(-)_{(P,n)}$ and the colocalization $(-)^M$ by $(-)^{(P,n)}$.

Thus, a map $f\colon X \to Y$ is a weak (P,n)-equivalence if and only if the induced homomorphisms $f_*\colon \pi_r(X;\mathbb{Z}/m) \to \pi_r(Y;\mathbb{Z}/m)$ are isomorphisms for $r \geq n+1$ and each $m \in P$. Note that, if $P_1 \subseteq P_2$ and $n_1 \geq n_2$, then every weak (P_2,n_2)-equivalence is a weak (P_1,n_1)-equivalence.

Our first aim is to provide an algebraic characterization of (P,n)-cofibrant spaces. We shall discuss primarily the cases when

$$P = \{p^k\} \quad \text{or} \quad P = \{p, p^2, p^3, \dots\},$$

where p is a prime and $k \geq 1$. In fact, Theorem 4.4 and Theorem 4.5 below will demonstrate that this is sufficiently general. Thus, let $M = M(\mathbb{Z}/p^k, n)$ or $M = \bigvee_{i=1}^{\infty} M(\mathbb{Z}/p^i, n)$, where p is a prime, $k \geq 1$, and $n \geq 2$.

Recall from [K-M, 3.10] that every torsion abelian group is the direct sum of its primary components, and every abelian p-group of finite exponent is a direct sum of cyclic groups. For a torsion abelian group G and a prime p, we denote by G_p the p-primary component of G.

LEMMA 4.1: *Let $f\colon X \to Y$ be a map between 1-connected spaces with torsion homotopy groups. Suppose that $\pi_r(X)_p = 0$ and $\pi_r(Y)_p = 0$ for $r \leq n-1$, where p is a prime. Then f induces isomorphisms $\pi_r(X;\mathbb{Z}/p^k) \cong \pi_r(Y;\mathbb{Z}/p^k)$ for $r \geq n+1$ if and only if the induced maps $\pi_r(X)_p \to \pi_r(Y)_p$ are isomorphisms for $r \geq n+1$ and $\mathrm{Tor}(\pi_n(X),\mathbb{Z}/p^k) \to \mathrm{Tor}(\pi_n(Y),\mathbb{Z}/p^k)$ is an isomorphism as well.*

Proof: In order to prove the first implication, let F be the homotopy fibre of f. The homotopy groups of F are torsion and $\pi_r(F)_p = 0$ for $r \leq n-2$.

Moreover, the assumption made implies that $\pi_r(F;\mathbb{Z}/p^k) = 0$ if $r \geq n+1$. Hence, $\pi_r(F;\mathbb{Z}/p^k) = 0$ for all r, except perhaps for $r = n$ and $r = n-1$. Now we exploit the exact sequence derived from (1.1),

$$0 \to \pi_r(F) \otimes \mathbb{Z}/p^k \to \pi_r(F;\mathbb{Z}/p^k) \to \operatorname{Tor}(\pi_{r-1}(F), \mathbb{Z}/p^k) \to 0, \tag{4.1}$$

together with the fact that the homotopy groups of F are torsion, to infer that $\pi_r(F)_p = 0$ for $r \neq n-1$. Thus, the map f induces isomorphisms $\pi_r(X)_p \cong \pi_r(Y)_p$ for all r, except perhaps for $r = n$, and the homomorphism $f_*\colon \pi_n(X)_p \to \pi_n(Y)_p$ is injective. This implies that $\operatorname{Tor}(\pi_n(X), \mathbb{Z}/p^k) \to \operatorname{Tor}(\pi_n(Y), \mathbb{Z}/p^k)$ is injective as well. In order to prove that the latter is surjective, consider the commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(X;\mathbb{Z}/p^k) & \longrightarrow & \operatorname{Tor}(\pi_n(X), \mathbb{Z}/p^k) \\ \cong\big\downarrow & & \big\downarrow \\ \pi_{n+1}(Y;\mathbb{Z}/p^k) & \longrightarrow & \operatorname{Tor}(\pi_n(Y), \mathbb{Z}/p^k), \end{array} \tag{4.2}$$

in which the horizontal maps are epimorphisms, and hence the right-hand map is an epimorphism too. The converse is proved using the exactness and naturality of the sequence (4.1). ∎

