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On the Tensor Product of Composition Algebras

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1. INTRODUCTION

Let $C_1 \otimes_F C_2$ be the tensor product of two composition algebras over a field F with $\mathrm{char}(F) \neq 2$. Brauer [8] and Albert [1–3] seemed to be the first mathematicians who investigated the tensor product of two quaternion algebras. Later their results were generalized to this more general

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situation by Allison [4–6] and to biquaternion algebras over rings by Knus [20].

In Section 2 we give some new results on the Albert form of these algebras. We also investigate the F-quadric defined by this Albert form, generalizing a result of Knus [21].

Since Allison regarded the involution $\sigma = \gamma_1 \otimes \gamma_2$ as an essential part of the algebra $C = C_1 \otimes_F C_2$, he only studied automorphisms of C which are compatible with σ. In Section 3 we determine, if $\operatorname{char}(F) \neq 2$, the automorphism group of a tensor product of octonion algebras. We also show that any automorphism of such a tensor product is compatible with the canonical tensor product involution. As a consequence, we determine the forms of a tensor product of octonion algebras. Furthermore, we show that any such algebra does not satisfy the Skolem–Noether Theorem.

Our results of Section 3 arise from a study of the generalized alternative nucleus of an algebra, since a tensor product of octonion algebras is generated by its generalized alternative nucleus. In Section 4, using Lie algebra-theoretic techniques, we classify finite dimensional simple unital algebras over an algebraically closed field of characteristic 0 which are generated by their generalized alternative nucleus, proving that such an algebra is the tensor product of a simple associative algebra and a symmetric tensor product of octonion algebras. This result is used in Section 5 to sketch a variation of the Allison–Smirnov proof of the classification of finite dimensional central simple structurable algebras over a field of characteristic 0.

Finally, in Section 6, we prove that if A is generated by its generalized alternative nucleus, then the associated bilinear form $(x, y) = \operatorname{trace}(L_x L_y)$ is associative.

Let F be a field and C a unital, nonassociative F-algebra. Then S is a *composition algebra* if there exists a nondegenerate quadratic form $n\colon C \to F$ such that $n(x \cdot y) = n(x)n(y)$ for all $x, y \in C$. The form n is uniquely determined by these conditions and is called the *norm* of C. We will write $n = n_C$. Composition algebras only exist in rank 1, 2, 4, or 8 (see [17]). Those of rank 4 are called *quaternion algebras* and those of rank 8 *octonion algebras*. A composition algebra C has a *canonical involution* γ given by $\gamma(x) = t(x)1_C - x$, where the *trace* map $t\colon C \to F$ is given by $t(x) = n(1, x)$.

An example of an eight-dimensional composition algebra is Zorn's algebra of vector matrices $\operatorname{Zor}(F)$ (see [22, p. 507] for the definition). The norm form of $\operatorname{Zor}(F)$ is given by the determinant and is a hyperbolic form.

Composition algebras are *quadratic*. That is, they satisfy the identities

$$x^2 - t(x)x + n(x)1_C = 0 \qquad \text{for all } x \in C,$$

$$n(1_C) = 1$$

and are *alternative* algebras; i.e., $xy^2 = (xy)y$ and $x^2y = x(xy)$ for all $x, y \in C$. In particular, $n(x) = \gamma(x)x = x\gamma(x)$ and $t(x)1_C = \gamma(x) + x$.

For any composition algebra D over F with $\dim_F(D) \leq 4$ and any $\mu \in F^\times$, the F-vector space $D \oplus D$ becomes a composition algebra via the multiplication

$$(u,v)(u',v') = (uu' + \mu\gamma(v')v, v'u + v\gamma(u'))$$

for all $u, v, u', v' \in D$, with norm

$$n((u,v)) = n_D(u) - \mu n_D(v).$$

This algebra is denoted by $\mathrm{Cay}(D, \mu)$. Note that the embedding of D into the first summand of $\mathrm{Cay}(D, \mu)$ is an algebra monomorphism. The norm form of $\mathrm{Cay}(D, \mu)$ is obviously isometric to $\langle 1, -\mu\rangle \otimes n_D$. Since two composition algebras are isomorphic if and only if their norm forms are isometric, we see that if C is a composition algebra whose norm form satisfies $n_C \cong \langle 1, -\mu\rangle \otimes n_D$ for some D then $C \cong \mathrm{Cay}(D, \mu)$. In particular, $\mathrm{Zor}(F) \cong \mathrm{Cay}(D, 1)$ for any quaternion algebra D since $\langle 1, -1\rangle \otimes n_D$ is hyperbolic. A composition algebra is *split* if it contains an isomorphic copy of $F \oplus F$ as a composition subalgebra, which is the case if and only if it contains zero divisors. Over algebraically closed fields any composition algebra of dimension ≥ 2 is split.

2. ALBERT FORMS

From now on we consider only the fields F with $\mathrm{char}(F) \neq 2$ unless stated otherwise. It is well known that any norm of a composition algebra is a 3-fold Pfister form, and conversely any 3-fold Pfister form is the norm of some composition algebra.

Let C be a composition algebra. Define $C' = \langle F1\rangle^\perp = \{x \in C : t(x) = n(x, 1) = 0\}$. Then $n' = n|_{C'}$ is the *pure norm* of C. Note that

$$\begin{aligned} C' &= \left\{x \in C : x = 0 \text{ or } x \notin F1_C \text{ and } x^2 \in F1_C\right\} \\ &= \{x \in C : \gamma(x) = -x\}. \end{aligned}$$

Moreover, C is split if and only if its norm n is hyperbolic, two composition algebras are isomorphic if and only if their norms are isometric, and C is a division algebra if and only if n is anisotropic.

We first investigate tensor products of two composition algebras. Following Albert, we associate to the tensor product $C = C_1 \otimes_F C_2$ of two composition algebras with $\dim(C_i) = r_i$ and $n_{C_i} = n_i$ the $(r_1 + r_2 - 2)$-dimensional form $n_1' \perp \langle -1\rangle n_2'$ of determinant -1. This definition, for

C_1 or C_2 an octonion algebra, was first given by Allison in [5]. In the Witt ring $W(F)$, obviously this form is equivalent to $n_1 - n_2$. Like the norm form of a composition algebra, this *Albert form* contains crucial information about the tensor product algebra C. For biquaternion algebras, this is well-known [1, Theorem 3; 19, Theorem 3.12]. We introduce some notation and terminology. If q is a quadratic form and if $\mathbb{H} = \langle 1, -1 \rangle$ is the hyperbolic plane, then $q = q_0 \perp i\mathbb{H}$ for some anisotropic form q_0 and integer i. The integer i is called the *Witt index* of q and is denoted by $i_W(q)$. In the proof of the following proposition, we use the notion of linkage of Pfister forms (see [12, Section 4]). Recall that two n-fold Pfister forms q_1 and q_2 are *r-linked* if there is an r-fold Pfister form h with $q_1 = h \otimes q_i'$ for some Pfister forms q_i'. Finally, we call a two-dimensional commutative F-algebra that is separable over F a *quadratic étale algebra*. Note that any quadratic étale algebra either is a quadratic field extension of F or is isomorphic to $F \oplus F$. Part of the following result has been proved in [15, Theorem 5.1].

PROPOSITION 2.1. *Let C_1 and C_2 be octonion algebras over F with norms n_1 and n_2, and let $i = i_W(N)$ be the Witt index of the Albert form $N = n_1' \perp \langle -1 \rangle n_2'$.*

(i) $i = 0 \Leftrightarrow C_1$ *and* C_2 *do not contain isomorphic quadratic étale subalgebras.*

(ii) $i = 1 \Leftrightarrow C_1$ *and* C_2 *contain isomorphic quadratic étale subalgebras, but no isomorphic quaternion subalgebras.*

(iii) $i = 3 \Leftrightarrow C_1$ *and* C_2 *contain isomorphic quaternion subalgebras, but* C_1 *and* C_2 *are not isomorphic.*

(iv) $i = 7 \Leftrightarrow C_1 \cong C_2$.

