Composition Algebras with Large Derivation Algebras

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The finite dimensional flexible composition algebras include the Hurwitz algebras (composition algebras with unit element), but also other interesting classes of algebras: the para-Hurwitz and the Okubo algebras. The above mentioned algebras present many symmetries, and this is reflected in their large derivation algebras. In the present paper we study the opposite question: What can be said about the composition algebras if we have some information about their derivation algebras? Our main result is the classification of all the composition algebras with such large derivation algebras. © 1997 Academic Press

1. INTRODUCTION

Throughout this paper, a *composition algebra* is an algebra A defined over a field F and equipped with a strictly nondegenerate quadratic form $q: A \to F$ verifying

$$q(xy) = q(x)q(y) \tag{1}$$

for any $x, y \in A$. The form q being strictly nondegenerate means that the associated bilinear form f defined by

$$f(x, y) = q(x + y) - q(x) - q(y)$$

is nondegenerate. Usually, the additional hypothesis of the existence of a unit element 1 ($1x = x1 = x$ for any x) is assumed, but WE WILL NOT DO SO.

In case a unit element exists, the algebras that appear are well known. They all have dimension 1, 2, 4, or 8, and are either the field F, if its characteristic is not 2, $F \oplus F$, the quadratic separable field extensions K of F, the generalized quaternion algebras over F, and the Cayley–Dickson algebras over F (see [S1, ZSSS]). These unital composition algebras will be called *Hurwitz algebras* in the sequel.

On the other hand, from any finite dimensional composition algebra, one can construct a Hurwitz one (see [K]). Just take an element $a \in A$ with $q(a) \neq 0$, then $b = (1/q(a))a^2$ verifies $q(b) = 1$, the right and left multiplications, R_b and L_b, by b are orthogonal transformations relative to q by (1) and the vector space A, with the new multiplication

$$x \cdot y = \left(R_b^{-1} x \right)\left(L_b^{-1} y \right) \tag{2}$$

still verifies $q(x \cdot y) = q(x)q(y)$ and b^2 is the unit element. Hence, again we conclude that the dimension of A is 1, 2, 4, or 8. We also conclude from here that any finite dimensional composition algebra can be obtained from a Hurwitz algebra C, with norm q, by means of a new multiplication

$$x * y = \varphi(x)\psi(y), \tag{3}$$

where φ and ψ are orthogonal transformations relative to q. Examples of infinite dimensional composition algebras (necessarily non-unital) have been obtained in [UW] over the real field, commutative examples over general fields appear in [EM4], and examples with a one-sided identity element in [EP3].

In dimension 2, the orthogonal transformations in (3) are easily described. This led Petersson to the classification of the two-dimensional composition algebras. For any such algebra A, there exists a Hurwitz algebra C, defined on the same vector space A, with multiplication denoted by juxtaposition and with canonical involution $x \mapsto \bar{x}$, such that the product $x * y$ in A is given by one of the following equations,

(i) $x * y = xy$ (ii) $x * y = \bar{x}y$ (iii) $x * y = x\bar{y}$ (iv) $x * y = u\bar{x}\bar{y}$,

$$\tag{4}$$

where u is a norm 1 element, that is, $q(u) = 1$ (see [P2] and extend easily the arguments there to cover the characteristic 2 case).

In particular, for an algebraically closed ground field, there are, up to isomorphism, only four different two-dimensional composition algebras, since case (iv) above can be reduced to $x * y = \bar{x}\bar{y}$.

In the same paper, Petersson proved that there are infinitely many isomorphism classes of composition algebras of dimension 4 and 8 over algebraically closed fields.

In dimension 4, orthogonal transformations in a generalized quaternion algebra can also be described in terms of the multiplication and canonical involution of the algebra. As a consequence (see [Sh], [S-R]), if A is any four-dimensional composition algebra with quadratic form q over a field of characteristic not 2, there is a generalized quaternion algebra C, defined on the same vector space A, and three elements a, b, c with $q(a)q(b)q(c) = 1$ such that, with the same conventions as above, the multiplication $x * y$ on A is given by one of the following:

$$\begin{array}{ll} \text{(i) } x * y = axbyc & \text{(ii) } x * y = a\bar{x}byc \\ \text{(iii) } x * y = axb\bar{y}c & \text{(iv) } x * y = a\bar{x}b\bar{y}c. \end{array} \tag{5}$$

Stampfli-Rollier [S-R] shows that some restrictions can be imposed on a, b, c so that isomorphism conditions can be given. No similar description is known for eight-dimensional composition algebras.

Okubo introduced in [O1] an interesting composition algebra which he called the algebra of *pseudo-octonions*, in connection with the SU(3) particle physics. This algebra is flexible and Lie-admissible. The forms of the pseudo-octonion algebras were called *Okubo* algebras in [EM1]. Other interesting composition algebras were introduced by Okubo and Myung [OM], called the para-Hurwitz algebras, which are related to type (iv) in (4) and (5). Given a Hurwitz algebra C with canonical involution $x \mapsto \bar{x}$, the new algebra defined on C with multiplication

$$x \cdot y = \bar{x}\bar{y} \tag{6}$$

is called the associated *para-Hurwitz* algebra. These algebras (Okubo and para-Hurwitz), under some restrictions, have appeared too in Petersson's research on "involutorial" algebras [P1]. The connection between these different approaches is clarified in [EP2].

Both para-Hurwitz and Okubo algebras are flexible (that is, $(xy)x = x(yx)$ for any x, y). Moreover, they satisfy

$$(xy)x = x(yx) = q(x)y \tag{7}$$

for any x, y. Over fields of characteristic $\neq 2, 3$, Okubo and Osborn [OO] showed that, under some restrictions which can be removed for algebraically closed fields, any composition algebra satisfying (7) is a para-

Hurwitz algebra or an Okubo algebra. Moreover, later on, Okubo [O3] proved that for algebraically closed fields of characteristic $\neq 2, 3$, the non-unital finite dimensional flexible composition algebras satisfy (7). More recently, Myung and the first author [EM2] simplified Okubo's arguments and classified Okubo algebras and the forms of para-Hurwitz algebras, thus finishing the classification of flexible composition algebras over fields of characteristic $\neq 2, 3$. Here, an algebra A over F will be called a *form* of the algebra B over the algebraic closure $\overline{F}$ of F in case $\overline{F} \otimes_F A$ is isomorphic to B. By the way, a form of a para-Hurwitz algebra is either a para-Hurwitz algebra or an algebra as in type (iv) of (4) by [EP2, Lemma 3.3] (see also [EM2, Sect. 4; EP1, Proposition 1.2]). In the proof of this result the concept of a *para-unit* (see [O2]) plays a key role (because a composition algebra is para-Hurwitz if and only if it contains a para-unit). Since this will happen in several parts of this paper too, we pause to give the definition.

An element e of a composition algebra is termed a *para-unit* if it verifies $q(e) = 1$ and $ex = xe = f(x, e)e - x$ for any $x \in A$.

In particular, any para-unit is an idempotent element ($e^2 = e$).

In [EP1], it is shown that if we assume only third-power associativity (characteristic $\neq 2, 3$) again only the flexible composition algebras appear.

Following an idea of Faulkner [F], it is proved in [EM4] that there is an equivalence between the categories of non-unital finite dimensional flexible composition algebras over fields of characteristic $\neq 2, 3$ and of certain separable alternative algebras (actually, this result can be extended to cover the characteristic 2 case). This will allow us to determine very easily in Section 3 the group of automorphisms and the Lie algebra of derivations of the flexible finite dimensional composition algebras. In dimension 4 or 8, these algebras have large automorphism groups and derivation algebras. This is equivalent to saying that they present much symmetry.

This suggests the idea of classifying composition algebras according to the different possibilities for their attached Lie algebra of derivations and of studying those composition algebras which present more symmetries (which is reflected in larger derivation algebras). We will show that if large derivation algebras are imposed we obtain an analogue of Petersson's results (4) in dimensions 4 or 8, with the exception of the appearance of the Okubo algebras and certain eight-dimensional composition algebras related to the so-called Color Algebra (see Section 5).

More precisely, we will determine all the possibilities for the Lie algebra of derivations for composition algebras of dimension 4 and all the possibilities with ''maximum toral rank'' in dimension 8. Then, we will classify those composition algebras with the largest derivation algebras and, in particular, those with the same derivation algebras as the flexible composition algebras.

The paper is structured as follows. Section 3 will be devoted to study analogues in dimensions 4 and 8 of the algebras that appear in (4). These algebras will appear throughout the paper. Section 4 will deal with the derivations of the flexible composition algebras and Sections 5 and 6 will be devoted to dimensions 4 and 8, respectively. In order to accomplish these results, the next section will provide an important preliminary result: the invariance of the quadratic form of a composition algebra under the Lie algebra of derivations.

In concluding this introduction, we note that some of our results have been announced in the monograph [M], where also detailed proofs and historical remarks about some of the above mentioned results can be found.

2. INVARIANCE OF THE QUADRATIC FORM

Of fundamental importance for our investigation will be the fact that if A is any finite dimensional composition algebra with quadratic form q and associated nondegenerate symmetric bilinear form f, then any derivation is skew symmetric relative to f. That is, for any $x \in A$ and any $d \in \mathrm{Der}\, A$ (the Lie algebra of derivations of A),

$$f(dx, x) = 0. \tag{8}$$

In other words, q is *invariant* under the action of $\mathrm{Der}\, A$. This is a linearized version of the fact, that any automorphism of A is orthogonal relative to q, which follows from [P2, Corollary to Proposition 1]. To prove (8) we need:

LEMMA 2.1.   *Let* $K = F[t]\, (t^2 = 0)$ *be the algebra of dual numbers and let* V *be a vector space over* F. *Assume that* v *and* w *are linearly independent vectors in* V *and* v', w' *arbitrary vectors. Then,* $v + tv'$ *and* $w + tw'$ *are linearly independent vectors of* $K \otimes_F V$ *over* K.

Proof.   For $\alpha = \alpha_0 + \alpha_1 t$ and $\beta = \beta_0 + \beta_1 t$ in K, $\alpha(v + tv') + \beta(w + tw') = 0$ if and only if $\alpha_0 v + \beta_0 w = 0$ and $\alpha_0 v' + \alpha_1 v + \beta_0 w' + \beta_1 w = 0$ if and only if $\alpha_0 = 0 = \beta_0$ and $\alpha_1 v + \beta_1 w = 0$ if and only if $\alpha = \beta = 0$.   ∎

LEMMA 2.2.   *Let* A *be a Hurwitz algebra of dimension* > 1 *with quadratic form* q *and associated nondegenerate symmetric bilinear form* f. *Let* K *be the algebra of dual numbers,* $\tilde{A} = K \otimes_F A$ *and assume that* $Q: \tilde{A} \to K$ *is a quadratic form over* K *permitting composition* $(Q(xy) = Q(x)Q(y)$ *for any* $x, y \in \tilde{A})$. *Assume there is no* $0 \neq x \in \tilde{A}$ *such that* $F(x, z) = 0$ *for any*

$z \in \tilde{A}$, where $F(x, y) = Q(x + y) - Q(x) - Q(y)$ is the associated bilinear form, then the quadratic form Q is the extension of q; that is, $Q(a + tb) = q(a) + tf(a, b)$ holds for any $a, b \in A$.

Proof. If $Q(xy) = Q(x)Q(y)$ for any $x, y \in \tilde{A}$, linearizing we obtain that $F(xy, xz) = Q(x)F(y, z)$ and $F(xy, zw) + F(xw, zy) = F(x, z)F(y, w)$ for any $x, y, z, w \in \tilde{A}$. Therefore, with $w = 1$ and $y = x$,

$$F(x^2, z) + F(x, zx) = F(x, z)F(x, 1),$$

and, since $F(x, zx) = Q(x)F(1, z)$, we arrive at

$$F(x^2 - F(x, 1)x + Q(x)1, z) = 0$$

for any $z \in \tilde{A}$. Thus,

$$x^2 - F(x, 1)x + Q(x)1 = 0$$

for any $x \in \tilde{A}$. But, if we keep the notations q and f for their extensions to $\tilde{A}$, we also have

$$x^2 - f(x, 1)x + q(x)1 = 0,$$

for any $x \in \tilde{A}$. Hence, for any $a \in A$ linearly independent with 1 and for any $b \in A$, Lemma 2.1 gives

$$F(a + tb, 1) = f(a + tb, 1) \qquad \text{and} \qquad Q(a + tb) = q(a + tb).$$

From this, we obtain that $Q(a) = q(a)$ and $F(a, b) = f(a, b)$ for any $a, b \in A$ and the result follows. ∎

Now, the announced result:

THEOREM 2.3. *Let A be a finite dimensional composition algebra over the field F with associated nondegenerate symmetric bilinear form f. Let d be any derivation of A. Then, for any $x \in A$*

$$f(dx, x) = 0.$$

Proof. Let $a \in A$ be any element with $q(a) \neq 0$. Then, the left multiplication by a is a bijection (it is a similarity of (A, q)). Hence, there exists an element $e \in A$ such that $ae = a$. From (1) it follows that $q(e) = 1$. As in (2) we consider the new product $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$ on the vector space A. The element $1 = e^2$ is the unit element of the Hurwitz algebra $(A, \cdot)$. Let $\tilde{A} = K \otimes_F A$ be as in Lemma 2.2. For any derivation of A, $d \in \text{Der } A$, we again denote by d its extension to a derivation of $\tilde{A}$. Let us consider the application $\varphi \colon \tilde{A} \to \tilde{A} \colon x \mapsto x + t(dx)$. Then, φ is an automorphism of $\tilde{A}$. We define now the K-quadratic form on $\tilde{A}$ by $Q(x) = q(\varphi(x))$ for any $x \in \tilde{A}$.