THEOREM 4.2: *Let X be a space, p a prime and $n \geq 2$.*

(1) *If $P = \{p^k\}$ with $k \geq 1$, then X has the weak homotopy type of a (P,n)-cofibrant space if and only if X is $(n-1)$-connected, $\pi_r(X)$ is p-torsion for all r and $\pi_n(X)$ is annihilated by p^k.*

(2) *If $P = \{p, p^2, p^3, \dots\}$, then X has the weak homotopy type of a (P,n)-cofibrant space if and only if X is $(n-1)$-connected and $\pi_r(X)$ is p-torsion for all $r \geq n$.*

Proof: In both cases, if X is (P,n)-cofibrant then the colocalization map $X^{(P,n)} \to X$ is a homotopy equivalence, by Corollary 3.4. In the construction of $X^{(P,n)}$ described at the end of Section 3, we see inductively that X^r is $(n-1)$-connected for all r. Hence $X^{(P,n)}$ is $(n-1)$-connected too. Since the class of p-torsion abelian groups is a Serre class [S] and it is closed under direct limits, it follows from a Mayer–Vietoris argument that the reduced singular homology groups $H_r(X^{(P,n)})$ are p-torsion for all r, and Serre's version of the Hurewicz theorem [S] ensures that the homotopy groups $\pi_r(X^{(P,n)})$ are p-torsion for all r as well. Moreover, $H_n(X^{(P,n)})$ is an epimorphic image of $H_n(X^0)$; hence, in case (1) the group $H_n(X^{(P,n)})$ is a $\mathbb{Z}/p^k$-module and therefore $\pi_n(X^{(P,n)})$ is also a $\mathbb{Z}/p^k$-module.

In order to prove the converse statements in (1) and (2), we need to show that the hypotheses made imply that the colocalization map $X^{(P,n)} \to X$ induces isomorphisms $\pi_r(X^{(P,n)}) \cong \pi_r(X)$ for all r. But this follows from Lemma 4.1. ∎

Notice that $M(\mathbb{Z}/p^2, n)$ is not (P,n)-cofibrant if $P = \{p\}$.

If $P = \{p\}$, then the homotopy category $\mathrm{Ho}(\mathrm{Top}_*^{(P,n)})$ is equivalent to the homotopy category of $(n-1)$-connected CW-complexes such that $\pi_n(X)$ is a $\mathbb{Z}/p$-vector space and $\pi_r(X)$ is p-torsion for $r \geq n+1$. This class of spaces was considered by Bousfield in [B94]. It would be interesting to develop algebraic models for their homotopy category; recent work of Goerss [G] has opened the way into this direction.

We next show that the case where P is any set of positive integers can be reduced to the special cases discussed above. We say that a prime p has finite height in the set P if there is a nonnegative integer h such that p^{h+1} does not divide any number $m \in P$. If this is the case, then the *height* of p in P is the minimum of such integers h; we shall denote it by $h(p)$. Otherwise, we say that p has infinite height in P. The following result generalizes Theorem 4.2.

THEOREM 4.3: *Let $n \geq 2$ and let P be an arbitrary set of positive integers. Then a space X has the weak homotopy type of a (P,n)-cofibrant space if and only if X is $(n-1)$-connected, $\pi_r(X)$ is P-torsion for all r and $\pi_n(X)_p$ is annihilated by $p^{h(p)}$ for each prime p which has finite height $h(p)$ in P.*

THEOREM 4.4: *For every space X and every set P of positive integers, let Q be the union of the sets $\{p, p^2, p^3, \dots\}$ for each prime p of infinite height in P, and $\{p^{h(p)}\}$ for each prime p of nonzero finite height $h(p)$ in P. Then $X^{(P,n)} \simeq X^{(Q,n)}$ for any $n \geq 2$.*

Proof: By Theorem 4.3, the classes of (P,n)-cofibrant spaces and (Q,n)-cofibrant spaces coincide. Hence, our claim follows from Corollary 3.3. ∎