Proof. By [12, Propositions 4.4 and 4.5], the Witt index of $n_1 \perp \langle -1 \rangle n_2$ is 2^r, where r is the linkage number of $n_1 \perp \langle -1 \rangle n_2$. Note that the Witt index of N is one less than the Witt index of $n_1 \perp \langle -1 \rangle n_2$ since $n_1 \perp \langle -1 \rangle n_2 = \mathbb{H} \perp N$. If $C_1 \cong C_2$, then $n_1 \cong n_2$, so $i = 7$. Conversely, if $i = 7$, then $n_1 \perp \langle -1 \rangle n_2$ is hyperbolic, so $n_1 \cong n_2$, which forces $C_1 \cong C_2$. If C_1 and C_2 are not isomorphic but contain a common quaternion algebra Q, then $C_i = \mathrm{Cay}(Q, \mu_i)$ for some i. Therefore, $n_1 = n_Q \otimes \langle 1, -\mu_1 \rangle$ and $n_2 = n_Q \otimes \langle 1, \mu_2 \rangle$. These descriptions show that n_1 and n_2 are 2-linked, so $i = 3$. Conversely, if $i = 3$, then n_1 and n_2 are 2-linked but not isometric. If $\langle\langle a, b \rangle\rangle$ is a factor of both n_1 and n_2, then $n_1 = \langle\langle a, b, c \rangle\rangle$ and $n_2 = \langle\langle a, b, d \rangle\rangle$ for some $c, d \in F^\times$. If $Q = (-a, -b)$, we get $C_1 = \mathrm{Cay}(Q, -c)$ and $\mathrm{Cay}(Q, -d)$, so C_1 and C_2 contain a common quaternion algebra. If C_1 and C_2 contain a common quadratic étale algebra $F[t]/(t^2 - a)$ but no common quaternion algebra, then $\langle 1, -a\rangle$ is a factor of n_1 and n_2, which means they are 1-linked. If n_1 and n_2 are 2-linked, then the previous step shows that C_1 and C_2 have a common quaternion subalgebra, which is false. Conversely, if n_1 and n_2 are 1-linked but not 2-linked, then C_1 and C_2 do not have a common quaternion subalgebra, and if $\langle\langle a\rangle\rangle$ is a common factor to n_1 and n_2, then C_1 and C_2 both contain the étale algebra $F[t]/(t^2 - a)$. ∎

PROPOSITION 2.2. *Let C_1 be an octonion algebra over F and C_2 be a quaternion algebra over F, with norms n_1 and n_2. Again consider the Witt index i of the Albert form $N = n_1' \perp \langle -1\rangle n_2'$.*

(i) $i = 0 \Leftrightarrow C_1$ *and* C_2 *do not contain isomorphic quadratic étale subalgebras.*

(ii) $i = 1 \Leftrightarrow C_1$ *and* C_2 *contain isomorphic quadratic étale subalgebras, but* C_2 *is not a quaternion subalgebra of* C_1.

(iii) $i = 3 \Leftrightarrow C_1 \cong \mathrm{Cay}(C_2, \mu)$ *for a suitable* $\mu \in F^\times$ *and* C_2 *is a division algebra.*

(iv) $i = 5 \Leftrightarrow C_1 \cong \mathrm{Zor}(F)$ *and* $C_2 \cong M_2(F)$.

Proof. In the case that both algebras C_1 and C_2 are division algebras, this is an immediate consequence of [15, Lemma 3.2]. If both C_1 and C_2 are split, then clearly N has Witt index 5. If C_2 is a division algebra and $C_1 = \mathrm{Cay}(C_2, \mu)$ for some μ, then n_2 is anisotropic and $N \perp \mathbb{H} = n_2 \otimes \langle 1, -\mu\rangle \perp \langle -1\rangle n_2 = 4\mathbb{H} \perp \langle -\mu\rangle n_2$, so N has Witt index 3. Note that the converse is easy, since if $i = 3$ then n_2 is isomorphic to a subform of n_1, which forces n_2 to be a factor of n_1. If $n_1 = \langle 1, a\rangle \otimes n_2$, then $C_1 \cong \mathrm{Cay}(C_2, -a)$, so C_2 is a subalgebra of C_1. If C_1 and C_2 contain a common quadratic étale algebra $F[t]/(t^2 - a)$ but C_2 is not a quaternion subalgebra of C_1, then n_1 and n_2 have $\langle\langle a\rangle\rangle$ as a common factor, so $i = 1$. Finally, if N is isotropic, there are $x_i \in C_i$, both skew, with $n_1(x_1) = n_2(x_2)$. Then, as $t_1(x_1) = 0 = t_2(x_2)$, the algebras $F[x_1]$ and $F[x_2]$ are isomorphic, so C_1 and C_2 share a common quadratic étale subalgebra. This finishes the proof. ∎

If C is a biquaternion algebra (i.e., $C \cong C_1 \otimes_F C_2$ for two quaternion algebras C_1 and C_2), then the Albert form $n_1' \perp \langle -1\rangle n_2'$ is determined up to similarity by the isomorphism class of the algebra C [19, Theorem 3.12]. Allison generalizes this result [5, Theorem 5.4] to tensor products of arbitrary composition algebras. However, he always considers the involution $\sigma = \gamma_1 \otimes \gamma_2$ as a crucial part of the algebra $C = C_1 \otimes_F C_2$. Allison proves that $(C_1 \otimes_F C_2, \gamma_1 \otimes \gamma_2)$ and $(C_3 \otimes_F C_4, \gamma_3 \otimes \gamma_4)$ are isotropic algebras if and only if they have similar Albert forms, for the cases that C_1,C_3 are octonion and C_2, C_4 are quaternion or octonion algebras.

The fact that any F-algebra isomorphism $\varphi\colon (C_1 \otimes_F C_2, \gamma_1 \otimes \gamma_2) \to (C_3 \otimes_F C_4, \gamma_3 \otimes \gamma_4)$ between arbitrary products of composition algebras yields an isometry $n_1' \perp \langle -1\rangle n_2' \cong \mu(n_3' \perp \langle -1\rangle n_4')$ for a suitable $\mu \in F^\times$ is easy to see. Also, since for $C = C_1 \otimes_F C_2$ the map $\langle\ ,\ \rangle\colon C \times C \to F$ given by $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = n_1(x_1, y_1) \otimes n_2(x_2, y_2)$ is a nondegenerate symmetric bilinear form on C such that $\langle \sigma(x), \sigma(y)\rangle = \langle x, y\rangle$, the equation $\langle zx, y\rangle = \langle x, \sigma(z)y\rangle$ holds (that is, $\langle\ ,\ \rangle$ is an *invariant form*; cf. [6, p. 144] or Section 6 below) and $\tau\colon C \times C \to k$, $\tau(x, y) = \langle x, \sigma(y)\rangle$ is an associative nondegenerate symmetric bilinear form which is proper, it follows easily that $n_1 \otimes n_2 \cong n_3 \otimes n_4$.

Suppose that we have two algebras that each are a tensor product of an octonion algebra and a quaternion algebra. We obtain a necessary and sufficient condition for when their Albert forms are similar. We use the notation $D(q)$ to denote the elements of $F^\times$ represented by a quadratic form q.

THEOREM 2.3. *Let C_1, C_2 be octonion algebras and Q_1, Q_2 quaternion algebras over F. Let N_1 and N_2 be the Albert forms of $C_1 \otimes_F Q_1$ and $C_2 \otimes_F Q_2$, respectively. If $N_1 \cong \mu N_2$ for some $\mu \in F^\times$, then $Q_1 \cong Q_2$. Moreover, there is a quaternion algebra Q and elements $c, d \in F^\times$ such that $C_1 \cong \mathrm{Cay}(Q, c)$, $C_2 \cong \mathrm{Cay}(Q, d)$, $\mathrm{Cay}(Q_1, \mu) \cong \mathrm{Cay}(Q, cd)$, and $-\mu c \in D(n_{C_2})$. Conversely, if there is a quaternion algebra Q and elements $c, d, \mu \in F^\times$ such that C_1, $Q_1 = Q_2$, and C_2 satisfy the conditions of the previous sentence, then $N_1 \cong \mu N_2$.*

Proof. Suppose that $N_1 \cong \mu N_2$ for some $\mu \in F^\times$. If $c\colon W(F) \to \mathrm{Br}(F)$ is the Clifford invariant, then $c(N_1) = c(\mu N_2) = c(N_2)$. Since c is trivial on $I^3(F)$, we have $c(N_1) = c(-n_{Q_1})$ and $c(N_2) = c(-n_{Q_2})$ (see [23, Chap. 5.3]). Therefore, $c(n_{Q_1}) = c(n_{Q_2})$. However, the Clifford invariant of the norm form of a quaternion algebra is the class of the quaternion algebra, by [23, Corollary V.3.3]. This implies that $c(N_1) = [Q_1]$ and $c(\mu N_2) = [Q_2]$. Since $[Q_1] = [Q_2]$, we get $Q_1 \cong Q_2$. As a consequence of this, $n_{Q_1} \cong n_{Q_2}$. Thus,

$$\begin{aligned} n_{C_1} &\cong -\langle 1, -\mu\rangle \otimes n_{Q_1} \cong n_{C_1} \perp \left(-n_{Q_1} \perp \mu n_{Q_1}\right) \\ &\cong \mu\left(n_{C_2} \perp -n_{Q_2}\right) \perp \mu n_{Q_1} \cong \mu n_{C_2} \perp \left(\mu n_{Q_1} \perp -\mu n_{Q_2}\right) \\ &\cong \mu n_{C_2} \perp 4\mathbb{H}. \end{aligned}$$

The forms n_{C_1} and $\langle 1, -\mu\rangle \otimes n_{Q_1}$ are Pfister forms. The line above shows that these Pfister forms are 2-linked, in the terminology of [12]. Therefore, there is a 2-fold Pfister form $\langle\langle -a, -b\rangle\rangle$ with $n_{C_1} \cong \langle\langle -a, -b, -c\rangle\rangle$ and $\langle 1, -\mu\rangle \otimes n_{Q_1} \cong \langle\langle -a, -b, -e\rangle\rangle$ for some $c, e \in F^\times$. An elementary calculation shows that

$$\langle\langle -a, -b, -c\rangle\rangle \perp -\langle\langle -a, -b, -e\rangle\rangle \cong 4\mathbb{H} \perp \langle -c, e\rangle \otimes \langle\langle -a, -b\rangle\rangle.$$