With $x, y \in \tilde{A}$, let $\hat{x} = R_e^{-1} x$, $\hat{y} = L_e^{-1} y$. Then,

$$q(\varphi(x)) = q(\varphi(\hat{x}e)) = q(\varphi(\hat{x})\varphi(e)) = q(\varphi(\hat{x}))q(\varphi(e)).$$

But, since $ae = a$,

$$f(da, a) = f((da)e + a(de), a) = f((da)e, ae) + f(a(de), ae)$$
$$= f(da, a) + q(a)f(de, e),$$

so $f(de, e) = 0$ and

$$q(\varphi(e)) = q(e + t(de)) = q(e) + tf(de, e) = q(e) = 1.$$

Therefore, $q(\varphi(x)) = q(\varphi(\hat{x}))$ and also, $q(\varphi(y)) = q(\varphi(\hat{y}))$. Now,

$$Q(x \cdot y) = q(\varphi(x \cdot y)) = q(\varphi(\hat{x}\hat{y})) = q(\varphi(\hat{x}))q(\varphi(\hat{y}))$$
$$= q(\varphi(x))q(\varphi(y)) = Q(x)Q(y).$$

By Lemma 2.2, for any $u \in A$,

$$q(u) = Q(u) = q(u + t(du)) = q(u) + tf(u, du),$$

so $f(du, u) = 0$ and we get the theorem. ∎

3. STANDARD COMPOSITION ALGEBRAS

Both Petersson's classification in dimension 2 (see (4)) and (6) inspire the next definition: let A be a Hurwitz algebra of dimension ≥ 2 with canonical involution $x \mapsto \bar{x}$. Then, the new algebras defined over A with respective multiplications

(i) $x \circ y = xy$,   (ii) $x \circ y = \bar{x}y$,   (iii) $x \circ y = x\bar{y}$,   or   (iv) $x \circ y = \bar{x}\bar{y}$
(9)

will be called the *standard composition algebras* associated to A.

These algebras will play a fundamental role in what follows. The four standard composition algebras associated to A are not isomorphic, since only in case (i) there is an identity element, only in case (ii) there is a left but not right identity element, and the same in case (iii) interchanging left and right. We can state a more general result over fields of characteristic not 2 which has its own independent interest. To do this, notice that by Kaplansky's argument mentioned in the Introduction, any finite dimensional composition algebra of dimension 2, 4, or 8 is obtained from a

Hurwitz algebra C, with norm q and canonical involution $x \mapsto \bar{x}$, with new multiplication as in (3). Since the canonical involution has determinant -1, (3) can be separated into four types,

$$x * y = \varphi(x)\psi(y) \qquad \text{(10-I)}$$

$$x * y = \varphi(\bar{x})\psi(y) \qquad \text{(10-II)}$$

$$x * y = \varphi(x)\psi(\bar{y}) \qquad \text{(10-III)}$$

$$x * y = \varphi(\bar{x})\psi(\bar{y}) \qquad \text{(10-IV)}$$

with φ and ψ in the special orthogonal group of (C, q).

We say that a finite dimensional composition algebra of dimension ≥ 2 $(A, \cdot)$ is, say, of type II in case there is a Hurwitz algebra C with $(A, \cdot)$ isomorphic to $(C, *)$, with $x * y$ as in (10-II).

Now the announced general result, which extends and is inspired by [S-R, 3.9 Satz]):

THEOREM 3.1. *Finite dimensional composition algebras of dimension ≥ 2 over a field of characteristic not 2 and of different types are not isomorphic.*

Proof. Assume that C is a Hurwitz algebra (multiplication denoted by juxtaposition) and form the algebra $(C, *)$ as in (10-II). Take any arbitrary $x \in C$ with $q(x) = 1$. Then, the linear maps L_x^*: $y \mapsto x * y$ and R_x^*: $y \mapsto y * x$ verify $L_x^* = L_{\varphi(\bar{x})}\psi$ and $R_x^* = R_{\psi(x)}\varphi j$, where j denotes the canonical involution of C and L_a and R_a are the left and right multiplications by the element a in C. However, for any $a \in C$ with $q(a) = 1$, L_a and R_a belong to the special orthogonal group of (C, q). Thus, $\det L_x^* = 1 = -\det R_x^*$. Therefore, for any $x \in C$ with $q(x) = 1$, types I–IV are determined by

Type I. $\det L_x^* = 1 = \det R_x^*$
Type II. $\det L_x^* = 1 = -\det R_x^*$
Type III. $\det L_x^* = -1 = -\det R_x^*$
Type IV. $\det L_x^* = -1 = \det R_x^*$

and the type is determined by the left and right multiplications by any element of norm 1. ∎

In particular, standard algebras of different types in (9) are not isomorphic.

We will often work first over algebraically closed fields by extending scalars and then will descend to the ground field. It is clear that any form

of a Hurwitz algebra is itself a Hurwitz algebra. More generally we have:

PROPOSITION 3.2. *Any form of a finite dimensional composition algebra is itself a composition algebra.*

Proof. Let A be a form of a composition algebra and let q be the nondegenerate quadratic form permitting composition of $\overline{A} = \overline{F} \otimes_F A$ (unique by [P2, Proposition 1]). Take $x \in A$ with $q(x) \neq 0$. Then $\overline{A}$ is a Hurwitz algebra with the new multiplication given by $a * b = (R_x^{-1}a)(L_x^{-1}b)$ and the form $\tilde{q}(a) = (1/q(x)^2)q(a)$. Since this new product is already defined on A, $(A, *)$ is a Hurwitz algebra and $\tilde{q}(a) \in F$ for any $a \in A$. Therefore, $\tilde{q}(x) = (1/q(x)) \in F$ and $q(a) \in F$ for any $a \in A$, so A, together with the restriction of q, is a composition algebra. ∎

As mentioned in the Introduction, the forms of para-Hurwitz algebras are known to be either para-Hurwitz or the two-dimensional algebras that appear in (4)(iv). Finally, to complete our knowledge of the forms of the standard composition algebras we prove:

PROPOSITION 3.3. *Any form of a standard composition algebra of type (9)(ii) or (9)(iii) is again a standard composition algebra (of the same type).*

Proof. Let A be a form of a standard composition algebra of type (9)(ii), so there is a multiplication $\cdot$ in $\overline{A} = \overline{F} \otimes_F A$, such that $(\overline{A}, \cdot)$ is a Hurwitz algebra with unit element e, norm q, and canonical involution $x \mapsto \bar{x}$ and the product in $\overline{A}$ is given by $xy = \bar{x} \cdot y$. Take $x \in A$ with $q(x) \neq 0$, then $ex = \bar{e} \cdot x = e \cdot x = x$, so $e = R_x^{-1}x \in A$. Also, for any $y \in A$, $\bar{y} = ye \in A$ and, therefore, A is closed under the Hurwitz product $\cdot$, so we obtain that A is the standard composition algebra of type (9)(ii) associated to the Hurwitz algebra $(A, \cdot)$. ∎

4. AUTOMORPHISMS AND DERIVATIONS OF THE FLEXIBLE COMPOSITION ALGEBRAS

In this section, we will always assume that the characteristic of the ground field F is $\neq 2, 3$.

According to [EM4], any flexible composition algebra is either a Hurwitz algebra or it is intimately connected to a separable alternative algebra of degree three. More precisely, if A is a finite dimensional separable alternative algebra of degree three over the ground field F, then any element of A satisfies the equation

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$$

for some linear form T (the trace), quadratic form S, and cubic form N (the norm). If F contains the cube roots of 1 or, equivalently, contains the element $\xi = \sqrt{-3}$, we can define a new multiplication on the set A_0 of elements of trace zero as

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega + 1}{3} T(ab)1 \qquad (11)$$

for $a, b \in A_0$ (see [EM4]). Here ω is a cube root of 1, $\omega = (-1 + \xi)/2$. The product in A can be recovered from (11), since $A = F1 \oplus A_0$ and for $a, b \in A_0$ [EM4, (4.4)],

$$ab = -\frac{S(a, b)}{3} 1 + \frac{1}{3} ((\omega^2 - 1) a * b - (\omega - 1) b * a), \qquad (12)$$

where $S(\ ,\ )$ is the symmetric bilinear form associated to S. The algebra $(A_0, *)$ is a flexible composition algebra.

Also, if F does not contain ξ and A is a finite dimensional separable alternative algebra of degree three over the field $K = F[\xi]$, equipped with a K/F-involution of the second kind J, and $A_0 = \{x \in A : T(x) = 0$ and $J(x) = -x\}$, then again we can define a product on A_0 by Eq. (11) and recover the product in A by Eq. (12), since $A = K1 \oplus KA_0$. The F-algebra $(A_0, *)$ is a flexible composition algebra.

To unify both situations, we take $K = F$ and $J = 1$ in case $\xi \in F$ and $K = F[\xi]$ and J of second kind in case $\xi \notin F$. Then, in both cases we shall speak of (A, J) with A a finite dimensional separable alternative algebra of degree 3 over K.

THEOREM 4.1 [EM4]. *Any finite dimensional flexible composition algebra over F either is Hurwitz or can be obtained as the algebra $(A_0, *)$ given by* (11) *for a pair (A, J) as above. Moreover, two such algebras $(A_0, *)$ and $(A'_0, *)$ are isomorphic if and only if so are the pairs (A, J) and (A', J'); that is, there is a K-isomorphism φ between A and A' such that $\varphi(x^J) = \varphi(x)^{J'}$ for any $x \in A$.*

The group of automorphisms and the Lie algebra of derivations of Hurwitz algebras are well known [J1, S1] so, in order to determine the group of automorphisms and Lie algebras of derivations of the flexible composition algebras, we need the next result, where $\mathrm{Aut}(A, J)$ denotes the group of automorphisms (over K) which commute with J and similarly for the Lie algebra of derivations $\mathrm{Der}(A, J)$.