THEOREM 4.5: *Let P be any set of positive integers and $n \geq 2$. Suppose that P is the union of a family of sets P_i such that the numbers in P_i are mutually prime with the numbers in P_j whenever $i \neq j$. Then, for each space X, we have*

$$X^{(P,n)} \simeq \bigvee_i X^{(P_i,n)}.$$

Proof: Since every weak (P,n)-equivalence is a weak (P_i,n)-equivalence, there is a map $X^{(P_i,n)} \to X^{(P,n)}$ for each i. These yield together a map

$$\bigvee_i X^{(P_i,n)} \longrightarrow X^{(P,n)}. \tag{4.3}$$

For each index i, the inclusion of $X^{(P_i,n)}$ into $\bigvee_i X^{(P_i,n)}$ induces an isomorphism in homology with coefficients in P_i. Hence, by [Ne, 3.10], it also induces an isomorphism in homotopy with coefficients in P_i, that is, it is a weak (P_i,n)-equivalence. Therefore, the natural map $\bigvee_i X^{(P_i,n)} \to X$ is a weak (P_i,n)-equivalence for all i, and hence it is a weak (P,n)-equivalence. It follows that (4.3) is a weak (P,n)-equivalence between (P,n)-cofibrant spaces, and thus it is a homotopy equivalence. ∎

We finally address the case where M is a wedge of Moore spaces of various dimensions. Observe that if $M_1 = M(\mathbb{Z}/p^{k_1}, n_1)$ and $M_2 = M(\mathbb{Z}/p^{k_2}, n_2)$ satisfy either $n_1 > n_2$ or $n_1 = n_2$ and $k_1 \le k_2$, then the classes of weak $(M_1 \vee M_2)$-equivalences and M_2-equivalences coincide, which implies that $X^{M_1 \vee M_2} \simeq X^{M_2}$, by Corollary 3.3. In order to generalize this fact, the following notation will be convenient. If k is an integer, then we write $M(p,k,n) = M(\mathbb{Z}/p^k, n)$; otherwise, $M(p,\infty,n) = \bigvee_{i=1}^{\infty} M(\mathbb{Z}/p^i, n)$.

Let X be a space and $W = \bigvee_{n\ge 2}\bigvee_{m\in P_n} M(\mathbb{Z}/m, n)$, where each P_n is a set of positive integers, possibly empty. For each prime p, let $n(p)$ be the smallest value of n such that p divides some number in P_n, or omit p from the indexing if it does not occur in W. Let $h(p)$ be the height of p in the set $P_{n(p)}$ (here we do not exclude the possibility that $h(p) = \infty$). Let $M = \bigvee_p M(p, h(p), n(p))$. Then

$$X^W \simeq X^M \simeq \bigvee_p X^{M(p,h(p),n(p))}. \tag{4.4}$$

Indeed, the first homotopy equivalence follows from the fact that the classes of weak W-equivalences and weak M-equivalences coincide, and the second equivalence is proved as in Theorem 4.5.

Let P be any set of primes and $M = \bigvee_{p\in P} M(p, k_p, n_p)$, where $n_p \ge 2$ and k_p is either a positive integer or ∞. Then one shows as in Theorem 4.2 that a space X has the weak homotopy type of an M-cofibrant space if and only if

(1) X is 1-connected,
(2) $\pi_r(X)$ is P-torsion for all $r \ge 1$,
(3) $\pi_r(X)_p = 0$ for $r < n_p$, and
(4) if k_p is finite, then $\pi_{n_p}(X)_p$ is annihilated by p^{k_p}.

As applications, we prove the following results.

THEOREM 4.6: *Let P be any set of primes. Let $P_1, \ldots, P_r$ be a finite partition of P into mutually disjoint subsets. Let $M_i = \bigvee_{p \in P_i} M(p, k_p, n_p)$, where $n_p \geq 2$ and k_p is either a positive integer or ∞. Then, for each space X, the inclusion*

$$\bigvee_i X^{M_i} \longrightarrow \prod_i X^{M_i} \tag{4.5}$$

is a weak homotopy equivalence.

Proof: Each projection $\prod_i X^{M_i} \to X^{M_i}$ induces isomorphisms on homotopy with coefficients in P_i and hence it is a weak $(P_i, 2)$-equivalence. Likewise, each inclusion $X^{M_i} \to \bigvee_i X^{M_i}$ induces isomorphisms on homology with coefficients in P_i, and hence it is also a weak $(P_i, 2)$-equivalence, by [Ne, 3.10]. Since the composite

$$X^{M_i} \longrightarrow \bigvee_i X^{M_i} \longrightarrow \prod_i X^{M_i} \longrightarrow X^{M_i}$$

is the identity for all i, the arrow (4.5) is a $(P_i, 2)$-equivalence for all i and hence it is a $(P, 2)$-equivalence. Finally, observe that the domain of (4.5) is $(P, 2)$-cofibrant and the codomain has the weak homotopy type of a $(P, 2)$-cofibrant space. ∎

This result remains true for an infinite partition of P into mutually disjoint subsets, provided we take $\prod_i X^{M_i}$ to be the *weak* product of the spaces X^{M_i}; thus, $\pi_n \left(\prod_i X^{M_i}\right) \cong \bigoplus_i \pi_n(X^{M_i})$ for all n. This fact, together with Theorem 4.5, shows that every n-connected space X (where $n \geq 1$) with torsion homotopy groups decomposes, up to weak homotopy equivalence, as a wedge $\bigvee_p X_p$ or also as a weak product $\prod_p X_p$, where each X_p is an n-connected, p-torsion CW-complex.