Therefore, $\mu n_{C_2} \cong \langle -c, e\rangle \otimes \langle\langle -a, -b\rangle\rangle$. Thus, $n_{C_2} \cong -\mu c\langle\langle -a, -b, -ce\rangle\rangle$. Since n_{C_2} and $\langle\langle -a, -b, -ce\rangle\rangle$ are Pfister forms, we get $n_{C_2} \cong \langle\langle -a, -b, -ce\rangle\rangle$. If we set $d = ce$ and let Q be the quaternion algebra $(a, b)_F$, then the isomorphisms $n_{C_1} \cong \langle\langle -a, -b, -c\rangle\rangle$ and $n_{C_2} \cong \langle\langle -a, -b, -d\rangle\rangle$ give $C_1 \cong \mathrm{Cay}(Q, c)$ and $C_2 \cong \mathrm{Cay}(Q, d)$. Moreover, $n_{C_2} \cong -\mu c n_{C_2}$, so $-\mu c \in D(n_{C_2})$. Finally, the isomorphism $\langle 1, -\mu\rangle \otimes n_{Q_2} \cong \langle\langle -a, -b, -e\rangle\rangle$ gives $\mathrm{Cay}(Q_1, \mu) \cong \mathrm{Cay}(Q, e) \cong \mathrm{Cay}(Q, cd)$.

It is a short calculation to show that if $C_1 = \mathrm{Cay}(Q, c)$, $C_2 = \mathrm{Cay}(Q, d)$, and $Q_1 = Q_2$ is a quaternion algebra with $\mathrm{Cay}(Q_1, \mu) \cong \mathrm{Cay}(Q, -dc)$, then $n'_{C_1} \perp \langle -1\rangle n'_{Q_1} \cong \mu(n'_{C_2} \perp \langle -1\rangle n'_{Q_2})$. ∎

The argument of the previous theorem does not work for a tensor product of two octonion algebras since the Albert form is an element of $I^3(F)$, whose Clifford invariant is trivial.

COROLLARY 2.4. *With the notation in the previous theorem, suppose that* $N_1 \cong \mu N_2$ *for some* $\mu \in F^\times$. *If one of* C_1 *and* C_2 *is split, then the other algebra is isomorphic to* $\mathrm{Cay}(Q_1, \mu)$.

Proof. We saw in the proof of the previous proposition that

$$n_{C_1} \perp -\langle 1, -\mu\rangle \otimes n_{Q_1} \cong \mu n_{C_2} \perp 4\mathbb{H}.$$

Suppose that C_2 is split. Then $n_{C_1} \perp -\langle 1, -\mu\rangle \otimes n_{Q_1}$ is hyperbolic, so $n_{C_1} \cong \langle 1, -\mu\rangle \otimes n_{Q_2}$. Therefore, $C_1 \cong \mathrm{Cay}(Q_1, \mu)$. On the other hand, if C_1 is split, then $n_{C_1} \cong 4\mathbb{H}$, so by Witt cancellation $-\mu n_{C_2} \cong \langle 1, -\mu\rangle \otimes n_{Q_1}$. Since n_{C_2} and $\langle 1, -\mu\rangle \otimes n_{Q_1}$ are both Pfister forms, this implies that $n_{C_2} \cong \langle 1, -\mu\rangle \otimes n_{Q_1}$, and so $C_2 \cong \mathrm{Cay}(Q_1, \mu)$. ∎

In Theorem 2.3 above, it is possible for $N_1 \cong \mu N_2$ without $C_1 \cong C_2$. Moreover, the quaternion algebra Q of the proposition need not be isomorphic to Q_1. We verify both of these claims in the following example.

EXAMPLE 2.5. In this example we produce nonisomorphic octonion algebras C_1 and C_2 and a quaternion algebra Q_1 that is not isomorphic to a subalgebra of either C_1 or C_2 and is such that the Albert forms of $C_1 \otimes_F Q_1$ and $C_2 \otimes_F Q_1$ are similar. To do this we produce nonisometric Pfister forms $\langle\langle x, y, z\rangle\rangle$ and $\langle\langle x, y, w\rangle\rangle$ and elements u, v, μ with $\langle\langle x, y, zw\rangle\rangle \cong \langle\langle u, v, \mu\rangle\rangle$ such that the Witt indexes of $\langle\langle x, y, z\rangle\rangle \perp -\langle\langle u, v, \mu\rangle\rangle$ and $\langle\langle x, y, w\rangle\rangle \perp -\langle\langle u, v, \mu\rangle\rangle$ are both 2 and $\mu z \in D(\langle\langle x, y, w\rangle\rangle)$. We then set $Q = (-x, -y)$, $C_1 = \mathrm{Cay}(Q, -z)$, $C_2 =$

$\mathrm{Cay}(Q, -w)$, and $Q_1 = (-u, -v)$. From Theorem 2.3, we have $N_1 \cong \mu N_2$. However, Proposition 2.2 shows that Q_1 is not isomorphic to a subalgebra of either C_1 or C_2. Moreover, C_1 and C_2 are not isomorphic since their norm forms are not isometric. Note that Q and C_2 are not isomorphic since Q_1 is not a subalgebra of C_1.

Let k be a field of characteristic not 2, and let $F = k(x, y, z, w)$ be the rational function field in four variables over k. Set $\mu = xyzw$, $n_1 = \langle\langle x, y, z\rangle\rangle$, and $n_2 = \langle\langle x, y, w\rangle\rangle$. By embedding F in the Laurent series field $k(x, y, z)((w))$, we see that n_1 and $n_2 = \langle\langle x, y\rangle\rangle \perp w\langle\langle x, y\rangle\rangle$ are not isomorphic over this field by Springer's theorem [23, Proposition VI.1.9], so n_1 and n_2 are not isomorphic over F. Also, $\mu z = z^2(xyw)$, which is clearly represented by n_2. Set $Q_1 = (-zw, -xzw)$. A short calculation shows that $\langle\langle x, y, zw\rangle\rangle = \langle\langle zw, xzw, \mu\rangle\rangle$. Finally, for the Witt indices, we have

$$\begin{aligned} n_1 \perp -n_{Q_1} &= \langle 1, x, y, xy, z, xz, yz, xyz\rangle \perp -\langle 1, zw, xzw, x\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, z, xz, yz, xyz, -zw, -xzw\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, z, xz, yz, xyz\rangle \perp w\langle -z, -xz\rangle. \end{aligned}$$

The Springer theorem shows that this form has Witt index 2. Similarly,

$$\begin{aligned} n_2 \perp -n_{Q_1} &= \langle 1, x, y, xy, w, xw, yw, xyw\rangle \perp -\langle 1, zw, xzw, x\rangle \\ &= 2\mathbb{H} \perp \langle y, xy, w, xw, yw, xyw, -zw, -xzw\rangle \\ &= 2\mathbb{H} \perp \langle y, xy\rangle \perp w\langle 1, x, y, xy, -z, -xz\rangle \end{aligned}$$

has Witt index 2.

For the remainder of this section we will also consider the case that $\mathrm{char}(F) = 2$. Let C_1 and C_2 be composition algebras over F of $\dim_F(C_i) = r_i \geq 2$, and let n_i be the norm form of C_i. Using the notation of [21], the subspace $Q(C_1, C_2) = \{u = x_1 \otimes 1 - 1 \otimes x_2 : t_1(x_1) = t_2(x_2)\}$ has dimension $r_1 + r_2 - 2$, and $Q(C_1, C_2) = \{z - (\gamma_1 \otimes \gamma_2)(z) : z \in C_1 \otimes_F C_2\}$ is the set of alternating elements of $C_1 \otimes_F C_2$ with respect to $\gamma_1 \otimes \gamma_2$. The nondegenerate quadratic form N: $Q(C_1, C_2) \to F$ given by $N(x_1 \otimes 1 - 1 \otimes x_2) = n_1(x_1) - n_2(x_2)$ is isometric to the Albert form $n_1' \perp \langle -1\rangle n_2'$ of $C_1 \otimes_F C_2$.

Let $V_N \subset \mathbb{P}^{r_1 + r_2 - 3}$ be the F-quadric defined via N. In the case that $\mathrm{char}(F) \neq 2$, V_N coincides with the open subvariety U_N of closed points $x_1 \otimes 1 - 1 \otimes x_2$ with $x_1 \notin F1$ and $x_2 \notin F1$. We now generalize [21, Proposition] in the following two propositions. We will make use of the following fact that comes from Galois theory: Let $F[z_i]$ be the commutative F-subalgebra of dimension two of C_i generated by $z_i \in C_i$ for $i = 1, 2$.