THEOREM 4.2. *Let (A, J) be as above. Then, with $(A_0, *)$ as in Theorem* 4.1, *the restriction map from A to A_0 gives isomorphisms between the group of automorphisms $\mathrm{Aut}(A, J)$ and $\mathrm{Aut}(A_0, *)$ and between the Lie algebras of derivations $\mathrm{Der}(A, J)$ and $\mathrm{Der}(A_0, *)$.*

Proof.   The result on automorphisms follows essentially from the argu-
ments in [EM4, Proof of the Main Theorem (I) and (II)]. As for deriva-
tions, if $d \in \mathrm{Der}(A, J)$, it is clear that $dA_0 \subseteq A_0$. Now, for $a, b \in A_0$

$$
\begin{aligned}
(da) * b + a * (db) &= \omega(da)b - \omega^2 b(da) - \frac{2\omega + 1}{3} T((da)b)1 \\
&\quad + \omega a(db) - \omega^2(db)a - \frac{2\omega + 1}{3} T(a(db))1 \\
&= \omega((da)b + a(db)) - \omega^2(b(da) + (db)a) \\
&= \omega d(ab) - \omega^2 d(ba) \\
&= d\left(\omega ab - \omega^2 ba - \frac{2\omega + 1}{3} T(ab)1 \right) \\
&= d(a * b)
\end{aligned}
$$

since $T(dx) = 0$ for any $x \in A$ [J2, Theorem 6.1]. Therefore, the restric-
tion map $d \mapsto d|_{A_0}$ gives an homomorphism from $\mathrm{Der}(A, J)$ to $\mathrm{Der}(A_0, *)$.
Since $A = K1 \oplus KA_0$ (and $KA_0 = A_0 \oplus \xi A_0$ in case $\xi \notin F$), given $d \in \mathrm{Der}(A_0, *)$, we can extend it to A by imposing $d(K1) = 0$ and $d(\xi a) = \xi da$ for $a \in A_0$. The quadratic form that permits composition in $(A_0, *)$ is
precisely the restriction of $S(\ ,\ )$ [EM4], so we may use (12) and Theorem
2.3 to check that this extension of d to A is a derivation in $\mathrm{Der}(A, J)$ and
the theorem follows.   ∎

The groups $\mathrm{Aut}(A, J)$ and the Lie algebras $\mathrm{Der}(A, J)$ are now easily
computed. We separate into two cases according to $\xi \in F$ or $\xi \notin F$. In the
first case the next theorem follows from known results:

THEOREM 4.3.   *Let A be a finite dimensional separable alternative algebra
of degree* 3 *over a field F containing the cube roots of* 1. *Then, one of the
following possibilities occurs*:

(i)   *A is a central simple associative algebra of degree three, so that
either A is the algebra of 3×3 matrices over F, or a division algebra over F.
In both cases $\mathrm{Aut} A \cong A^\times / F^\times$, where $A^\times$ denotes the group of invertible
elements of A (in particular, $\mathrm{Aut} \mathrm{Mat}_{3 \times 3}(F) \cong PGL(3, F)$) and $\mathrm{Der} A \cong A_0$,
the Lie algebra of trace zero elements of A under commutation. This is a
central simple Lie algebra of type A_2.*

(ii)   *$A = F \oplus C$, where C is a Hurwitz algebra of dimension* 2, 4, *or* 8.
*Then $\mathrm{Aut} A$ is naturally isomorphic to $\mathrm{Aut} C$ unless $C = F \oplus F$, where
$\mathrm{Aut} A$ is the symmetric group on three elements. Moreover, $\mathrm{Der} A$ is naturally
isomorphic to $\mathrm{Der} C$, which is trivial if $\dim C = 2$, a central simple Lie
algebra of type A_1 if $\dim C = 4$ and a central simple Lie algebra of type G_2 in
case $\dim C = 8$.*

(iii)   *A/F is a cubic field extension. Then* Der $A = 0$ *since A/F is separable, and* Aut A *is either the cyclic group of three elements, if this is a Galois extension* (*so, with our hypotheses on F, $A \cong F[X]/(X^3 - \alpha)$ for some $\alpha \in A$, $\alpha \notin F$), or otherwise,* Aut A *is the trivial group.*

Case (i) in the theorem above corresponds to the Okubo algebras, case (ii) to the para-Hurwitz algebras, and case (iii) to forms of para-Hurwitz algebras in dimension 2. The same can be said after the next theorem (see [EM4]). Notice that for $C = F \oplus F$ in (ii), the corresponding flexible composition algebra is the para-Hurwitz algebra obtained from C (see (6)). This algebra contains three para-units: $(1, 1)$, (ω, ω^2), and (ω^2, ω), and the action of the automorphism group is just the action by permutations on the set of para-units.

THEOREM 4.4. *Let F be a field not containing the cube roots of 1, $K = F[\xi]$, and A a separable alternative algebra of degree 3 over K, equipped with a K/F-involution of second kind. Then, one of the following possibilities occurs*:

(*i*)   *A is a central simple associative algebra of degree three, then* $\mathrm{Aut}(A, J) \cong \{b \in A: \ b^J b \in F^\times\}/K^\times$ *and* $\mathrm{Der}(A, J) \cong \{x \in A_0: \ x^J = -x\}$, *the Lie algebra of trace zero elements of A which are skew relative to J under commutation. Again, this is a central simple Lie algebra of type A_2.*

(ii)   *$A = K \oplus C$, where C is a Hurwitz algebra of dimension 2, 4, or 8 over K, $J|_K$ is the nontrivial F-automorphism in K and $C^J = C$. Then, if $\tilde{C} = \{x \in C: x^J = \bar{x}\}$, where $x \mapsto \bar{x}$ is the canonical conjugation in C, $\tilde{C}$ is a Hurwitz algebra over F, and $\mathrm{Aut}(A, J) \cong \mathrm{Aut}\, \tilde{C}$, unless $A = K \oplus K \oplus K$ and J does not permute the copies of K. In this case, $\mathrm{Aut}(A, J)$ is the symmetric group on three elements. Moreover, $\mathrm{Der}(A, J) \cong \mathrm{Der}\, \tilde{C}$. Again, this implies that either $\mathrm{Der}(A, J)$ is the trivial algebra if $\dim_K C = 2$, a central simple Lie algebra of type A_1 for $\dim_K C = 4$, and a central simple Lie algebra of type G_2 for $\dim_K C = 8$.*

(iii)   *A is a cubic field extension of K, so $J \in \mathrm{Aut}_F A$. Then $\mathrm{Der}(A, J) = 0$ and $\mathrm{Aut}(A, J)$ is either the cyclic group of order 3 in case A/F is a cyclic extension, or trivial otherwise.*

Proof.   For (i) notice that any automorphism or derivation of A (as an algebra over K) is inner. Then, if $\varphi \in \mathrm{Aut}\, A$, there is a $b \in A^\times$ such that $\varphi(x) = bxb^{-1}$ for any x. But, if $\varphi \in \mathrm{Aut}(A, J)$, then $bx^J b^{-1} = \varphi(x^J) = \varphi(x)^J = (bxb^{-1})^J = (b^J)^{-1}x^J b^J$ for any $x \in A$. Thus, $b^J b$ is an element of the center of A and fixed by J. Therefore $b^J b \in F^\times$. Also notice that the inner derivation ad a, with $T(a) = 0$, commutes with J if and only if $a^J = -a$.

Now, if $A = K \oplus C$, with C a Hurwitz algebra of dimension ≥ 2 and $C \neq K \oplus K$, it is clear that $K^J = K$ and $C^J = C$. But even in the case that $C = K \oplus K$, so that $A = K \oplus K \oplus K$, since the order of J is 2, at least one of the copies of K is fixed by J. Therefore, we can always assume that $A = K \oplus C$ with $K^J = K$ and $C^J = C$. Case (ii) now follows easily.

Finally, assume that A/K is a field extension of degree 3, so that A/F is a field extension of degree 6. Since $J \in \mathrm{Aut}_F A$, let L denote the fixed field by J, so that A/L is a quadratic field extension. Then $A \cong L \otimes_F K$ and $\mathrm{Aut}(A, J)$ is isomorphic (by restriction) to $\mathrm{Aut}_F L$. This latter group is the cyclic group of order 3 in case L/F is a Galois extension and trivial otherwise. Besides, in case $\mathrm{Aut}(A, J)$ is the cyclic group of order 3, then $\mathrm{Aut}_F A$ is generated by J and by $\mathrm{Aut}(A, J)$, so that $\mathrm{Aut}_F A$ is the cyclic group of order 6 and A/F is a cyclic extension. Conversely, if A/F is cyclic, then $\mathrm{Aut}(A, J)$ is a normal subgroup of $\mathrm{Aut}_F A$ and L is a cyclic extension of F. ∎

The exceptional case in part (ii) of the above theorem occurs in case $A = K \oplus K \oplus K$ and J does not permute the copies of K. If this is the case and C is the sum of the last two copies of K, $\tilde{C} = \{(a, a^J): a \in K\}$, which is isomorphic to K. The para-units of the para-Hurwitz algebra associated to K are precisely the cube roots of 1 and, under the isomorphisms, $\mathrm{Aut}(A, J)$ is the symmetric group on these para-units.

In case (iii), notice that if $\mathrm{Aut}_F L$ is the trivial group, then $\{\sigma \in \mathrm{Aut}_F A: \sigma J = J\sigma\} = \{1, J\}$, so that $\mathrm{Aut}_F A$ is either the cyclic group of order 2 generated by J or the symmetric group on three elements.

COROLLARY 4.5. *Let A be a finite dimensional flexible composition algebra over F. Then the Lie algebra of derivations* Der A *is either*:

- *trivial if* $\dim A \leq 2$,

- *a three-dimensional central simple Lie algebra if* $\dim A = 4$, *that is, if A is either a generalized quaternion algebra or its para-Hurwitz counterpart*,

- *a central simple Lie algebra of type* A_2 *if A is an Okubo algebra, or*

- *a central simple Lie algebra of type* G_2 *if A is either a Cayley−Dickson algebra or its para-Hurwitz counterpart*.

5. THE FOUR-DIMENSIONAL CASE

It is clear that the only one-dimensional composition algebra over the field F is F itself if the characteristic is not 2 and its Lie algebra of derivations is trivial. With the exception of fields of characteristic three, this is also the case for two-dimensional composition algebras:

THEOREM 5.1. *Let A be a composition algebra over F. If* $\dim A \le 2$, *then* $\operatorname{Der} A = 0$, *unless* $\dim A = 2$, *F has characteristic three, and A is as in part* (iv) *of* (4) (*that is, a form of the two-dimensional para-Hurwitz algebra over the algebraic closure of F*). *In this case* $\operatorname{Der} A = Fd$, *and d acts semisimply and nonsingularly on A.*

Proof. We can assume that $\dim A = 2$ and, extending scalars, that F is algebraically closed. Then, A has a basis $\{u, v\}$ with $q(u) = q(v) = 0$ and $f(u, v) = 1$. If $d \in \operatorname{Der} A$, by Theorem 2.3, $f(du, u) = 0 = f(dv, v) = f(du, v) + f(u, dv)$, so that $du = \alpha u$, $dv = \beta v$, and $\alpha = -\beta \in F$. Thus, if $\operatorname{Der} A \ne 0$, then $\operatorname{Der} A = Fd$ with $du = u$ and $dv = -v$. In particular, d is a bijection.

In this case, $d(uv) = (du)v + u(dv) = uv - uv = 0$, so $uv = 0$. Analogously, $vu = 0$. Therefore, A is commutative. Moreover, $d(u^2) = 2u(du) = 2u^2$ and also $d(v^2) = 2v(dv) = -2v^2$. This forces the characteristic of F to be 3 and $u^2 = \lambda v$, $v^2 = \eta u$. Now,

$$q\big((u + v)^2\big) = q(\lambda v + \eta u) = \lambda \eta f(u, v) = \lambda \eta,$$

and also,

$$q\big((u + v)^2\big) = q(u + v)^2 = f(u, v) = 1.$$

Thus, $\lambda \eta = 1$. If μ is a cube root of $\lambda^{-1} = \eta$ and we consider $a = \mu u$, $b = \mu^{-1} v$, then $a^2 = b$, $b^2 = a$, $ab = ba = 0$, and $f(a, b) = 1$. We check that A is the two-dimensional para-Hurwitz algebra ($a + b$ is a para-unit). ∎

From now on in this section the characteristic of the ground field F will be assumed to be different from 2. This being the case, given a composition algebra A with corresponding quadratic form q, for any $x, y \in A$, we put $(x, y) = \frac{1}{2} f(x, y)$, so that $q(x) = (x, x)$ holds for any $x \in A$.

In order to study the four-dimensional case, first we will pay attention to the nilpotent derivations:

LEMMA 5.2. *Let A be a four-dimensional composition algebra over the field F and let $0 \ne d \in \operatorname{Der} A$ be a nilpotent derivation. Then $d^2 \ne 0 = d^3$ and $\ker d$ is not totally isotropic (that is, there are elements $e \in \ker d$ with $q(e) \ne 0$).*

Proof. If d^3 were not 0, there would exist a basis $\{v, dv, d^2v, d^3v\}$ of A. But since $d^4 = 0$, (8) implies

$$(d^3v, d^3v) = (d^3v, d^2v) = (d^3v, dv) = 0,$$

and also

$$(d^3 v, v) = -(d^2 v, dv) = -(d(dv), dv) = 0,$$

so $(d^3 v, v) = 0$ too, which is a contradiction with the nondegeneracy of $(\ ,\ )$. Therefore, $d^3 = 0$.

Assume that $d^2 = 0$, then either there is a basis $\{v, dv, w, dw\}$ or $\{v, dv, x, y\}$ with $dx = dy = 0$. In the last case, by (8), dv is contained in the radical of $(\ ,\ )$, a contradiction. In the former case, $\ker d = F - \operatorname{span}\langle dv, dw \rangle$ is a subalgebra of A. Also, for any $x, y \in A$

$$0 = d^2(xy) = (d^2 x)y + 2(dx)(dy) + x(d^2 y) = 2(dx)(dy),$$

so the multiplication in $\ker d$ is trivial. But, given $x \in \ker d$ and $y \in A$, $d(xy) = x(dy) \in (\ker d)^2 = 0$. Thus, xy, and also yx, belongs to $\ker d$ and $\ker d$ is an ideal of A. This is a contradiction since A is simple [P2, Corollary to Proposition 2].