Given arbitrary spaces X and Y, the natural map $X^{(P,n)} \times Y^{(P,n)} \to X \times Y$ is a weak (P, n)-equivalence. Hence, there is a map

$$(X \times Y)^{(P,n)} \longrightarrow X^{(P,n)} \times Y^{(P,n)}, \tag{4.6}$$

which is also a weak (P, n)-equivalence. Since the domain of (4.6) is (P, n)-cofibrant and the codomain has the weak homotopy type of a (P, n)-cofibrant space (by Theorem 4.3), the map (4.6) is a weak homotopy equivalence. As above, this result remains true for infinite weak products.

5. Calculating (P,n)-CW-approximations

Fix a set P of positive integers and an integer $n \geq 2$. Recall from Theorem 3.6 that, for every space X, the colocalization $X^{(P,n)}$ is closely related to the homotopy fibre of the localization map $X \to X_{(P,n)}$. The space $X_{(P,n)}$ is constructed from X by means of a sequence of push-outs involving $(n-1)$-connected spaces, in the process of factoring the map $X \to \star$ into a (P,n)-cofibration followed by a trivial (P,n)-fibration. Therefore, we have

$$\pi_r(X) \cong \pi_r(X_{(P,n)}) \qquad \text{for } r \leq n-1,$$

and $\pi_r(X_{(P,n)}; \mathbb{Z}/m) = 0$ for $r \geq n+1$ and $m \in P$, since $X_{(P,n)}$ is weakly (P,n)-equivalent to a point. By (1.1), this implies that the homotopy groups $\pi_r(X_{(P,n)})$ are uniquely P-divisible for $r \geq n+1$ and $\pi_n(X_{(P,n)})$ is P-torsion-free. Moreover, if we denote by $\mathbb{Z}[P^{-1}]$ the smallest subring of the rationals containing $1/m$ for all $m \in P$, then

$$\pi_r(X_{(P,n)}) \cong \pi_r(X) \otimes \mathbb{Z}[P^{-1}] \qquad \text{for } r \geq n+1, \tag{5.1}$$

while $\pi_n(X_{(P,n)})$ is isomorphic to the quotient of $\pi_n(X)$ by its P-torsion subgroup; cf. [B94, 5.2]. We shall use the fact that the P-torsion subgroup of an abelian group A is isomorphic to $\text{Tor}(A, \mathbb{Z}[P^{-1}]/\mathbb{Z})$, since $\mathbb{Z}[P^{-1}]/\mathbb{Z}$ is a direct sum of groups $\mathbb{Z}/p^\infty$, where p ranges over all primes dividing the numbers in P.

THEOREM 5.1: *The homotopy fibre F of the map η: $X \to X_{(P,n)}$ is weakly equivalent to a (P,n)-cofibrant space if and only if the two following conditions are satisfied for every prime p which has finite height $h(p)$ in P:*

1. *The p-torsion subgroup of $\pi_n(X)$ is annihilated by $p^{h(p)}$;*
2. *$\pi_{n+1}(X) \otimes \mathbb{Z}/p^\infty = 0$.*

Proof: We infer from the homotopy exact sequence associated to $F \to X \to X_{(P,n)}$ that F is always $(n-1)$-connected and its homotopy groups are P-torsion. Thus, if no prime has finite height in P, then F is weakly equivalent to a (P,n)-cofibrant space by Theorem 4.3. In the general case, it follows from (5.1) that there is a short exact sequence for $r \geq n$,

$$0 \to \pi_{r+1}(X) \otimes (\mathbb{Z}[P^{-1}]/\mathbb{Z}) \to \pi_r(F) \to \text{Tor}(\pi_r(X), \mathbb{Z}[P^{-1}]/\mathbb{Z}) \to 0, \tag{5.2}$$

which splits because the kernel is a divisible group. Look at the case $r = n$ and observe that, for any abelian group A, the group $A \otimes \mathbb{Z}/p^\infty$ is p-divisible and hence it cannot be annihilated by any power of p unless it is zero. This proves our claim. ∎

Note that

$$X^{(P,n)} \to X \to X_{(P,n)} \tag{5.3}$$

is a homotopy fibre sequence if and only if conditions (1) and (2) of Theorem 5.1 are fulfilled for every prime which has finite height in P. Of course, this restriction disappears if all primes dividing the numbers in P have infinite height, e.g. if P is multiplicatively closed. In that case, (5.3) is a homotopy fibre sequence for all spaces X.