Then there exists an isomorphism $\alpha\colon F[z_1] \xrightarrow{\sim} F[z_2\}$ such that $\alpha(z_1) = z_2$ if and only if $n_1(z_1) = n_2(z_2)$ and $t_1(z_1) = t_2(z_2)$.

PROPOSITION 2.6. *There exists a bijection Φ between the set of F-rational points of U_N and the set of triples (K_1, K_2, α), where K_i is a two-dimensional commutative subalgebra of C_i and where $\alpha\colon K_1 \xrightarrow{\sim} K_2$ is an F-algebra isomorphism,*

$$\Phi\colon \{P \in U_N : P \text{ an } F\text{-rational point}\} \xrightarrow{\sim} \{(K_1, K_2, \alpha) : K_1, K_2, \alpha \text{ as above}\}$$

$$P = z_1 \otimes 1 - 1 \otimes z_2 \mapsto \left(F[z_1], F[z_2], \begin{array}{c} \alpha\colon F[z_1] \xrightarrow{\sim} F[z_2] \\ z_1 \mapsto z_2 \end{array} \right).$$

Proof. Any F-rational point $P \in U_N$ corresponds with an element $x_1 \otimes 1 - 1 \otimes x_2 \in Q(C_1, C_2)$ with $t_1(x_1) = t_2(x_2)$ and $n_1(x_1) = n_2(x_2)$. Then there exists an F-algebra isomorphism $\alpha\colon F[x_1] \xrightarrow{\sim} F[x_2]$ with $x_1 \mapsto x_2$. For $x_1 \otimes 1 - 1 \otimes x_2 = z_1 \otimes 1 - 1 \otimes z_2$ it can easily be verified that

$$\left(F[x_1], F[x_2], \begin{array}{c} \alpha\colon F[x_1] \xrightarrow{\sim} F[x_2] \\ x_1 \mapsto x_2 \end{array} \right)$$

$$= \left(F[z_1], F[z_2], \begin{array}{c} \beta\colon F[z_1] \xrightarrow{\sim} F[z_2] \\ z_1 \mapsto z_2 \end{array} \right).$$

Therefore, the mapping Φ is well defined.

Given a triple (K_1, K_2, α), there are elements $z_i \in C_i'$ such that $K_i = F[z_i]$ and $\alpha\colon F[z_1] \xrightarrow{\sim} F[z_2]$ with $z_1 \mapsto z_2$. By the remark before the proposition, we have $n_1(z_1) = n_2(z_2)$ and $t_1(z_1) = t_2(z_2)$; thus $N(z_1 \otimes 1 - 1 \otimes z_2) = 0$ and the triple defines the F-rational point $P \in U_N$ corresponding to $z_1 \otimes 1 - 1 \otimes z_2$. So Φ is surjective.

To prove injectivity, suppose that $\Phi(x_1 \otimes 1 - 1 \otimes x_2) = \Phi(z_1 \otimes 1 - 1 \otimes z_2)$. Then $F[x_1] = F[z_1]$, $F[x_2] = F[z_2]$, and the maps $\alpha\colon F[x_1] \xrightarrow{\sim} F[x_2]$, $x_1 \mapsto x_2$, and $\beta\colon F[z_1] \xrightarrow{\sim} F[z_2]$, $z \mapsto z_2$, are equal. Since $F[x_i] = F[z_i]$, we write $x_1 = a + bz_1$ and $x_2 = c + dz_2$ with $a, b, c, d \in F$. We have $a = c$ since $t_1(x_1) = t_2(x_2)$. Therefore, we may replace x_1 with bz_1 and x_2 with dz_2 without changing $x_1 \otimes 1 - 1 \otimes x_2$. Thus, $x_1 \otimes 1 - 1 \otimes x_2 = bz_1 \otimes 1 - 1 \otimes dz_2$, and $n_1(x_1) = n_2(x_2)$, $n_1(z_1) = n_2(z_2)$ imply that $n_1(x_1) = b^2 n_1(z_1)$ and $n_2(x_2) = d^2 n_2(z_2)$. Therefore, $b^2 = d^2$, so $b = \pm d$. Now $x_2 = \alpha(x_1) = \alpha(bz_1) = bz_2$ yields $b = d$, and we get $x_1 \otimes 1 - 1 \otimes x_2 = b(z_1 \otimes 1 - 1 \otimes z_2)$ which shows that Φ is injective.

In the case that $\operatorname{char}(F) = 2$, the set $U_N = \{x_1 \otimes 1 - 1 \otimes x_2 : x_1 \notin F1, x_2 \notin F1\}$ is a proper open subvariety of V_N. The proof of the previous proposition shows that Φ again is a bijection between the F-rational points of U_N and the triples (K_1, K_2, α), where K_i is a two-dimensional commutative F-subalgebra of C_i and $\alpha\colon K \xrightarrow{\sim} L$ is an F-algebra isomorphism. We can say more in this situation.

PROPOSITION 2.7. *Let* $\operatorname{char}(F) = 2$. *There exists an F-rational point in V_N if and only if there exists a triple (K_1, K_2, α) such that K_i is a quadratic étale subalgebra of C_i and $\alpha\colon K_1 \xrightarrow{\sim} K_2$ is an F-algebra isomorphism. In addition, there exists an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$ if and only if there exists a triple (K_1, K_2, α) such that K_1 and K_2 are purely inseparable quadratic extensions and $\alpha\colon K_1 \xrightarrow{\sim} K_2$ is an F-algebra isomorphism.*

Proof. As pointed out before the proposition, there is a bijection between F-rational points in U_N and triples (K_1, K_2, α) with $K_i \subseteq C_i$ commutative subalgebras of dimension 2 over F. To prove the first statement, only one half needs further argument. Suppose V_N has an F-rational point. Since V_N is a quadric hypersurface, V_N is then birationally equivalent to $\mathbb{P}^{r_1+r_2-3}$. The F-rational points of projective space are dense, so there is an F-rational point in U_N. Therefore, we get a triple (K_1, K_2, α) with K_i a quadratic étale subalgebra of C_i.

For the second statement, an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$ corresponds with an element $x_1 \otimes 1 - 1 \otimes x_2$ such that $n_1(x_1) = n_2(x_2)$ and $t_1(x_1) = t_2(x_2) = 0$, so $F[x_i]$ is a purely inseparable extension. There exists an isomorphism $\alpha\colon F[x_1] \xrightarrow{\sim} F[x_2]$ with $\alpha(x_1) = x_2$ and thus a triple $(F[x_1], F[x_2], \alpha)$. Conversely, if there is a triple (K_1, K_2, α) with K_i purely inseparable, there are $x_i \in C_i$ with $t_i(x_i) = 0$ such that $K = F[x_1]$, $L = F[x_2]$, and $\alpha\colon F[x_1] \xrightarrow{\sim} F[x_2]$, $x_1 \mapsto x_2$, so $n_1(x_1) = n_2(x_2)$ and $x_1 \otimes 1 - 1 \otimes x_2$ defines an F-rational point in $V_N \cap \{t_1(x_1) = 0\}$.

3. THE AUTOMORPHISM GROUP OF A TENSOR PRODUCT OF OCTONION ALGEBRAS

In this section we compute the automorphism group, the derivation algebra, and the forms of a tensor product of a finite number of octonion algebras over a field F with $\operatorname{char}(F) \neq 2$. Let $C = C_1 \otimes_F \cdots \otimes_F C_n$ be the tensor product of octonion algebras. As we will see, the subspace $C_1 \otimes_F F \otimes_F \cdots \otimes_F F + \cdots + F \otimes_F \cdots \otimes_F F \otimes_F C_n$ is responsible for many properties of C, so our first goal is to characterize it.

Recall that the *associative nucleus* of an algebra A is $\mathrm{N}(A) = \{a \in A : (a, A, A) = (A, a, A) = (A, A, a) = 0\}$, where $(x, y, z) = (xy)z - x(yz)$ denotes the usual associator. The *commutative nucleus* of A is the set $\mathrm{K}(A) = \{a \in A : [(a, A] = 0\}$.

DEFINITION 3.1. The subspace

$$\mathrm{N}_{\mathrm{alt}}(A) := \{a \in A : (a, x, y) = -(x, a, y) = (x, y, a)\ \forall x, y \in A\}$$

will be called the generalized alternative nucleus of A.

Remark 3.2. The *alternative nucleus* was introduced by Thedy [35] as

$$\{a \in A : (x, a, x) = 0 \text{ and } (a, x, y) = (x, y, a) = (y, a, x) \text{ for all } x, y \in A\}$$

and is a subalgebra of A. The generalized alternative nucleus differs from this nucleus and, in general, it may not be closed under products, although it possesses an interesting algebraic structure (see Proposition 4.6).