Hence $d^2 \neq 0 = d^3$ and there is a basis $\{v, dv, d^2 v, w\}$ of A with $w \in \ker d$. The associated matrix of d is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (8), $(v, dv) = (dv, d^2 v) = (dv, w) = (d^2 v, w) = 0$ and $(dv, dv) = -(v, d^2 v)$. In particular, this forces $q(w) \neq 0$, as required. ∎

LEMMA 5.3. *Let A be a four-dimensional composition algebra over F and let $0 \neq N$ be a subalgebra of $\operatorname{Der} A$ consisting entirely of nilpotent derivations. Then the dimension of N is 1.*

Proof. Let us consider the subalgebra $A_0 = \{x \in A : Nx = 0\}$, which is not zero by Engel's Theorem. If there is an element $a \in A_0$ with $q(a) \neq 0$, then with $b = (1/q(a))a^2 \in A_0$, we may form the Hurwitz algebra $(A, \cdot)$ with multiplication as in (2). Since $Nb = 0$, the elements in N commute with L_b and R_b and hence become derivations of $(A, \cdot)$. But the derivation algebra of the four-dimensional Hurwitz algebra $(A, \cdot)$ is a three-dimensional simple Lie algebra, and since N becomes a nilpotent subalgebra, the dimension is at most 1.

Let us see now that A_0 cannot be totally isotropic. Otherwise, and if the dimension of A_0 were 2, then for any $0 \neq d \in N$, dA would be orthogonal to A_0 by the linearization of (8) and this implies $dA \subseteq A_0$ and $d^2 = 0$, a contradiction with Lemma 5.2. Finally, if A_0 were totally isotropic and $\dim A_0 = 1$, by Lemma 5.2 the dimension of N would be ≥ 2. Besides, A_0

would be contained in its orthogonal subspace $A_0^\perp$ relative to q and a basis of isotropic vectors $\{a, u, v, z\}$ might be so chosen that $A_0 = Fa$, $A_0^\perp = F - \text{span}\langle a, u, v \rangle$, $(a, z) = 1 = (u, v)$, and $(u, z) = (v, z) = 0$. By (8), $dA \subseteq A_0^\perp$ for every $d \in N$ and du is orthogonal to u and a. Since d is nilpotent there is an $\alpha \in F$ such that $du = \alpha a$, similarly $dv = \beta a$ and dz is orthogonal to z so $dz = \gamma u + \delta v$. Now,

$$0 = (du, z) + (u, dz) = \alpha + \delta,$$
$$0 = (dv, z) + (v, dz) = \beta + \gamma,$$

so dim $N = 2$ and the linear map given by $da = 0 = dv$, $du = a$, and $dz = -v$ must belong to N. But $d^2 = 0$, a contradiction with Lemma 5.2. ∎

Now we turn our attention to the semisimple derivations:

LEMMA 5.4. *Let A be a four-dimensional composition algebra over an algebraically closed field F and let H be a nonzero subalgebra of* Der A *of diagonalizable derivations. Then there is a diagonalizable $d \in H$ such that $H = Fd$. Moreover,* dim ker $d = 2$ *and* ker d *is a composition subalgebra of A.*

Proof. Let $0 \neq d \in H$. For any eigenvalue α of d, let A_α be the corresponding eigenspace. It is clear that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for any α, β. From (8) it immediately follows that $(A_\alpha, A_\beta) = 0$ if $\alpha + \beta \neq 0$, so by nondegeneracy of (,), dim $A_\alpha = $ dim $A_{-\alpha}$ for any eigenvalue α.

Let α be a nonzero eigenvalue of d, then $A = A_\alpha \oplus A_{-\alpha} \oplus A_*$. Take $0 \neq a_{\pm\alpha} \in A_{\pm\alpha}$ with $(a_\alpha, a_{-\alpha}) \neq 0$. Then $q(a_\alpha + a_{-\alpha}) \neq 0$, so the left multiplication $L_{a_\alpha + a_{-\alpha}}$ is a bijection and $0 \neq (a_\alpha + a_{-\alpha})A_\alpha \subseteq A_{2\alpha} \oplus A_0$, so either $A_0 \neq 0$ or $A_{2\alpha} \neq 0$.

Assume first that the characteristic of F is not 3. Then, if $A_{2\alpha} \neq 0$, $A = A_\alpha \oplus A_{-\alpha} \oplus A_{2\alpha} \oplus A_{-2\alpha}$. The same argument now shows that $A_{4\alpha} \neq 0$, so necessarily the characteristic is 5. But, since $0 \neq (a_\alpha + a_{-\alpha})A_\alpha \subseteq A_{2\alpha}$, it follows that $a_{-\alpha}A_\alpha \subseteq A_0 = 0$ and $a_\alpha^2 \neq 0$. In the same way, with $a_{\pm2\alpha} \in A_{\pm2\alpha}$ such that $(a_{2\alpha}, a_{-2\alpha}) \neq 0$, we obtain $a_{-2\alpha}^2 \neq 0$. Since $(a_\alpha a_{2\alpha}, a_\alpha^2) = (a_\alpha, a_\alpha)(a_{2\alpha}, a_\alpha) = 0$ and $A_{2\alpha} = F(a_\alpha)^2$, it follows that $a_\alpha a_{2\alpha} = 0$. Also, $(a_\alpha a_{-2\alpha}, a_{-2\alpha}^2) = 0$ and $a_\alpha a_{-2\alpha} = 0$. But then $a_\alpha(a_{2\alpha} + a_{-2\alpha}) = 0$, and this is a contradiction since $q(a_{2\alpha} + a_{-2\alpha}) \neq 0$. Therefore, we conclude that $A_0 \neq 0$, so necessarily $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ with dim $A_0 = 2$ and dim $A_{\pm\alpha} = 1$. Now in Der A, ad $d: d' \mapsto [d, d']$ is also diagonalizable, so if dim $H \geq 2$, there is a $d' \in H$ linearly independent with d and such that $[d, d'] = \lambda d'$. Then, we also have the decomposition in eigenspaces relative to d': $A = A_0' \oplus A_\beta' \oplus A_{-\beta}'$, with dim $A_0' = 2$ and dim $A_{\pm\beta}' = 1$. But $0 = [d, d'](A_0') = d'(dA_0')$. Hence, $dA_0' \subseteq A_0'$ and, by

Theorem 5.1, since A'_0 is a composition subalgebra of A, we conclude that $dA'_0 = 0$, so $A'_0 = A_0$. By orthogonality, $A_\alpha \oplus A_{-\alpha} = A'_\beta \oplus A'_{-\beta}$ and it follows easily that d and d' are proportional. Hence, if the characteristic is not 3 we are finished.

Assume now that the characteristic of F is 3. As above, $A = A_\alpha \oplus A_{-\alpha} \oplus A_*$, where A_* is a direct sum of eigenspaces relative to d. In case $A_* = A_\beta \oplus A_{-\beta}$ with α and β linearly independent over the prime field $\mathbb{Z}_3$, then again with $a_{\pm \alpha} \in A_{\pm \alpha}$ with $(a_\alpha, a_{-\alpha}) \neq 0$, we obtain $0 \neq (a_\alpha + a_{-\alpha})A_\beta \subseteq A_{\alpha+\beta} \oplus A_{\beta-\alpha} = 0$, and this is a contradiction. Therefore, the only possibilities are either $A = A_\alpha \oplus A_{-\alpha}$ with dim $A_{\pm \alpha} = 2$ or $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ with dim $A_0 = 2$ and dim $A_{\pm \alpha} = 1$. In the first case $A_\alpha A_{-\alpha} = A_{-\alpha} A_\alpha = 0$, so for any $0 \neq x_\alpha \in A_\alpha$, an element $x_{-\alpha} \in A_{-\alpha}$ can be taken with $q(x_\alpha + x_{-\alpha}) = 2(x_\alpha, x_{-\alpha}) \neq 0$ and, since the left multiplication by $x_\alpha + x_{-\alpha}$ is a bijection, $A = A_\alpha \oplus A_{-\alpha} = (x_\alpha + x_{-\alpha})(A_\alpha \oplus A_{-\alpha}) = x_{-\alpha}A_{-\alpha} \oplus x_\alpha A_\alpha$, so $A_{-\alpha} = x_\alpha A_\alpha$. With a similar argument we conclude that for any $0 \neq x_\alpha \in A_\alpha$ and $0 \neq y_{-\alpha} \in A_{-\alpha}$, $x_\alpha A_\alpha = A_\alpha x_\alpha = A_{-\alpha}$ and $y_{-\alpha}A_{-\alpha} = A_{-\alpha}y_{-\alpha} = A_\alpha$. Let $0 \neq b_{\pm \alpha} \in A_{\pm \alpha}$ with $(b_\alpha, b_{-\alpha}) = 0$. Then, $(b_\alpha x_\alpha, b^2_{-\alpha}) = 2(b_\alpha, b_{-\alpha})(x_\alpha, b_{-\alpha}) - (b_\alpha b_{-\alpha}, b_{-\alpha}x_\alpha) = 0$ for any $x_\alpha \in A_\alpha$. In consequence $(A_{-\alpha}, b^2_{-\alpha}) = 0$, $b^2_{-\alpha} = 0$ and, since the kernel of the restriction to $A_{-\alpha}$ of the left multiplication by $b_{-\alpha}$ is 0, we get $b_{-\alpha} = 0$, a contradiction.

So we are left with the only possibility: $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ with dim $A_0 = 2$ and dim $A_{\pm \alpha} = 1$. As in the characteristic not 3 situation, if dim $H > 1$, there is another derivation d' linearly independent with d and such that $[d, d'] = \lambda d'$ for some $\lambda \in F$. Moreover, with the same reasoning as above, if $A = A'_0 \oplus A'_\beta \oplus A'_{-\beta}$ is the corresponding decomposition with respect to d', we obtain that $dA'_0 \subseteq A'_0$ and, according to Theorem 5.1, either $dA'_0 = 0$, which leads to contradiction again, or $d|_{A'_0}$ has two eigenvectors with opposite eigenvalues. In this situation, $A'_0 = A_\alpha \oplus A_{-\alpha}$, but then $A_0 A'_0 + A'_0 A_0 = A_0(A_\alpha + A_{-\alpha}) + (A_\alpha + A_{-\alpha})A_0 \subseteq A_\alpha + A_{-\alpha} = A'_0$ and A'_0 is an ideal of A, a contradiction with the simplicity of A. ∎

Recall (see for instance [W, Corollary 2.4.14]) that given a derivation d of an algebra over a field F which splits or is separable over F, then the semisimple and nilpotent parts d_s and d_n of d are again derivations.

THEOREM 5.5. *Let A be a four-dimensional composition algebra and* Der A *its Lie algebra of derivations. Then the following conditions are equivalent*:

(a) *A is a standard composition algebra.*

(b) *The dimension of* Der A *is* ≥ 2.

(c) Der A *is a central simple three-dimensional Lie algebra.*

Proof. Let C be a four-dimensional Hurwitz algebra (that is, a generalized quaternion algebra) with canonical involution $x \mapsto \bar{x}$, and let $(A, \circ)$ be a standard composition algebra associated to C, so that as vector spaces $A = C$ and the multiplication in A is given by one of the following:

(i) $x \circ y = xy$,   (ii) $x \circ y = \bar{x}y$,

(iii) $x \circ y = x\bar{y}$,   or   (iv) $x \circ y = \bar{x}\bar{y}$

as in (9). In the first two cases, if e is the unit element of C, the left multiplication in A by e, L_e°, is the identity mapping. Hence, for any $d \in \mathrm{Der}\, A$, $0 = [d, L_e^\circ] = L_{de}^\circ$, and $de = 0$. We argue similarly for (iii) with R_e°. In the para-Hurwitz case $Fe = \{x \in A : x \circ y = y \circ x \text{ for any } y \in A\}$, so Fe is invariant under any $d \in \mathrm{Der}\, A$ and again we arrive at $de = 0$. Therefore, in the four cases $(\mathrm{Der}\, A)e = 0$ and this forces any derivation to commute with the canonical involution of C. As a consequence, $\mathrm{Der}\, A = \mathrm{Der}\, C$, which is known to be a three-dimensional central simple Lie algebra. Therefore condition (a) implies condition (c). Obviously, condition (c) implies (b).