The following result answers a question left open in [DF92, 6.4], where it was asked if F and $X^{(P,n)}$ differ at most in one homotopy group.

THEOREM 5.2: *Let X be any space and $P = \{p^k\}$, where p is a prime. Let F be the homotopy fibre of the localization map η: $X \to X_{(P,n)}$. Then there is a homotopy fibre sequence*

$$X^{(P,n)} \to F \to K(\pi, n)$$

where $\pi = \pi_n(F)/\operatorname{Tor}(\pi_n(F), \mathbb{Z}/p^k)$.

Proof: Since F is $(n-1)$-connected, we have $H^n(F;\pi) \cong \operatorname{Hom}(\pi_n(F), \pi)$, and hence we may pick a map g: $F \to K(\pi, n)$ inducing the natural projection $\pi_n(F) \to \pi$. Let F' be the homotopy fibre of g. Then $\pi_r(F') \cong \pi_r(F)$ for $r \geq n+1$, and $\pi_n(F') \cong \operatorname{Tor}(\pi_n(F), \mathbb{Z}/p^k)$. Therefore, the map $F' \to F$ is a weak (P, n)-equivalence and F' has the weak homotopy type of a (P, n)-cofibrant space. This shows that F' is weakly equivalent to $X^{(P,n)}$. ∎

Now the homotopy groups of $X^{(P,n)}$ can easily be computed in terms of the homotopy groups of X, for any $n \geq 2$ and any set P of positive integers. In the case $P = \{p, p^2, p^3, \dots\}$, the homotopy groups of $X^{(P,n)}$ are isomorphic to those of F, and the latter can be read directly from the split exact sequence (5.2). The case $P = \{p^k\}$ is covered by Theorem 5.2. Finally, by resorting to Theorem 4.4 and Theorem 4.5, one can compute $X^{(P,n)}$ for other sets P of positive integers.

Example 5.3: Let $P = \{p^k\}$, where p is a prime. Then, for any abelian group A and $d \geq 1$, we have

$$K(A,d)^{(P,n)} \simeq \begin{cases} \star & \text{if } d \leq n-1; \\ K(\operatorname{Tor}(A, \mathbb{Z}/p^k),\, n) & \text{if } d = n; \\ K(B,\, n) \times K(T_pA,\, n+1) & \text{if } d = n+1; \\ K(A \otimes \mathbb{Z}/p^\infty,\, d-1) \times K(T_pA,\, d) & \text{if } d \geq n+2, \end{cases}$$

where $B = \operatorname{Tor}(A \otimes \mathbb{Z}/p^\infty, \mathbb{Z}/p^k) \cong A/(p^kA + T_pA)$ and we denote by T_pA the p-torsion subgroup of A. To check this, consider the homotopy fibre F of $\eta\colon K(A,d) \to K(A,d)_{(P,n)}$ and use Theorem 5.2. If $d \geq n+1$, then

$$K(A,d)_{(P,n)} \simeq K(A \otimes \mathbb{Z}[1/p],\, d)$$

and F is in fact a product

$$F \simeq K(A \otimes \mathbb{Z}/p^\infty,\, d-1) \times K(T_pA,\, d);$$

cf. [B82, § 4]. If $d = n$, then $F \simeq K(T_pA,\, n)$.

Example 5.4: Let $P = \{p, p^2, p^3, \dots\}$, where p is a prime. Using similar arguments as in the previous example, for any abelian group A and $d \geq 1$, we have

$$K(A,d)^{(P,n)} \simeq \begin{cases} \star & \text{if } d \leq n-1; \\ K(T_pA,\, n) & \text{if } d = n; \\ K(A \otimes \mathbb{Z}/p^\infty,\, d-1) \times K(T_pA,\, d) & \text{if } d \geq n+1. \end{cases}$$

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