PROPOSITION 3.3. *Let A_1, A_2 be unital algebras with $\mathrm{N}(A_1) = F = \mathrm{K}(A_2)$ or $\mathrm{N}(A_2) = F = \mathrm{K}(A_1)$. Then $\mathrm{N}_{\mathrm{alt}}(A_1 \otimes_F A_2) = \mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F F + F \otimes_F \mathrm{N}_{\mathrm{alt}}(A_2)$.*

Proof. By symmetry we can assume that $\mathrm{N}(A_1) = F = \mathrm{K}(A_2)$. Let $a = \sum a_i \otimes a'_i \in \mathrm{N}_{\mathrm{alt}}(A_1 \otimes_F A_2)$ with a'_i linearly independent. The identities defining the generalized alternative nucleus with x replaced with $x \otimes 1$ and y with $y = y \otimes 1$ show that $a_i \in \mathrm{N}_{\mathrm{alt}}(A_1)$. A similar argument with a_i linearly independent (in $\mathrm{N}_{\mathrm{alt}}(A_1)$) shows that $\mathrm{N}_{\mathrm{alt}}(A_1 \otimes_F A_2) \subseteq \mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F \mathrm{N}_{\mathrm{alt}}(A_2)$. Now the identity $(a, x, y) = -(x, a, y)$ with x replaced with $x \otimes x'$ and y with $y \otimes 1$ leads to $\sum (a_i, x, y) \otimes a'_i x' = -\sum (x, a_i, y) \otimes x' a'_i$. Since $a_i \in \mathrm{N}_{\mathrm{alt}}(A_1)$, this implies that $[\sum (a_i, x, y) \otimes a'_i, 1 \otimes x'] = 0$. But, by hypothesis, the centralizer of A_2 in $A_1 \otimes_F A_2$ is $A_1 \otimes_F F$, hence $\sum (a_i, x, y) \otimes a'_i \in A_1 \otimes_F F$. By choosing $a'_1 = 1$ and a'_i linearly independent, we get that $(a_i, x, y) = 0$ if $i \geq 2$, and since $a_i \in \mathrm{N}_{\mathrm{alt}}(A_1)$, it follows that $(x, a_i, y) = 0 = (x, y, a_i)$, too. Therefore, $a_i \in \mathrm{N}(A_1) = F$ if $i \geq 2$ and $\mathrm{N}_{\mathrm{alt}}(A_1 \otimes_F A_2) \subset \mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F F + F \otimes_F \mathrm{N}_{\mathrm{alt}}(A_2)$. The other inclusion is obvious. ∎

In general, in a tensor product of algebras $A_1 \otimes_F \cdots \otimes_F A_n$ we will identify the factors A_i with the subalgebra $F \otimes_F \cdots \otimes_F A_i \otimes_F \cdots \otimes_F F$ without mention; thus, for instance, we will write $A_1 \otimes_F \cdots \otimes_F A_n = \prod_i A_i$.

COROLLARY 3.4. $\mathrm{N}_{\mathrm{alt}}(C_1 \otimes_F \cdots \otimes_F C_n) = C_1 + \cdots + C_n$.

Proof. It is well known [37, p. 41] that $\mathrm{N}(C_i) = F = \mathrm{K}(C_i)$ and that $\mathrm{N}(A_1 \otimes_F A_2) = \mathrm{N}(A_1) \otimes_F \mathrm{N}(A_2)$ and $\mathrm{K}(A_1 \otimes_F A_2) = \mathrm{K}(A_1) \otimes_F \mathrm{K}(A_2)$. ∎

Recall that in any algebra with product denoted by $[\ ,\ ]$ the element $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$ is called the *Jacobian* of x, y, z. The algebra is called a *Malcev algebra* if it is anticommutative and $J(x, y, [x, z]) = [J(x, y, z), x]$. One important example of a simple Malcev algebra is the algebra of elements of zero trace in an octonion algebra with the product given by the commutator [24, 26, 29, 30]. Therefore $\mathrm{N}_{\mathrm{alt}}(C)$ is a Malcev algebra, and $\mathrm{N}_{\mathrm{alt}}(C) = F1 \oplus C_1' \oplus \cdots \oplus C_n'$ with C_i' minimal ideals that are simple Malcev algebras and F1 the center. The derived algebra of $\mathrm{N}_{\mathrm{alt}}(C)$ is

$$\mathrm{N}_{\mathrm{alt}}'(C) = \left[\mathrm{N}_{\mathrm{alt}}(C), \mathrm{N}_{\mathrm{alt}}(C)\right] = C_1' \oplus \cdots \oplus C_n'.$$

Remark 3.5. Let $\varphi_0\colon C_1' \to C_2'$ be an isomorphism of Malcev algebras. Since $[a,[a,b]] = -4n_1(a)b + 2n_1(a,b)a$, we have $n_2(\varphi_0(a), \varphi_0(b)) = n_1(a,b)$, so we can define $\varphi\colon C_1 \to C_2$ by $\alpha 1 + a \mapsto \alpha 1 + \varphi_0(a)$, which is an isomorphism because of the identity $2ab = [a,b] - n_1(a,b)$. That is, any isomorphism from C_1' onto C_2' is the restriction of an isomorphism between C_1 and C_2. Moreover, given an automorphism $\sigma \in \mathrm{Aut}(F)$ then any σ-semilinear isomorphism φ_0 between C_1' and C_2' is induced by a σ-semilinear isomorphism $\varphi\colon \alpha 1 + a \mapsto \sigma(\alpha)1 + \varphi_0(a)$ between C_1 and C_2. In the same way, any derivation of C_1' is the restriction of a derivation of C_1. Something similar holds for $C_1 \otimes_F \cdots \otimes_F C_n$. Let φ_0 be an automorphism of $C_1' \oplus \cdots \oplus C_n'$. Since C_i' are the minimal ideals there exists a permutation $\pi \in \Sigma_n$ such that $\varphi_0(C_i') = C_{\pi(i)}'$. Therefore, by the previous, $\varphi_0|_{C_i'}$ is the restriction of an isomorphism $\varphi_i\colon C_i \to C_{\pi(i)}$ and we can define an automorphism φ of $C_1 \otimes_F \cdots \otimes_F C_n$ such that the restriction to C_i is φ_i. Hence, any automorphism of $C_1' \oplus \cdots \oplus C_n'$ is the restriction of an automorphism of $C_1 \otimes_F \cdots \otimes_F C_n$. The same is true for σ-semilinear automorphisms.

PROPOSITION 3.6. *The restriction map gives the isomorphisms*

$$\mathrm{Aut}(C) \cong \mathrm{Aut}\left(\mathrm{N}_{\mathrm{alt}}'(C)\right),$$
$$\mathrm{Der}(C) \cong \mathrm{Der}\left(\mathrm{N}_{\mathrm{alt}}'(C)\right) \cong \mathrm{Der}(C_1) \oplus \cdots \oplus \mathrm{Der}(C_n).$$

Proof. Any automorphism (resp. derivation) of C leaves $\mathrm{N}_{\mathrm{alt}}'(C)$ invariant, so it induces an automorphism (resp. derivation) of $\mathrm{N}_{\mathrm{alt}}'(C)$. Since $\mathrm{N}_{\mathrm{alt}}'(C)$ generates C as an algebra, the restriction map induces monomorphisms $\mathrm{Aut}(C) \to \mathrm{Aut}(\mathrm{N}_{\mathrm{alt}}'(C))$ and $\mathrm{Der}(C) \to \mathrm{Der}(\mathrm{N}_{\mathrm{alt}}'(C))$. In the case of

$\mathrm{Aut}(C)$ this monomorphism is also an epimorphism by Remark 3.5. In the case of $\mathrm{Der}(C)$, given $d \in \mathrm{Der}(\mathrm{N}'_{\mathrm{alt}}(C))$ we have

$$d(C'_i) = d([C'_i, C'_i]) \subseteq [d(C'_i), C'_i] + [C'_i, d(C'_i)] \subseteq C'_i$$

so $\mathrm{Der}(\mathrm{N}'_{\mathrm{alt}}(C)) = \mathrm{Der}(C'_1) \oplus \cdots \oplus \mathrm{Der}(C'_n)$. Since any derivation d_i of C'_i is induced by a derivation d_i of C_i, and $d_i \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \cdots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes d_n$ is a derivation of C, it follows that $\mathrm{Der}(C'_1) \oplus \cdots \oplus \mathrm{Der}(C'_n) = \mathrm{Der}(C)$. ▮

Let σ be the tensor product of the canonical involutions of the C_i and $\mathrm{Aut}(C, \sigma)$ the automorphisms of C that commute with σ.

COROLLARY 3.7. *With the previous notation*, $\mathrm{Aut}(c) = \mathrm{Aut}(C, \sigma)$.

Remark 3.8. We can write $C \cong C_1^{\otimes n_1} \otimes_F \cdots \otimes_F C_m^{\otimes n_m}$ with $C_i^{\otimes n_i}$ the tensor product of n_i copies of C_i and $C_i \not\cong C_j$ if $i \neq j$ and $n_1 + \cdots + n_m = n$. Thus, $\mathrm{N}'_{\mathrm{alt}}(C) \cong n_1 C'_1 \oplus \cdots \oplus n_m C'_m$ with $n_i C'_i$ isomorphic to the direct sum of n_i copies of C'_i, and $C'_i \not\cong C'_j$ if $i \neq j$. Since $\mathrm{Aut}(n_i C'_i)$ is the wreath product of $\mathrm{Aut}(C'_i)$ and the symmetric group Σ_{n_i}, we obtain that $\mathrm{Aut}(C^{\otimes n_i}) \cong \mathrm{Aut}(C_i)_{n_i}$, the wreath product of $\mathrm{Aut}(C_i)$ and Σ_{n_i}, and thus $\mathrm{Aut}(C) \cong \mathrm{Aut}(C_1)_{n_1} \times \cdots \times \mathrm{Aut}(C_m)_{n_m}$.