So assume that $\dim \mathrm{Der}\, A \geq 2$ and since we can extend scalars, that F is algebraically closed. By Lemmas 5.3 and 5.4 and the remark preceding the theorem, in $\mathrm{Der}\, A$ there are both diagonalizable and nilpotent derivations. By Lemma 5.4 there is a diagonalizable derivation h such that $A = A_0 \oplus A_1 \oplus A_{-1}$ with $A_\mu = \{x \in A : hx = \mu x\}$. Since $d \mapsto [h, d]$ is diagonalizable too in $\mathrm{Der}\, A$, there is also a decomposition in eigenspaces relative to the adjoint action of h: $\mathscr{L} = \mathrm{Der}\, A = \oplus_{\lambda \in \Lambda} \mathscr{L}_\lambda$, with $\mathscr{L}_\lambda = \{d \in \mathrm{Der}\, A : [h, d] = \lambda d\}$. Moreover, $\mathscr{L}_\lambda A_\mu \subseteq A_{\lambda + \mu}$ for any λ, μ, so λ is restricted to be $0, \pm 1, \pm 2$. Since $\dim A_{\pm 1} = 1$ and because of Theorem 5.1 and Lemma 5.4 we get that $\mathscr{L}_0 = Fh$.

In case the characteristic of F is not 3, for any $d \in \mathscr{L}_2$, $dA_{-1} \subseteq A_1$ and $(dA_{-1}, A_{-1}) = 0$ by (8). Therefore $dA_{-1} = 0$ and $d = 0$. Hence $\mathscr{L}_{\pm 2} = 0$. In case the characteristic is 3 then $\mathscr{L}_{\pm 2} = \mathscr{L}_{\mp 1}$ and for any $d \in \mathscr{L}_2 = \mathscr{L}_{-1}$ again $dA_{-1} = 0$, so $d^3 = 0$ since $dA_1 \subseteq A_0$ and $dA_0 \subseteq A_{-1}$. Hence, in any characteristic ($\neq 2$) $\mathscr{L} = Fh \oplus \mathscr{L}_1 \oplus \mathscr{L}_{-1}$ and $dA_{\pm 1} = 0$ for any $d \in \mathscr{L}_{\pm 1}$, so d is nilpotent.

Without loss of generality, we can assume that there is a nonzero derivation $d \in \mathscr{L}_1$. Then by Lemma 5.2 and since $(A_1, A_1) = 0$, $dA_1 = 0$, and $\dim(\ker d \cap A_0) = 1$, there is an element $e \in A_0$ with $de = 0$ and $q(e) \neq 0$. But $de^2 = 0$ so $e^2 \in Fe$ and we may assume that $e^2 = e$ and thus $q(e) = 1$. But $dA_1 = 0$ and $d^2 \neq 0$ by Lemma 5.2, hence if $0 \neq a \in A_{-1}$, $da \neq 0 \neq d^2a$. Besides, $(e, da) = -(de, a) = 0$, so $\{e, da\}$ is an orthogonal basis of A_0, $A_1 = Fd^2a$, and the orthogonal subspace to Fe is F-span$\langle a, da, d^2a \rangle$. Now, $ea \in A_{-1}$ so $ea = \mu a$ for some $\mu \in F$, thus

$$eda = d(ea) = d(\mu a) = \mu da \quad \text{and} \quad ed^2a = d^2(ea) = \mu d^2a,$$

but

$$0 \neq q(da) = q(e)q(da) = q(eda) = q(\mu da) = \mu^2 q(da),$$

so $\mu = \pm 1$. Therefore the restriction of L_e to the orthogonal subspace to e is plus or minus the identity and the same for the restriction of the right multiplication R_e. Then, with the new multiplication $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$, $(A, \cdot)$ is a Hurwitz algebra with unit element e, and, since $xy = (R_e x)(L_e y)$, A is a standard composition algebra associated to $(A, \cdot)$. ∎

From the previous results it is clear what the possibilities are for the Lie algebra of derivations in the four-dimensional case:

THEOREM 5.6. *Let A be a four-dimensional composition algebra. Then either*:

   (i)   Der A *is a three-dimensional simple Lie algebra and this happens if and only if A is a standard composition algebra*, *or*

   (ii)   dim Der $A = 1$, Der $A = Fd$ *with semisimple action of d, and* dim ker $d = 2$, *or*

   (iii)   dim Der $A = 1$, Der $A = Fd$ *with $d^2 \neq 0 = d^3$, and* ker d *is not totally isotropic*.

   (iv)   Der $A = 0$.

All the possibilities in Theorem 5.6 actually arise. The corresponding composition algebras can be constructed starting from a Hurwitz algebra with unit element 1 and considering the multiplication $x * y = \varphi(x)\psi(y)$ for suitable orthogonal transformations φ and ψ.

Remark that the largest possible derivation algebras are the three-dimensional simple Lie algebras, so the classification of the four-dimensional composition algebras with the largest possible derivation algebra is analogous to Petersson's classification in dimension 2. Also notice that the results in this section strengthen [M, Theorem 9.14 and 9.15].

6. THE EIGHT-DIMENSIONAL CASE

Along this section, there will naturally appear some $\mathbb{Z}_3$-gradations of the split Cayley−Dickson algebra, so we will start by considering what these $\mathbb{Z}_3$-gradations look like. This will permit us to clarify the relationship among flexible composition algebras which are not Hurwitz and certain algebras introduced in [P1], as was mentioned in the Introduction.

Let C be the split Cayley−Dickson algebra over F, thought of as Zorn's vector matrix algebra (see [S1, Chap. III]). There is a basis $\{e_1, e_2, u_1,$

$u_2, u_3, v_1, v_2, v_3\}$ of C in which the multiplication table is:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

$$(13)$$

Such a basis will be called a *canonical basis* of C. The elements e_1 and e_2 are orthogonal idempotents of C and the u_i's and v_i's span the other Peirce subspaces.

Given a Cayley–Dickson algebra A, several nontrivial $\mathbb{Z}_3$-gradations $A = A_0 \oplus A_1 \oplus A_2$ ($A_i \neq 0$ for any i, $A_i A_j \subseteq A_{i+j}$, indices modulo 3) will appear in the sequel. Hence the next lemma will be very useful. It was proved in [EP2] with an additional hypothesis.

LEMMA 6.1. *Let A be a Cayley–Dickson algebra and let $A = A_0 \oplus A_1 \oplus A_2$ be a nontrivial $\mathbb{Z}_3$-gradation of A. Then A is split, $1 \in A_0$, and there is a canonical basis of A such that either*:

(i) $A_0 = F\text{-span}\langle e_1, e_2 \rangle$, $A_1 = F-\text{span}\langle u_1, u_2, u_3 \rangle$, *and* $A_2 = F-\text{span}\langle v_1, v_2, v_3 \rangle$, *or*

(ii) $A_0 = F\text{-span}\langle e_1, e_2, u_1, v_1 \rangle$, $A_1 = F\text{-span}\langle u_2, v_3 \rangle$, *and* $A_2 = F\text{-span}\langle u_3, v_2 \rangle$.

Proof. It is clear that $1 \in A_0$. Since $x^2 - f(x,1)x + q(x)1 = 0$ for any $x \in A$, for $x \in A_i$, $i = 1, 2$, one gets $x^2 = 0 = f(x,1)x = q(x)$, hence $\bar{x} = f(x,1)1 - x = -x \in A_i$, so $\bar{A}_j = A_j$ for any $j = 0, 1, 2$ and for $x \in A_i$ and $y \in A_j$ we have $f(x, y) = x\bar{y} + y\bar{x} \in A_{i+j} \cap F1$. Thus $f(A_i, A_j) = 0$ unless $i + j = 0 \pmod 3$. This is the additional hypothesis used in [EP2, Proposition 3.4], whence the result follows. ∎

Given a Hurwitz algebra C and an antiautomorphism φ of C such that φ^3 equals the canonical involution of C, then Petersson constructed in [P1] a new algebra C_φ on the same vector space C but with new multiplication

$$x * y = \varphi(x)\varphi^{-1}(y). \tag{14}$$

Then $(x * y) * x = x * (y * x) = q(x)y$, where q is the norm in C, so this construction gives flexible composition algebras. In this situation, since φ is an antiautomorphism of C, it commutes with the canonical involution $x \mapsto \bar{x}$ of C and we get the automorphism ψ of C given by $\psi(x) = \varphi(\bar{x}) = \overline{\varphi(x)}$. Moreover, $\psi^3 = 1$ and, conversely, given an automorphism ψ of C of order 3, the antiautomorphism given by $\varphi(x) = \psi(\bar{x}) = \overline{\psi(x)}$ verifies that φ^3 is the canonical involution. The composition algebra with multiplication $x * y = \varphi(x)\varphi^{-1}(y) = \psi(\bar{x})\psi^{-1}(\bar{y})$ as in (14) will also be denoted by C_ψ.

THEOREM 6.2 [EP2, Theorem 3.5]. *Let C be a Hurwitz algebra over a field of characteristic $\neq 3$, let $\psi \in \text{Aut}\, C$ such that $\psi^3 = 1$, and let $C_0 = \{x \in C : \psi(x) = x\}$. Then, if $\psi = 1$ (and this is always the case if $\dim C \leq 2$), C_ψ is the para-Hurwitz algebra associated to C. Otherwise, with $\psi \neq 1$ either*:

   (i) $\dim C_0 = 2$ *and C_ψ is a para-Hurwitz algebra, or*
   (ii) $\dim C_0 = 4$, $\dim C = 8$, *and C_ψ is an Okubo algebra.*

It must be remarked here that not all the Okubo algebras over arbitrary fields of characteristic $\neq 2, 3$ are constructed as C_ψ for suitable C and ψ, since all these algebras contain idempotents [EP2, Theorem 2.5] and this is no longer true for general Okubo algebras by [EM4, Proposition 7.4], which also shows that, under the restrictions considered in [P1] about F containing the cube roots of 1, only the split Okubo algebra (the one obtained from $sl(3, F)$ by means of (11)) appears in this construction.

We come back to the study of derivations.

LEMMA 6.3. *Let A be an eight-dimensional composition algebra with a nontrivial $\mathbb{Z}_3$-gradation $A = A_0 \oplus A_1 \oplus A_2$ such that $\dim A_0 = 2$. Then, if the subalgebra A_0 contains a one-sided unit element e such that the left and right multiplications in A by e commute, any derivation of A annihilates e.*

Proof. Nothing is lost if we extend scalars, so we will assume the ground field to be algebraically closed. The $\mathbb{Z}_3$-gradation in A induces a $\mathbb{Z}_3$-gradation in $\mathscr{L} = \text{Der}\, A$, so that $\mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1 \oplus \mathscr{L}_2$ with $\mathscr{L}_i = \{d \in \mathscr{L} : dA_j \subseteq A_{i+j}\}$ (indices modulo 3). We consider the Hurwitz algebra $(A, \cdot)$ with $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$. Since $q(e) = 1$ and $e \in A_0$, $R_e^{-1}A_i = A_i = L_e^{-1}A_i$ for any i and the $\mathbb{Z}_3$-gradation in A is also a $\mathbb{Z}_3$-gradation in $(A, \cdot)$, so that there is a canonical basis of $(A, \cdot)$ with $A_0 = Fe_1 + Fe_2$, $A_1 = F\text{-span}\langle u_1, u_2, u_3\rangle$, and $A_2 = F\text{-span}\langle v_1, v_2, v_3\rangle$. But e is the unit element of $(A, \cdot)$ so $e = e_1 + e_2$. For any $d \in \mathscr{L}_0$, $d|_{A_0} \in \text{Der}\, A_0 = 0$ by Theorem

5.1 and so $de = 0$. Now take $d \in \mathcal{L}_1$ and three possibilities arise:

(i) e is a two-sided unit of A_0: Then $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$, $i, j = 1, 2$. Hence, $de_i = de_i^2 = (de_i)e_i + e_i(de_i) = ((de_i)e) \cdot e_i + e_i \cdot (ede_i)$. For $i = 1$, since $d \in \mathcal{L}_1$, we get $de_1 = ede_1$, so $(de_1)e = (ede_1)e$. For $i = 2$, $de_2 = (de_2)e$, so $ede_2 = e((de_2)e)$. Then $0 = d(e_1 e_2) = ((de_1)e) \cdot e_2 + e_1 \cdot (e(de_2)) = (de_1)e + e(de_2) = (e(de_1))e + e((de_2)e) = e(de)e$, since $[L_e, R_e] = 0$, and this forces $de = 0$.

(ii) e is a left unit of A_0, but not a right unit: Then, A_0 is a standard composition algebra of type (9)(ii), so $e_i^2 = 0$, $i = 1, 2$, $e_1 e_2 = e_2$, and $e_2 e_1 = e_1$. Hence, $de_1 = d(e_2 e_1) = ((de_2)e) \cdot (ee_1) + (e_2 e) \cdot (ede_1) = ((de_2)e) \cdot e_1 + e_1 \cdot (e(de_1)) = e(de_1)$, and $de_2 = d(e_1 e_2) = ((de_1)e) \cdot (ee_2) + (e_1 e) \cdot (e(de_2)) = (de_1)e$. Now, $0 = de_2^2 = (e_2 e) \cdot (ede_2) + ((de_2)e) \cdot (ee_2) = e(de_2) + (de_2)e = e(de_1)e + (de_2)e = (de_1)e + (de_2)e = (de)e$ and $de = 0$ again.

(iii) If e is a right unit of A_0, but not a left unit, everything works the same.

Therefore $\mathcal{L}_1 e = 0$ and similarly we prove $\mathcal{L}_2 e = 0$, thus $(\mathrm{Der}\, A)e = 0$, as required. ∎

LEMMA 6.4. *Let A be a finite dimensional composition algebra with associated quadratic form permitting composition q and let $e \in A$ with $q(e) = 1$. Let $\cdot$ be the Hurwitz product given by $x \cdot y = (R_e^{-1} x)(L_e^{-1} y)$ defined on A with unit element e^2. Then,*

$$\{d \in \mathrm{Der}\, A: de = 0\} = \{d \in \mathrm{Der}(A, \cdot): [d, L_e] = [d, R_e] = 0\},$$

where L_e and R_e are the left and right multiplications by e in A.

Proof. For $d \in \mathrm{Der}\, A$ with $de = 0$, $[d, L_e] = L_{de} = 0 = R_{de} = [d, R_e]$. Hence, $[d, L_e^{-1}] = [d, R_e^{-1}] = 0$ and it is clear now that $d \in \mathrm{Der}(A, \cdot)$. Conversely, any $d \in \mathrm{Der}(A, \cdot)$ with $[d, L_e] = [d, R_e] = 0$ is a derivation of A since $xy = (R_e x) \cdot (L_e y)$ for any $x, y \in A$. Moreover, since e^2 is the unit of $(A, \cdot)$, $0 = de^2 = d(L_e e) = L_e(de)$, but $q(e) = 1$, so L_e is a bijection and $de = 0$. ∎

From now on, we will assume that *the characteristic of the ground field F is $\neq 2, 3$.*

Our first purpose is to show that under some restrictions, given an eight-dimensional composition algebra, either it is an Okubo algebra or there is an idempotent e with $q(e) = 1$ such that $(\mathrm{Der}\, A)e = 0$. In the first case the derivation algebra is known by Section 4. In the last case, Lemma 6.4 tells us that $\mathrm{Der}\, A$ is a specific subalgebra of the derivation algebra of a Hurwitz algebra, and this latter algebra is well known.

Let A be an eight-dimensional composition algebra over F, which we assume for a while to be algebraically closed. Let $\mathscr{L} = \operatorname{Der} A$ and let H be a Cartan subalgebra of $\mathscr{L}$. Then A decomposes in a direct sum of weight spaces,

$$A = \bigoplus_{\alpha \in \Pi} A_\alpha, \qquad (15)$$

where Π is a subset of mappings from H into F.

By Theorem 2.3, if $(\ , \ ) = \frac{1}{2}f$ is the bilinear symmetric form permitting composition, $(A_\alpha, A_\beta) \neq 0$ if and only if $\beta = -\alpha$, and by the nondegeneracy of $(\ , \ )$ this forces $\dim A_\alpha = \dim A_{-\alpha}$. Also $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for any $\alpha, \beta \in \Pi$.

PROPOSITION 6.5. *Let A be an eight-dimensional composition algebra over an algebraically closed field F, let $\mathscr{L}$ be its Lie algebra of derivations, and let H be a Cartan subalgebra of $\mathscr{L}$ (although it would suffice H to be a nilpotent subalgebra). Assume that there are two linearly independent weights in the decomposition* (15). *Then, there are two linearly independent weights α, β such that* (15) *becomes*

$$A = A_0 \oplus \left(A_\alpha \oplus A_\beta \oplus A_{-(\alpha+\beta)} \right) \oplus \left(A_{-\alpha} \oplus A_{-\beta} \oplus A_{\alpha+\beta} \right), \quad (16)$$

where $\dim A_0 = 2$, $\dim A_\gamma = 1$ *for any* $\gamma \neq 0$. *Moreover, A_0 together with $A_1 = A_\alpha \oplus A_\beta \oplus A_{-(\alpha+\beta)}$ and $A_2 = A_{-\alpha} \oplus A_{-\beta} \oplus A_{\alpha+\beta}$ forms a $\mathbb{Z}_3$-gradation of A. Besides, H is a two-dimensional abelian subalgebra.*

Proof. Let α and β be linearly independent weights in (15). Then

$$A = \left(A_\alpha \oplus A_{-\alpha} \right) \oplus \left(A_\beta \oplus A_{-\beta} \right) \oplus A_*,$$

where A_* is the sum of the other weight spaces and it is also the orthogonal complement to $(A_\alpha \oplus A_{-\alpha}) \oplus (A_\beta \oplus A_{-\beta})$. In case $A_0 = 0$, as in Lemma 5.4 we conclude that $\pm 2\alpha, \pm 2\beta \in \Pi$, but then also $0 \neq (A_\alpha \oplus A_{-\alpha})A_\beta$, so $\alpha + \beta$ or $-\alpha + \beta \in \Pi$ and there is not enough room for so many weights. Hence, $A_0 \neq 0$ and again either $\alpha + \beta$ or $\beta - \alpha \in \Pi$. In case $\alpha - \beta \in \Pi$, we change β by $-\beta$ to get the decomposition (16). The $\mathbb{Z}_3$-gradation is clear and for any $d \in H$, $d|_{A_0}$ is a derivation of a two-dimensional composition algebra. Hence $d|_{A_0} = 0$ by Theorem 5.1, so any $d \in H$ acts diagonally on A and the last assertion follows easily. ∎

Therefore, the maximum number of linearly independent weights for the action of the Cartan subalgebra H of $\mathscr{L} = \operatorname{Der} A$ on A is two. In case this maximum is attained we will say that the *toral rank* of $\mathscr{L}$ on A is two. Notice also that the roots of H in $\mathscr{L}$ are differences of weights.

Now, with A, H, $\mathscr{L}$, and F as in Proposition 6.5, A_0 is a two-dimensional composition algebra over the algebraically closed field F, so by (4) and the comments that follow it, A_0 is a standard composition algebra. Let e be an idempotent of A_0 with $q(e) = 1$. Hence, e is the unit element if A_0 is Hurwitz, the one-sided unit element in cases (4)(ii) and (iii) or a para-unit in the para-Hurwitz case. Moreover, since $\dim A_\gamma = 1$ for any $\gamma \neq 0$ and $A_0 A_\gamma + A_\gamma A_0 \subseteq A_\gamma$, the left and right multiplications by e, L_e, and R_e, are diagonalizable. Consider the Hurwitz product on A: $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$ with unit element e. By Lemma 6.1 and its proof, there is a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of $(A, \cdot)$ with $A_0 = Fe_1 + Fe_2$, $e = e_1 + e_2$, $A_\alpha = Fu_1$, $A_\beta = Fu_2$, $A_{-(\alpha+\beta)} = Fu_3$, $A_{-\alpha} = Fv_1$, $A_{-\beta} = Fv_2$, and $A_{\alpha+\beta} = Fv_3$. In the basis $\{e, e_1 - e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ the matrices associated to L_e and R_e are diagonal and present the form

$$\mathrm{diag}\bigl(1, \pm 1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1^{-1}, \epsilon_2^{-1}, \epsilon_3^{-1}\bigr)$$

since both L_e and R_e are orthogonal transformations.

THEOREM 6.6. *Let A be an eight-dimensional composition algebra over F, $\overline{F}$ the algebraic closure of F, and $\overline{A} = \overline{F} \otimes_F A$. Assume that $\mathrm{Der}\,\overline{A}$ has toral rank 2 on $\overline{A}$. Then either*:

(i) *there is an element $e \in A$ with $q(e) = 1$ (q being the quadratic form permitting composition) such that $(\mathrm{Der}\,A)e = 0$, or*

(ii) *A is an Okubo algebra.*

Proof. We can assume that F is algebraically closed since $\overline{F} \otimes_F \{x \in A: (\mathrm{Der}\,A)x = 0\} = \{x \in \overline{A}: (\mathrm{Der}\,\overline{A})x = 0\}$ and if $(\mathrm{Der}\,A)x = 0$ and $q(x) \neq 0$, then with $e = (1/q(x))x^2$, $q(e) = 1$, and $(\mathrm{Der}\,A)e = 0$.

We have then a decomposition (16) relative to certain Cartan subalgebra H of $\mathscr{L} = \mathrm{Der}\,A$ and take e as in the paragraph preceding the theorem. Then, since L_e and R_e are simultaneously diagonalizable, they commute. So in case A_0 is not para-Hurwitz, then $(\mathrm{Der}\,A)e = 0$ by Lemma 6.3. Thus, assume A_0 is para-Hurwitz and none of the three para-units of A_0 is annihilated by $\mathscr{L}$. Let z be a para-unit of A_0; without loss of generality there is a $d \in \mathscr{L}_\alpha$ (root space) such that $0 \neq dz \in A_\alpha$. Then, the composition subalgebra $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ has a nonzero nilpotent derivation, namely the restriction of d, and at least a nonzero semisimple derivation (the restriction of $h \in H$ with $\alpha(h) \neq 0$). By Theorem 5.5, $A(\alpha) = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ is a standard composition algebra. Then, the only element in $A(\alpha)$, up to scalars, annihilated by $\mathrm{Der}\,A(\alpha)$ is the distinguished element of $A(\alpha)$, in this case the unique para-unit of $A(\alpha)$, which belongs to A_0; let us denote it by e. Thus, $\mathscr{L}_{\pm\alpha}e = 0$, where $\mathscr{L}_\alpha$ is the corresponding root space relative to the Cartan subalgebra H in $\mathscr{L}$, and in particular, $de = 0$.

Define now the Hurwitz algebra $(A, \cdot)$ by $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$ with unit element e and consider a canonical basis as in the paragraph preceding the theorem.

Since $dz \neq 0 = de$, we have $de_1 = -de_2 \neq 0$ and can assume $de_1 = u_1$ (recall that $d \in \mathscr{L}_\alpha$). Since $de = 0$, $d \in \text{Der}(A, \cdot)$ and $du_2 = d(e_1 \cdot u_2) = (de_1) \cdot u_2 + e_1 \cdot (du_2) = u_1 \cdot u_2 = v_3$, since $du_2 \in \mathscr{L}_\alpha A_\beta \subseteq A_{\alpha+\beta} = Fv_3$ and $e_1 \cdot v_3 = 0$. Let the matrices associated to L_e and R_e in the basis $\{e, e_1 - e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ be given by $\text{diag}(1, -1, \alpha_l, \beta_l, \gamma_l, \alpha_l^{-1}, \beta_l^{-1}, \gamma_l^{-1})$ and $\text{diag}(1, -1, \alpha_r, \beta_r, \gamma_r, \alpha_r^{-1}, \beta_r^{-1}, \gamma_r^{-1})$. Since e is the para-unit of $A(\alpha)$, $\alpha_l = \alpha_r = -1$. Now, $\gamma_l^{-1}v_3 = e(du_2) = d(eu_2) = \beta_l(du_2) = \beta_l v_3$, so $\beta_l = \gamma_l^{-1}$ and also $\beta_r = \gamma_r^{-1}$. On the other hand, our hypothesis is that there are derivations which do not annihilate e and this can be taken in root spaces of $\mathscr{L}$ with respect to H. Without loss of generality, there is $d' \in \mathscr{L}_\beta$ such that $d'e = u_2$. Then

$$d'e_2 = d'e_1^2 = (d'e_1)e_1 + e_1(d'e_1)$$
$$= ((d'e_1)e) \cdot e_2 + e_2 \cdot (e(d'e_1)) = (d'e_1)e$$
$$d'e_1 = d'e_2^2 = (d'e_2)e_2 + e_2(d'e_2) \qquad (17)$$
$$= ((d'e_2)e) \cdot e_1 + e_1 \cdot (e(d'e_2)) = e(d'e_2)$$
$$0 = d'(e_2e_1) = ((d'e_2)e) \cdot e_2 + e_1 \cdot (e(d'e_1)) = (d'e_2)e + e(d'e_1).$$

Hence, $e(d'e_1) = -(d'e_2)e$ and $L_e^3(d'e_1) = -L_e^2((d'e_2)e) = -L_e((e(d'e_2))e) = -L_e((d'e_1)e) = -L_e(d'e_2) = -d'e_1$, where we have used that $[L_e, R_e] = 0$. Similarly $R_e^3(d'e_1) = -d'e_1$, $L_e^3(d'e_2) = -d'e_2$, and $R_e^3(d'e_2) = -d'e_2$. Since $d'e_1, d'e_2 \in A_\beta = Fu_2$, they are eigenvectors of L_e and R_e which are not zero, we conclude that β_l and β_r are cube roots of -1. If, for instance, $\beta_l = -1$ $(ed'e_1 = -d'e_1)$, then $d'e_1 = e(ed'e_1) = -e(d'e_2)e = -(d'e_1)e = -d'e_2$ by (17), but this forces $d'e = d'(e_1 + e_2) = 0$, a contradiction. By the same token $\beta_r \neq -1$.