The uniqueness of the decomposition $\mathrm{N}'_{\mathrm{alt}}(C) \cong C'_1 \oplus \cdots \oplus C'_n$ gives the following uniqueness of the factorization of C.

PROPOSITION 3.9. *Let A_1, A_2 be unital algebras such that $C_1 \otimes_F \cdots \otimes_F C_n \cong A_1 \otimes_F A_2$. Then there exists a partition $\{1, \dots, n\} = \Lambda_1 \cup \Lambda_2$ such that $A_1 \cong \bigotimes_{i \in \Lambda_1} C_i$ and $A_2 \cong \bigotimes_{j \in \Lambda_2} C_j$. In particular, the factors $\{C_1, \dots, C_n\}$ in $C_1 \otimes_F \cdots \otimes_F C_n$ are uniquely determined up to order and isomorphism.*

Proof. Since the associative and commutative nuclei of $C = C_1 \otimes_F \cdots \otimes_F C_n$ are each the base field, $\mathrm{N}(A_i) = F = \mathrm{K}(A_i)$ for $i = 1, 2$. By Proposition 3.3,

$$\mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F + F \otimes_F \mathrm{N}_{\mathrm{alt}}(A_2) \cong C_1 + \cdots + C_n = F1 \oplus C'_1 \oplus \cdots \oplus C'_n.$$

The C'_i are minimal ideals; therefore, there exists a partition $\{1, \dots, n\} = \Lambda_1 \cup \Lambda_2$ such that $\mathrm{N}_{\mathrm{alt}}(A_i) \cong F1 \oplus \bigoplus_{j \in \Lambda_i} C'_j$. Thus, the image in $A_1 \otimes_F A_2$ of the subalgebra $\bigotimes_{j \in \Lambda_i} C_j$ generated by $\bigoplus_{j \in \Lambda_i} C'_j$ is contained in A_i. Since the two algebras have the same dimension, they must be equal. ▮

COROLLARY 3.10. *Two algebras $C_1 \otimes_F \cdots \otimes_F C_n$ and $\tilde{C}_1 \otimes_F \cdots \otimes_F \tilde{C}_m$ are isomorphic if and only if $n = m$ and there exists a permutation τ such that C_i is isomorphic to $\tilde{C}_{\tau(i)}$.*

EXAMPLE 3.11. It is well known that if the Albert forms of two biquaternion algebras are similar, then the algebras are isomorphic. We give an example to show that the analogue of this result is false for octonion algebras. Let C_1 and C_2 be nonisomorphic octonion F-algebras and consider the tensor products $C_1 \otimes_F C_1$ and $C_2 \otimes_F C_2$. Then their Albert forms are $n_{C_1'} \perp -n_{C_1'}$ and $n_{C_2'} \perp -n_{C_2'}$, respectively. Therefore, these forms are isomorphic as they are both hyperbolic. However, $C_1 \otimes_F C_1$ is not isomorphic to $C_2 \otimes_F C_2$ by the previous corollary since C_1 and C_2 are not isomorphic.

The following result points out the special role played by the involution σ.

COROLLARY 3.12. *The only involution of $C_1 \otimes_F \cdots \otimes_F C_n$ which commutes with all automorphisms is σ.*

Proof. Let σ' be another involution of $C = C_1 \otimes_F \cdots \otimes_F C_n$ commuting with $\operatorname{Aut}(C)$. The elements fixed by $\operatorname{Aut}(C_i) \subseteq \operatorname{Aut}(C)$ are exactly $\hat{C}_i = \otimes_{j \neq i} C_j$. Therefore, $\sigma'(\hat{C}_i) = \hat{C}_i$. Looking at the centralizer of $\hat{C}_i$ yields $\sigma'(C_i) = C_i$. The automorphism $\sigma'\sigma$ induces an automorphism of C_i which commutes with $\operatorname{Aut}(C_i)$. That is, $\sigma'\sigma = \mathrm{id}$ [17] and $\sigma' = \sigma$. ∎

We will call σ the *canonical involution* of $C_1 \otimes_F \cdots \otimes_F C_n$.

COROLLARY 3.13. *Let $\varphi: C = C_1 \otimes_F \cdots \otimes_F C_n \to \tilde{C} = \tilde{C}_1 \otimes_F \cdots \otimes_F \tilde{C}_n$ be an isomorphism. If σ and σ' are the canonical involutions of C and $\tilde{C}$ respectively, then $\varphi\sigma = \sigma'\varphi$.*

Proof. Since $\operatorname{Aut}(\tilde{C}) = \varphi \operatorname{Aut}(C) \varphi^{-1}$, then $\varphi\sigma\varphi^{-1}$ commutes with $\operatorname{Aut}(\tilde{C})$. Therefore, $\sigma' = \varphi\sigma\varphi^{-1}$. ∎ ∎

We now show that the Skolem-Noether theorem does not hold for C.

COROLLARY 3.14. *There exist simple F-subalgebras B and B' of C and an F-algebra isomorphism $f: B \to B'$ such that there is no F-algebra automorphism φ of C with $\varphi|_B = f$.*

Proof. Let Q_i be a quaternion subalgebra of C_i for $i = 1,2$ and let f be an F-algebra automorphism of $A = Q_1 \otimes_F Q_2$ that is not compatible with $\sigma|_A$; such maps exist since we can take f to be the inner automorphism of an element $t \in A$ with $\sigma(t)t \notin F$. The condition $\sigma(t)t \in F$ is precisely the condition needed to ensure that f is compatible with $\sigma|_A$. For example, we can take $t = 1 + i_1 i_2 \in A = Q_1 \otimes_F Q_2$ (where the standard generators of Q_r are i_r and j_r). If f extends to an automorphism φ of C, then $\varphi(A) = A$, so φ is compatible with σ. This forces $\varphi|_A = f$ to be compatible with $\sigma|_A$, and f is chosen so that this does not happen. ∎

We devote the remainder of this section to computing the forms of tensor products of octonions, that is, F-algebras A such that $A_K = K \otimes_F$

$A \cong C_1 \otimes_K \cdots \otimes_K C_n$ for some extension K/F and octonion algebras C_i over K. We will denote $C_1 \otimes_K \cdots \otimes_K C_n$ by T.

Since $K \otimes_F \mathrm{N}'_{\mathrm{alt}}(A) \cong \mathrm{N}'_{\mathrm{alt}}(K \otimes_F A) \cong C'_1 \oplus \cdots \oplus C'_n$, the algebra $\mathrm{N}'_{\mathrm{alt}}(A)$ is separable. It is worth noting that for a finite dimensional separable F-algebra R and a field extension K/F such that $K \otimes_F R \cong R_1 \oplus \cdots \oplus R_n$, with R_i central simple K-algebras, there exists a subfield K_0 of K such that K_0/F is a finite Galois extension and $K_0 \otimes_F R \cong \tilde{R}_1 \oplus \cdots \oplus \tilde{R}_n$ with $\tilde{R}_i$ central simple K_0-algebras and $R_i \cong K \otimes_{K_0} \tilde{R}_i$.

LEMMA 3.15. *Let A be a form of a tensor product of octonion algebras over F. There exists a finite Galois extension $F \subseteq K_0 \subseteq K$ such that A_{K_0} is the tensor product of octonion algebras over K_0.*

Proof. Let K_0 be a finite Galois extension of F contained in K such that $K_0 \otimes_F \mathrm{N}'_{\mathrm{alt}}(A) = \tilde{C}'_1 \oplus \cdots \oplus \tilde{C}'_n$ with $\tilde{C}_i$ octonion algebras over K_0 and $K \otimes_{K_0} \tilde{C}'_i \cong C'_i$. By Remark 3.5, this isomorphism is induced by an isomorphism $K \otimes_{K_0} \tilde{C}_i \cong C_i$. Thus we have an isomorphism $K \otimes_{K_0} (\tilde{C}_1 \otimes_{K_0} \cdots \otimes_{K_0} \tilde{C}_n) \cong T$ which restricts to an isomorphism $\tilde{C}_1 \otimes_{K_0} \cdots \otimes_{K_0} \tilde{C}_n \cong K_0 \otimes_F A$. ∎

This proposition allows us to assume in the following that K/F is a finite Galois extension. We denote the F-subalgebra generated by S by $\mathrm{alg}_F\langle S\rangle$ and the subspace spanned by S by $\mathrm{span}_F\langle S\rangle$. Since

$$\begin{aligned} K \otimes_F \mathrm{alg}_K\langle \mathrm{N}'_{\mathrm{alt}}(A)\rangle &\cong \mathrm{alg}_F\langle \mathrm{N}'_{\mathrm{alt}}(K \otimes_F A)\rangle = \mathrm{alg}_K\langle C'_1 \oplus \cdots \oplus C'_n\rangle \\ &= T = K \otimes_F A, \end{aligned}$$

we obtain A from $\mathrm{N}'_{\mathrm{alt}}(A)$ since $A = \mathrm{alg}_F\langle \mathrm{N}'_{\mathrm{alt}}(A)\rangle$.