Then, $\beta_l = -\omega$, where ω is a cube root of 1, $\omega \neq 1$, so $\gamma_l = \beta_l^{-1} = -\omega^2$. On the other hand, $d'e = d'e^2 = (R_e + L_e)d'e$, so $\beta_r - \omega = 1$. Since β_r is also a primitive cube root of -1, we obtain $\beta_r = -\omega^2$ and $\gamma_r = \beta_r^{-1} = -\omega$. As a conclusion, the matrices associated to L_e and R_e in the basis $\{e, e_1 - e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ are respectively $\text{diag}(1, -1, -\omega, -\omega^2, -1, -\omega^2, -\omega)$ and $\text{diag}(1, -1, -1, -\omega^2, -\omega, -1, -\omega, -\omega^2)$, so $xy = (xe) \cdot (ye) = \psi(\bar{x})\psi^{-1}(\bar{y})$, where ψ is the automorphism of $(A, \cdot)$ such that $\psi = 1$ on $A_0 = F\text{-span}\langle e_1, e_2, u_1, v_1 \rangle$, $\psi = \omega 1$, on $\tilde{A}_1 = F\text{-span}\langle u_2, v_3 \rangle$, and $\psi = \omega^2 1$ on $\tilde{A}_2 = F\text{-span}\langle u_3, v_2 \rangle$. From Theorem 6.2, we conclude that A is the Okubo algebra. ∎

Remark . The proof of Theorem 6.6 shows that if A is an eight-dimensional composition algebra over an algebraically closed field, other than an Okubo algebra, and H is a Cartan subalgebra of Der A with associated weight decomposition as in (16), then the element e in part (i) of the preceding theorem can be chosen to be a distinguished element (unit, one-sided unit, or para-unit) of the two-dimensional composition subalgebra A_0. In particular, it can be chosen to be an idempotent.

With A as in the theorem, in case there is an element $e \in A$ with $q(e) = 1$ and (Der $A)e = 0$, Lemma 6.4 proves that Der A is a subalgebra of Der$(A, \cdot)$, where $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$, so that $(A, \cdot)$ is a Hurwitz algebra. Then, Der$(A, \cdot)$ is known to be a central simple Lie algebra of type G_2. The largest possible derivation algebras in this situation appear when Der $A = $ Der$(A, \cdot)$. As in the four-dimensional case (Theorem 5.5), we have:

THEOREM 6.7. *Let A be an eight-dimensional composition algebra and* Der *A its Lie algebra of derivations. Then A is a standard composition algebra if and only if* Der *A is a central simple Lie algebra of type* G_2.

Proof. By Section 3, we again can assume that F is algebraically closed. If C is a Hurwitz algebra, the same argument as in the proof of Theorem 5.5 shows that the derivation algebra of any of the standard composition algebras associated to C is the same derivation algebra of C. Now, assume that A is an eight-dimensional composition algebra with Der A the central simple Lie algebra of type G_2 over F. Take H a Cartan subalgebra, since any root of H is a difference of weights (the action of Der A on A is faithful) it follows that the toral rank of Der A on A is two and by the remark following Theorem 6.6, since the derivation algebra of any Okubo algebra is not of type G_2, we conclude that there is an idempotent $e \in A$ such that $q(e) = 1$ and (Der $A)e = 0$. We pass to the Hurwitz algebra $(A, \cdot)$ with $x \cdot y = (R_e^{-1}x)(L_e^{-1}y)$ and unit element e, so by our hypotheses and Lemma 6.4, Der $A = $ Der$(A, \cdot)$. But Der$(A, \cdot)$ acts irreducibly on $V = (Fe)^\perp$, the orthogonal complement to Fe under q. By Schur's Lemma, $L_e|_V$ and $R_e|_V$ are scalars, and since they are orthogonal transformations, we get $L_e|_V = \pm 1$ and $R_e|_V = \pm 1$. That is, both L_e and R_e are either 1 or the canonical involution of $(A, \cdot)$. Since $xy = (R_e x) \cdot (L_e y)$ we obtain that A is a standard composition algebra associated to $(A, \cdot)$. ∎

Notice that, similarly to the situation in the four-dimensional case, the composition algebras analogous to the ones in Petersson's classification in dimension 2 appear when we impose a central simple derivation algebra of type G_2.

Assume for a while that the field F is algebraically closed, A is an eight-dimensional composition algebra, $\mathcal{L} = \text{Der}\, A$ its Lie algebra of derivations, H a Cartan subalgebra of $\mathcal{L}$ so that we have a weight space decomposition as in (16), and A is not an Okubo algebra. From the remark after Theorem 6.6, there is a distinguished element in A_0 (unit, one-sided unit, or para-unit) with $(\text{Der}\, A)e = 0$. We use this e to define the Hurwitz algebra $(A, \cdot)$ as in the paragraph preceding Theorem 6.6. Then, the matrix of L_e in basis $\{e, e_1 - e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is $\text{diag}(1, \pm 1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1^{-1}, \epsilon_2^{-1}, \epsilon_3^{-1})$ and similarly for R_e. The type of A (see Section 3) is determined by the signs in the second place of the diagonal of these matrices. Therefore, it seems more natural to consider, instead of $(A, \cdot)$, the associated standard composition algebra of the same type as A. Hence, if $x \mapsto \bar{x}$ is the canonical involution of $(A, \cdot)$, we consider the standard algebra $(A, \circ)$ with $x \circ y = x \cdot y$, $\bar{x} \cdot y$, $x \cdot \bar{y}$ or $\bar{x} \cdot \bar{y}$ according to the type of A. For instance, we choose $x \circ y = \bar{x} \cdot y$ in case $e(e_1 - e_2) = e_1 - e_2$ and $(e_1 - e_2)e = e_2 - e_1$. The multiplication in A is then given by

$$xy = \varphi(x) \circ \psi(y),$$

where $\varphi(x) = xe$ or $\bar{x}e$ $(= \overline{xe}$ since R_e commutes with the canonical involution) and $\psi(x) = ex$ or $e\bar{x}$ $(= \overline{ex})$ according to the type. From Lemma 6.4 we conclude that

$$\text{Der}\, A = \{d \in \text{Der}(A, \cdot) : [d, L_e] = [d, R_e] = 0\}$$
$$= \{d \in \text{Der}(A, \circ) : [d, \varphi] = [d, \psi] = 0\}.$$

Moreover, in the chosen basis of A, the matrices of φ and ψ are

$$\varphi \leftrightarrow \text{diag}(1, 1, \mu_1, \mu_2, \mu_3, \mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}),$$
$$\psi \leftrightarrow \text{diag}(1, 1, \nu_1, \nu_2, \nu_3, \nu_1^{-1}, \nu_2^{-1}, \nu_3^{-1})$$

for suitable nonzero scalars μ_i's and ν_i's. A lengthy computation, which we omit, gives:

PROPOSITION 6.8.   *With the hypotheses above*:

(i)   Der A *is the central simple Lie algebra of type* G_2 *if and only if* $\varphi = \psi = 1$, *so that* $A = (A, \circ)$.

(ii)   Der A *is isomorphic to* $sl(3, F)$ (*the central simple Lie algebra of type* A_2) *if and only if* $\mu_1 = \mu_2 = \mu_3$, $\nu_1 = \nu_2 = \nu_3$, *and at least one of* μ_1 *and* ν_1 *are not equal to* 1. *In this case* $\{x \in A : (\text{Der}\, A)x = 0\}$ *equals* A_0 *in* (16), *so it is two-dimensional and the orthogonal complement of* A_0 *relative to* q *splits into the direct sum of the two irreducible dual modules of* Der $A \cong$ $sl(3, F)$, *namely the subspaces* A_1 *and* A_2 *in the* $\mathbb{Z}_3$-*gradation considered in Proposition* 6.5.

(iii)   *Otherwise,* $\dim \operatorname{Der} A \leq 6$ *and* $\operatorname{Der} A$ *is either isomorphic to a direct sum of two copies of* $sl(2, F)$, *or to a direct sum of a copy of* $sl(2, F)$ *and a one-dimensional center or it is a two-dimensional abelian Lie algebra.*

Since we want to consider those composition algebras with derivation algebras as large as the ones of the flexible composition algebras, we must finally turn our attention to those composition algebras which, after extension of scalars, become the algebras in Proposition 6.8(ii). For the latter algebras, the last assertion in Proposition 6.8(ii) tells us that the centralizer $\mathscr{C}$ of the action of $\operatorname{Der} A$ in the orthogonal complement $W = A_0^{\perp}$ is two-dimensional. Moreover, since $(\operatorname{Der} A)A_0 = 0$, it follows that

$$\{L_x|_W : x \in A_0\} = \mathscr{C} = \{R_x|_W : x \in A_0\}.$$

THEOREM 6.9.   *Let A be an eight-dimensional composition algebra such that* $\operatorname{Der} A$ *is a central simple Lie algebra of type A_2 and A is not an Okubo algebra. Then, there is a Cayley–Dickson algebra C, with multiplication denoted by juxtaposition, a two-dimensional Hurwitz subalgebra K of C and two orthogonal transformations φ and ψ of C which are the identity on the orthogonal complement W of K (so that $\varphi|_K$ and $\psi|_K$ are orthogonal transformations of K) such that A is isomorphic to $(C, \circ)$, where*

$$x \circ y = \varphi(x)\psi(y) \tag{18}$$

for all $x, y \in C$. Moreover, under the isomorphism K corresponds to the subalgebra $\{x \in A : (\operatorname{Der} A)x = 0\}$.

Conversely, given $C, K, W, \varphi, \psi,$ and $x \circ y$ as above, $\operatorname{Der}(C, \circ)$ is a central simple Lie algebra of type A_2 with the only exceptions of $(\varphi, \psi) = (1, 1), (-j, 1), (1, -j),$ or $(-j, -j),$ where j is the canonical involution of C. For these four cases $(C, \circ)$ is a standard composition algebra and $\operatorname{Der}(C, \circ)$ is a central simple Lie algebra of type G_2.

Proof.   Let A be an eight-dimensional composition algebra as in the theorem. From Proposition 6.8 and the comments following it, if $A_0 = \{x \in A : (\operatorname{Der} A)x = 0\}$, then A_0 is two-dimensional, and if W is the orthogonal complement to A_0, the centralizer of the action of $\operatorname{Der} A$ on W is $\{L_x|_W : x \in A_0\} = \{R_x|_W : x \in A_0\}$. Hence, there are elements $u, v \in A_0$ with $R_u|_W = 1 = L_v|_W$ and this forces $q(u) = q(v) = 1$. We consider now the composition algebra $C = (A, \cdot)$ with multiplication

$$x \cdot y = \left(R_u^{-1}x\right)\left(L_v^{-1}y\right)$$

for any $x, y \in A$. C is a Hurwitz algebra with unit $1 = vu$. Then, $xy = \varphi(x) \cdot \psi(y)$ with $\varphi = R_u$ and $\psi = L_v$, which verify $\varphi|_W = \psi|_W = 1$ and $K = (A_0, \cdot)$ is a Hurwitz subalgebra of C.

For the converse, let C, K, W, φ, and ψ be as in the theorem and let $x \circ y$ be defined by (18). Let $\mathrm{Der}_K C = \{d \in \mathrm{Der}\, C : d(K) = 0\}$, which is a central simple Lie algebra of type A_2 (see [EM5, Theorem 4.11]). By the hypotheses on φ and ψ, it is clear that $\mathrm{Der}_K C \subseteq \mathrm{Der}(C, \circ)$. Now, by Proposition 6.8, either $\mathrm{Der}(C, \circ) = \mathrm{Der}_K C$ or $(C, \circ)$ is a standard composition algebra. Thus, we must study under what conditions this last situation is verified.