PROPOSITION 3.16. *The map*

$$\begin{aligned} \{F\text{-forms of } C_1 \otimes_K \cdots \otimes_K C_n\} &\to \{F\text{-forms of } C'_1 \oplus \cdots \oplus C'_n\} \\ A &\mapsto \mathrm{N}'_{\mathrm{alt}}(A) \end{aligned}$$

is a bijection with inverse given by $N' \mapsto \mathrm{alg}\langle N'\rangle$. Moreover, if A and B are F-forms of $C_1 \otimes_K \cdots \otimes_K C_n$ then $A \cong B$ if and only if $\mathrm{N}'_{\mathrm{alt}}(A) \cong \mathrm{N}'_{\mathrm{alt}}(B)$.

Proof. It is clear that $\mathrm{N}'_{\mathrm{alt}}(A)$ is an F-form of $C'_1 \oplus \cdots \oplus C'_n$ if A is an F-form of $C_1 \otimes_K \cdots \otimes_K C_n$. Conversely, let N' be an F-form of $C'_1 \oplus \cdots \oplus C'_n$ and $\{U_\sigma : \sigma \in \mathrm{Gal}(K/F)\}$ the semilinear automorphisms of $C'_1 \oplus \cdots \oplus C'_n$ such that N' is the set of fixed elements. By Remark 3.5, we can assume that U_σ is the restriction of a σ-semilinear automorphism $\tilde{U}_\sigma$ of $C_1 \otimes_K \cdots \otimes_K C_n$. The algebra A of fixed elements by $\{\tilde{U}_\sigma : \sigma \in \mathrm{Gal}(K/F)\}$ is an F-form of $C_1 \otimes_K \cdots \otimes_K C_n$ containing N'. In fact, since

N' extends to $C'_1 \oplus \cdots \oplus C'_n$, we get $N' = \mathrm{N}'_{\mathrm{alt}}(A)$, and thus $\mathrm{alg}_F\langle N'\rangle = A$ is an F-form of $C_1 \otimes_K \cdots \otimes_K C_n$.

If $A \cong B$ then $\mathrm{N}'_{\mathrm{alt}}(A) \cong \mathrm{N}'_{\mathrm{alt}}(B)$. Conversely, if $\varphi_0\colon \mathrm{N}'_{\mathrm{alt}}(A) \to \mathrm{N}'_{\mathrm{alt}}(B)$ is an isomorphism, it induces an automorphism $\tilde{\varphi}_0$ of $C'_1 \oplus \cdots \oplus C'_n$ that, by Remark 3.5, is the restriction of an automorphism $\tilde{\varphi}$ of $C_1 \otimes_K \cdots \otimes_K C_n$. Since

$$\tilde{\varphi}(A) = \tilde{\varphi}(\mathrm{alg}_F\langle \mathrm{N}'_{\mathrm{alt}}(A)\rangle) = \mathrm{alg}_F\big\langle \tilde{\varphi}(\mathrm{N}'_{\mathrm{alt}}(A))\big\rangle = \mathrm{alg}_F\big\langle (\mathrm{N}'_{\mathrm{alt}}(B))\big\rangle = B,$$

it follows that $A \cong B$. ∎

This proposition allows us to construct easily the forms of a tensor product of octonion algebras. First, observe that if $\mathrm{N}'_{\mathrm{alt}}(A) = N_1 \oplus N_2$ then $K \otimes_F N_i \cong \bigoplus_{j \in \Lambda_i} C'_j$, $i = 1, 2$ for some partition $\{1, \ldots, n\} = \Lambda_1 \cup \Lambda_2$. By Proposition 3.16, $A_i = \mathrm{alg}_F\langle N_i\rangle$ is an F-form of $\bigotimes_{j \in \Lambda_i} C_j$ and hence $A \cong A_1 \otimes_F A_2$. Therefore, it is enough to construct the forms of a tensor product of octonion algebras with simple generalized alternative nucleus.

Let A be an F-algebra with $\mathrm{N}'_{\mathrm{alt}}(A)$ simple and K a finite Galois extension such that $K \otimes_F A \cong C_1 \otimes_K \cdots \otimes_K C_n$ for some octonion algebras C_i over K. Since $\mathrm{N}'_{\mathrm{alt}}(A)$ is simple, the centroid $\Gamma = \Gamma(\mathrm{N}'_{\mathrm{alt}}(A))$ is a finite separable extension of F. In fact,

$$K \otimes_F \Gamma \cong \Gamma\big(K \otimes_F \mathrm{N}'_{\mathrm{alt}}(A)\big) \cong \Gamma\big(C'_1 \oplus \cdots \oplus C'_n\big) \cong K \oplus \cdots \oplus K$$

implies that Γ is an extension of degree n and that we have n different F-monomorphisms $\sigma_i\colon \Gamma \to K$. Every σ_i allows us to define a right Γ-vector space structure on K by $\alpha \circ \gamma = \alpha\sigma_i(\gamma)$, $\alpha \in K$, $\gamma \in \Gamma$. We denote this new vector space by K^{σ_1}. Now,

$$\begin{aligned} K \otimes_F \mathrm{N}'_{\mathrm{alt}}(A) &\cong K \otimes_F \big(\Gamma \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A)\big) \cong (K \otimes_F \Gamma) \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A) \\ &\cong (K^{\sigma_1} \oplus \cdots \oplus K^{\sigma_n}) \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A) \\ &\cong \big(K^{\sigma_1} \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A)\big) \oplus \cdots \oplus \big(K^{\sigma_n} \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A)\big) \\ &\cong C'_1 \oplus \cdots \oplus C'_n \end{aligned}$$

implies that, up to order, $K^{\sigma_i} \otimes_\Gamma \mathrm{N}'_{\mathrm{alt}}(A) \cong C'_i$. Therefore, we can think of the Γ-algebra $\mathrm{N}'_{\mathrm{alt}}(A)$ as a form of C'_i. By the arguments in [24, pp. 240–241], for instance, we can conclude that $\mathrm{N}'_{\mathrm{alt}}(A) \cong C'$ for some octonion algebra C over Γ. Under the isomorphism $K \otimes_F C' \cong (K^{\sigma_1} \otimes_\Gamma C') \oplus \cdots \oplus (K^{\sigma_n} \otimes_\Gamma C')$ we identify $x \in C'$ with $(1 \otimes x) + \cdots + (1 \otimes x)$. Since we can view $(K^{\sigma_1} \otimes_\Gamma C') \oplus \cdots \oplus (K^{\sigma_n} \otimes_\Gamma C')$ as the generalized alternative nucleus of $(K^{\sigma_1} \otimes_\Gamma C') \otimes_K \cdots \otimes_K (K^{\sigma_n} \otimes_\Gamma C')$, the algebra A corresponds with the F-subalgebra generated by $(1 \otimes_\Gamma x) \otimes_K \cdots \otimes_K (1 \otimes_\Gamma 1) + \cdots + (1 \otimes_\Gamma 1) \otimes_K \cdots \otimes_K (1 \otimes_\Gamma x)$. Conversely, given an octonion Γ-algebra C with Γ a finite separable extension of F, it is easy to check that the algebra A constructed as above is a form of a tensor product of octonion algebras with a simple generalized alternative nucleus.

4. THE GENERALIZED ALTERNATIVE NUCLEUS

The generalized alternative nucleus is responsible for many properties of the tensor product of octonion algebras. In this section we pay special attention to this nucleus. We classify the simple finite dimensional unital algebras which are generated by their generalized alternative nucleus. Our methods rely on the representation theory of some Lie algebras; therefore, in this section we will assume that $\mathrm{char}(F) = 0$ and that F is algebraically closed. We make free use of Lie algebra terminology and refer the reader to the books of Humphreys [16] and Jacobson [18] for definitions and results.

Let C be an octonion algebra over F and $\mathrm{Sym}^n(C)$ the symmetric tensors of $C \otimes_F \cdots \otimes_F C$, the tensor product of n copies of C. The elements $a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a$ with $a \in C$ lie in $\mathrm{N}_{\mathrm{alt}}(C \otimes_F \cdots \otimes_F C)$ and generate $\mathrm{Sym}^n(C)$. Therefore $\mathrm{Sym}^n(C)$ is an algebra generated by its generalized alternative nucleus. However, $\mathrm{Sym}^n(C)$ is no longer simple. The contraction $\mathrm{Sym}^n(C) \to \mathrm{Sym}^{n-2}(C)$ induced by $x \otimes \cdots \otimes x \mapsto n(x)x \otimes \cdots \otimes x$ is an epimorphism whose nucleus we will denote by $\mathrm{T}_n(C)$, $n \geq 2$. We recover the Kantor–Smirnov structurable algebra when $n = 2$ [33, 7]. We will see that $\mathrm{T}_n(C)$ is a unital simple algebra generated by $\mathrm{N}_{\mathrm{alt}}(\mathrm{T}_n(C))$. We set $\mathrm{T}_1(C) = C$ and $\mathrm{T}_0(C) = F$. We now give our classification result.