If $(C, \circ)$ is a Hurwitz algebra with unit element e, then $\mathrm{Der}(C, \circ)e = 0$, so $(\mathrm{Der}_K C)e = 0$ and $e \in K$. For any $x \in W$ with $q(x) \neq 0$, if 1 denotes the unit element of C, $1x = x = e \circ x = \varphi(e)x$, so $\varphi(e) = 1$ and, similarly, $\psi(e) = 1$. Besides, for any $x \in K$, $x = e \circ x = \varphi(e)\psi(x) = 1\psi(x) = \psi(x)$, so $\psi = 1$, and also $\varphi = 1$.

If $(C, \circ)$ is a standard composition algebra of type (9)(ii) and e is its left unit, as above we conclude that $\varphi(e) = 1$ and $\psi = 1$. Then, for any $x \in W$, $-x = x \circ e = \varphi(x)e = xe$, so $e = -1$. Now, if $x \in K$ and $t(x) = 0$, $-x = x \circ e = \varphi(x)e = -\varphi(x)$ and $\varphi = -j$. In the same way, if $(C, \circ)$ is standard of type (9)(iii), then $\varphi = 1$ and $\psi = -j$.

Finally, if $(C, \circ)$ is para-Hurwitz with para-unit e, again $e \in K$. For any $x \in W$ with $q(x) \neq 0$, $-x = e \circ x = \varphi(e)x$, so $\varphi(e) = -1$, and also $\psi(e) = -1$. Now, with $x \in K$ such that $(x, e) = 0$, $-x = e \circ x = \varphi(e)\psi(x) = -\psi(x)$, so $\psi(x) = x = \varphi(x)$ and since φ and ψ are orthogonal, $\varphi(e) = \pm e = \psi(e)$, so that $e = \pm 1$. Since $\varphi = \psi$ are not 1 (otherwise $C, \circ$) would be Hurwitz), we arrive at $e = 1$ and $\varphi = \psi = -j$. ∎

The next lemma extends [EM5, Theorem 4.11,(ii)] and will help us to set the notation too:

LEMMA 6.10. *Let C be a Cayley–Dickson algebra, K a two-dimensional Hurwitz subalgebra of C, and let W be the orthogonal complement to K. Define a multiplication in W by*

$$x * y = \text{projection of } xy \text{ on } W \tag{19}$$

for any $x, y \in W$, and let $\mathrm{Aut}(C, K)$ be the group of automorphisms φ of C such that $\varphi(K) = K$. Then, the restriction map $\varphi \mapsto \varphi|_W$ is an isomorphism between $\mathrm{Aut}(C, K)$ and $\mathrm{Aut}(W, *)$ (the group of automorphisms of $(W, *)$).

Proof. With the same notation as in [EM5], for $a, b \in W$, we have $ab = a * b - \sigma(a, b)$, with $\sigma: W \times W \to K$ a K-hermitian form. If $\varphi \in \mathrm{Aut}(C, K)$ and $a, b \in W$, $\varphi(a * b) = \varphi(ab + \sigma(a, b)) = \varphi(a)\varphi(b) + \varphi(\sigma(a, b)) = \varphi(a) * \varphi(b) + (\varphi(\sigma(a, b)) - \sigma(\varphi(a), \varphi(b)))$, so that $\varphi(a * b) = \varphi(a) * \varphi(b)$ and $\varphi(\sigma(a, b)) = \sigma(\varphi(a), \varphi(b))$. In particular, $\varphi|_W \in \mathrm{Aut}(W, *)$. Conversely, given $\psi \in \mathrm{Aut}(W, *)$, then by [EM5, Theorem 2.5], there exists $s: K \to K$ ($x \mapsto x^s$) automorphism (so that s equals either 1 or the conjugation) such that $\sigma(\psi(a), \psi(b)) = \sigma(a, b)^s$, and we can extend ψ to C by $\psi(\alpha + a) = \alpha^s + \psi(a)$ for any $\alpha \in K$ and $a \in W$. By [EM5, (3.2)], this extension belongs to $\mathrm{Aut}(C, K)$. ∎

From [EM5, Theorem 2.5] it follows that for $\varphi \in \mathrm{Aut}(C, K)$ as in Lemma 6.10, $\varphi|_K = 1$ if and only if $\varphi|_W$ is K-linear (W is a left K-module in a natural way) and $\varphi|_K$ is the conjugation in K if and only if $\varphi|_W$ is s-semilinear, with $s = \varphi|_K$ the nontrivial F-automorphism of K. Moreover, [EM5, Theorem 4.9] tells us that the group $\mathrm{Aut}(C, K)$ is the semidirect product of $\mathrm{Aut}_K C$ (the group of automorphisms of C fixing element-wise K) and a cycle group of order 2 generated by an element $\varphi \in \mathrm{Aut}(C, K)$ such that $\varphi|_K$ is the nontrivial F-automorphism of K.

The algebra $(W, *)$ in Lemma 6.10 is a "vector color algebra," which consists of the elements of trace zero of a form of the color algebra introduced in [D-KD] in connection with the Gell-Mann quark model and which has been studied by several authors [E, EM3, EM5, S2, S3].

As to the problem of isomorphism between composition algebras in Theorem 6.9, we have:

THEOREM 6.11.   *Let (C_1, K_1) and (C_2, K_2) be two pairs of Cayley−Dickson algebras and Hurwitz subalgebras of dimension* 2. *Let φ_i, ψ_i $(i = 1, 2)$ be orthogonal transformations of C_i such that $\varphi_i|W_i = \psi_i|W_i = 1$, where W_i is the orthogonal complement to K_i $(i = 1, 2)$. Define the new composition algebras $(C_i, \circ)$ by means of* (18) *and let $\rho: C_1 \to C_2$ be a linear map. Then ρ is an isomorphism between $(C_1, \circ)$ and $(C_2, \circ)$ if and only if ρ is an isomorphism between the Cayley−Dickson algebras C_1 and C_2 and $\varphi_2 = \rho \varphi_1 \rho^{-1}$ and $\psi_2 = \rho \psi_1 \rho^{-1}$.*

Proof.   In case the $(C_i, \circ)$ are standard, it is clear that any isomorphism between $(C_1, \circ)$ and $(C_2, \circ)$ is an isomorphism between C_1 and C_2, so $\rho j_1 = j_2 \rho$, where j_i is the canonical involution of C_i $(i = 1, 2)$ and the result follows in this case.

So assume that $(C_i, \circ)$ $(i = 1, 2)$ is not standard. If ρ is an isomorphism between $(C_1, \circ)$ and $(C_2, \circ)$ then, since by Theorem 6.9, $K_i = \{x \in C_i: \mathrm{Der}(C_i, \circ)x = 0\}$ $(i = 1, 2)$, we get that $\rho(K_1) = K_2$ and then also $\rho(W_1) = W_2$. Therefore, $\rho|_{W_1}$ gives an isomorphism between the algebras $(W_1, *)$ and $(W_2, *)$ defined by (19). The proof of Lemma 6.10 gives that there is a unique isomorphism $\tilde{\rho}$ between the Cayley−Dickson algebras C_1 and C_2 such that $\tilde{\rho}|_{W_1} = \rho|_{W_1}$ (see also [EM5, Theorem 3.1]). But for any $\alpha \in K_1$, there are $a, b \in W_1$ such that $ab = \alpha + a * b$. Then, $\rho(\alpha)$ is the projection on K_2 of $\rho(ab) = \rho(a \circ b) = \rho(a) \circ \rho(b) = \tilde{\rho}(a) \circ \tilde{\rho}(b) = \tilde{\rho}(a)\tilde{\rho}(b) = \tilde{\rho}(ab)$, which is $\tilde{\rho}(\alpha)$. Hence, $\rho = \tilde{\rho}$ is an isomorphism between C_1 and C_2. Now, take a nonisotropic vector $y \in W_1$, then for any $x \in C_1$, $\rho(\varphi_1(x))\rho(y) = \rho(\varphi_1(x))\rho(\psi_1(y)) = \rho(\varphi_1(x)\psi_1(y)) = \rho(x \circ y) = \rho(x) \circ \rho(y) = (\varphi_2(\rho(x))(\psi_2(\rho(y)) = \varphi_2(\rho(x))\rho(y)$. Since $\rho(y)$ is not isotropic, we conclude that $\rho \varphi_1 = \varphi_2 \rho$ and, in the same way, $\rho \psi_1 = \psi_2 \rho$, as required. The converse is clear. ∎

Now, the problem of isomorphism is solved by:

THEOREM 6.12.   *Let C be a Cayley−Dickson algebra, K a two-dimensional Hurwitz subalgebra of C, φ_1, φ_2, ψ_1, ψ_2 orthogonal transformations of C with $\varphi_i|_W = \psi_i|_W = 1$ $(i = 1, 2)$, where W is the orthogonal complement to K. Define new multiplications as in (18) by*

$$x \circ y = \varphi_1(x)\psi_1(y) \qquad and \qquad x \diamond y = \varphi_2(x)\psi_2(y).$$

Denote by $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\psi}_1$, $\hat{\psi}_2$ the restrictions to K. Then, $(C, \circ)$ is isomorphic to $(C, \diamond)$ if and only if either $(\hat{\varphi}_1, \hat{\psi}_1) = (\hat{\varphi}_2, \hat{\psi}_2)$ or $(\hat{\varphi}_1, \hat{\psi}_1) = (s\hat{\varphi}_2 s, s\hat{\psi}_2 s)$, where s is the nontrivial F-automorphism of K (the restriction of the canonical involution of C).

Proof.   For the converse, if for instance, $(\hat{\varphi}_1, \hat{\psi}_1) = (s\hat{\varphi}_2 s, s\hat{\psi}_2 s)$, we take an element $\rho \in \text{Aut}(C, K)$ such that $\rho|_K = s$ (this is always possible by the remarks after Lemma 6.10). Then, $\varphi_2 = \rho\varphi_1\rho^{-1}$, $\psi_2 = \rho\psi_1\rho^{-1}$, and for any $x, y \in C$,

$$\rho(x \circ y) = \rho(\varphi_1(x)\psi_1(y)) = (\rho\varphi_1(x))(\rho\psi_1(y))$$
$$= (\varphi_2\rho(x))(\psi_2\rho(y)) = \rho(x)\diamond\rho(y).$$

Now, if $\rho: (C, \circ) \to (C, \diamond)$ is an isomorphism and $(C, \circ)$ is standard, then so is $(C, \diamond)$ and of the same type. We necessarily have then by Theorem 6.9 that $(\varphi_1, \psi_1) = (\varphi_2, \psi_2)$, so $(\hat{\varphi}_1, \hat{\psi}_1) = (\hat{\varphi}_2, \hat{\psi}_2) = (s\hat{\varphi}_2 s, s\hat{\psi}_2 s)$ and we are done. Otherwise, by the proof of Theorem 6.11, $\rho(K) = K$, $\rho(W) = W$, and $\varphi_2 = \rho\varphi_1\rho^{-1}$, $\psi_2 = \rho\psi_1\rho^{-1}$. Since $\varphi_i|_W = \psi_i|_W = 1$ $(i = 1, 2)$ and $\rho|_K$ is either 1 or s, by restricting to K we obtain either $(\hat{\varphi}_1, \hat{\psi}_1) = (\hat{\varphi}_2, \hat{\psi}_2)$ or $(\hat{\varphi}_1, \hat{\psi}_1) = (s\hat{\varphi}_2 s, s\hat{\psi}_2 s)$.   ∎

COROLLARY 6.13.   *Assume that F is algebraically closed and C is the Cayley−Dickson algebra over F. Let $K = F \oplus F$ be a two-dimensional Hurwitz subalgebra of C. Then, for each type I−IV, there are infinite classes of isomorphism of composition algebras constructed as in Theorem 6.9 from the pair (C, K).*

Proof.   From Theorem 6.12, it is enough to notice that the special orthogonal group of K is the group of diagonal matrices $\{\text{diag}(\alpha, \alpha^{-1}): 0 \neq \alpha \in F\}$, which is infinite.   ∎

In [P2], it is proved that there are infinite classes of isomorphism of composition algebras. The corollary above shows that this is indeed the case even if we assume a large derivation algebra.

In conclusion, notice that we have proved that the class of eight-dimensional composition algebras with derivation algebra as large as the deriva-

tion algebra of the flexible composition algebras consists of the standard composition algebras, the Okubo algebras, and some algebras constructed from pairs (C, K) as in Theorem 6.9.

The number of possibilities for the derivation algebra of an eight-dimensional composition algebra is very large and we have only touched upon the toral rank two case. Much more information is contained in [Pé].

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