THEOREM 4.1. *Any simple finite dimensional unital algebra over an algebraically closed field of characteristic zero which is generated by its generalized alternative nucleus is isomorphic to the tensor product of a simple associative algebra and* $\mathrm{T}_n(C)$ *for some* n.

Recall from [28] that a *ternary derivation* of an algebra A is a triple $(d_1, d_2, d_3) \in \mathrm{End}_F(A) \times \mathrm{End}_F(A) \times \mathrm{End}_F(A)$ such that

$$d_1(xy) = d_2(x)y + xd_3(y) \tag{1}$$

for any $x, y \in A$. The Lie algebra of ternary derivations is denoted by $\mathrm{Tder}(A)$. If $d_1 = d_2 = d_3$ then (1) says that d_1 is a derivation, and in that case we will say that (d_1, d_2, d_3) *represents a derivation*. Let $T_a = L_a + R_a$. It is worth noting that

$$a \in \mathrm{N}_{\mathrm{alt}}(A) \Leftrightarrow (L_a, T_a, -L_a) \text{ and } (R_a, -R_a, T_a) \in \mathrm{Tder}(A). \tag{2}$$

The following identities will be useful.

LEMMA 4.2. *Let* $a, b \in \mathrm{N}_{\mathrm{alt}}(A)$ *and* $x \in A$. *Then*

(i) $L_{ax} = L_a L_x + [R_a, L_x]$, $L_{xa} = L_x L_a + [L_x, R_a]$.

(ii) $R_{ax} = R_x R_a + [R_x, L_a]$, $R_{xa} = R_a R_x + [L_a, R_x]$.

(iii) $[L_a, R_b] = [R_a, L_b]$.

(iv) $[L_a, L_b] = L_{[a,b]} - 2[R_a, L_b]$, $[R_a, R_b] = -R_{[a,b]} - 2[L_a, R_b]$.

(v) *The map* $D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ *is a derivation of* A, $D_{a,b} = \mathrm{ad}_{[a,b]} - 3[L_a, R_b]$ *and* $2D_{a,b} = \mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b]$, *where* $\mathrm{ad}_a \colon x \mapsto [a, x]$.

Proof. Parts (i) and (ii) follow from the identities $(a, x, y) = (x, y, a)$, $(x, a, y) = -(x, y, a)$, $(y, a, x) = -(a, y, x)$, and $(y, x, a) = (a, y, x)$. Part (iii) follows from $(b, x, a) = -(a, x, b)$, while (iv) is an easy consequence of parts (i), (ii), and (iii). Now, by (2) we have that $([L_a, L_b], [T_a, T_b], [L_a, L_b])$, $([L_a, R_b], -[T_a, R_b], -[L_a, T_b])$, and $([R_a, R_b], [R_a, R_b], [T_a, T_b])$ lie in $\mathrm{Tder}(A)$. Adding up these elements and using (ii), we obtain a ternary derivation that represents the derivation $D_{a,b}$. From (iv) we get $D_{a,b} = \mathrm{ad}_{[a,b]} - 3[L_a, R_b]$. Finally,

$$\begin{aligned}\mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b] &= \mathrm{ad}_{[a,b]} + [L_a, L_b] + [R_a, R_b] - 2[L_a, R_b] \\ &= 2\big(\mathrm{ad}_{[a,b]} - 3[L_a, R_b]\big) = 2D_{a,b}.\end{aligned}$$

∎

As we saw in the case of tensor products of octonions, $\mathrm{N}_{\mathrm{alt}}(A)$ may not be a subalgebra of A. The natural product on $\mathrm{N}_{\mathrm{alt}}(A)$ seems to be the commutator $[a, b] = ab - ba$.

PROPOSITION 4.3. *Given* $a, b \in \mathrm{N}_{\mathrm{alt}}(A)$ *then* $[a, b] \in \mathrm{N}_{\mathrm{alt}}(A)$. *Moreover,* $(\mathrm{N}_{\mathrm{alt}}(A), [\ ,\ ])$ *is a Malcev algebra.*

Proof. By Lemma 4.2(iv), $L_{[a,b]} = [L_a, L_b] + 2[R_a, L_b]$ and $R_{[a,b]} = -[R_a, R_b] - 2[L_a, R_b]$, thus by (2) we obtain

$$\big(L_{[a,b]}, [T_a, T_b] + 2[-R_a, T_b], [L_a, L_b] + 2[T_a, -L_b]\big) \in \mathrm{Tder}(A).$$

Since

$$T_{[a,b]} = [L_a, L_b] - [R_a, R_b] = [T_a, T_b] + 2[-R_a, T_b]$$

and

$$-L_{[a,b]} = -[L_a, L_b] - 2[R_a, L_b] = [L_a, L_b] + 2[T_a, -L_b],$$

it follows that $(L_{[a,b]}, T_{[a,b]}, -L_{[a,b]}) \in \mathrm{Tder}(A)$. Similarly, $(R_{[a,b]}, -R_{[a,b]}, T_{[a,b]}) \in \mathrm{Tder}(A)$. Therefore, $[a, b] \in \mathrm{N}_{\mathrm{alt}}(A)$. The same arguments as those in [27, p. 9] show that $(\mathrm{N}_{\mathrm{alt}}(A), [\ ,\ ])$ is a Malcev algebra. ∎

In the following we will always assume that A is simple, finite dimensional and generated by $\mathrm{N}_{\mathrm{alt}}(A)$. In our discussion, the Lie algebra $T(A)$ generated by $\{L_a, R_a : a \in \mathrm{N}_{\mathrm{alt}}(A)\}$ will play a prominent role.

Given a subset S of an algebra we say that $\delta(x)$ is the *degree* of x on S if x can be written as $x = p(s_1, \dots, s_m)$ with $s_1, \dots, s_n \in S$ and $p(x_1, \dots, x_n)$ some nonassociative polynomial (constants are allowed) of degree $\delta(x)$, and if there is no other such expression for a polynomial of degree $< \delta(x)$. By convention the degree of 0 is set to $-\infty$.

LEMMA 4.4. *If $S \subseteq \mathrm{N}_{\mathrm{alt}}(A)$ then the degree of x on S is the same as the degree of L_x, R_x on* $\mathrm{span}_F\langle L_a, R_a : a \in S\rangle$.

Proof. We proceed by induction to see that the degree of L_x and R_x is $\leq \delta(x)$. The case $\delta(x) = -\infty$ is trivial. If $\delta(x) = 0$ then $0 \neq x \in F$ and therefore $\delta(L_x) = 0 = \delta(R_x)$. Now let x be a monomial of degree $n > 1$, so $x = x_1 x_2$ with $\delta(x_i) < \delta(x)$. By induction $\delta(L_{x_1}) < \delta(x)$, and therefore $x = a x_0$ or $x = x_0 a$ with $a \in S$ and $\delta(x_0) < \delta(x)$. By Lemma 4.2 (i and ii) and by the hypothesis of induction we get $\delta(L_x)$, $\delta(R_x) \leq \delta(x)$. Finally, since $x = L_x(1) = R_x(1)$, it follows that $\delta(x) \leq \delta(L_x), \delta(R_x)$. ∎

PROPOSITION 4.5. *A is an irreducible* $\mathrm{T}(A)$*-module and* $\mathrm{T}(A) = \mathrm{T}'(A) \oplus F\,\mathrm{id}$, *with* $\mathrm{T}'(A) = [\mathrm{T}(A), \mathrm{T}(A)]$ *a semisimple Lie algebra.*

Proof. By Lemma 4.4, the multiplication algebra of A is generated by the left and right multiplication maps L_a, R_a for $a \in \mathrm{N}_{\mathrm{alt}}(A)$. Therefore, any $\mathrm{T}(A)$-submodule is an ideal, hence A irreducible. Since A is irreducible and faithful, $\mathrm{T}'(A)$ is semisimple and $\mathrm{T}(A)$ is the direct sum of $\mathrm{T}'(A)$ and the center [19, p. 47]. But any element in the center commutes with the multiplication algebra of A and therefore lives in the centroid of A. Since A is simple and F is algebraically closed, we conclude that the center is F id. ∎

Recall that a Malcev algebra is semisimple if 0 is the only Abelian ideal [24].

PROPOSITION 4.6. $\mathrm{N}'_{\mathrm{alt}}(A)$ *is a semisimple Malcev algebra.*

Proof. Let I be an ideal of $\mathrm{N}'_{\mathrm{alt}}(A)$ and consider $T_I = \mathrm{span}_F\langle L_a, R_a, D_{a,c} : a \in I,\ c \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$. By Lemma 4.2 (iv), $T_I \subseteq T'(A)$. Moreover, Part (v) of the same lemma shows that $[L_a, R_b] = [R_a, L_b] \in T_I$ if $a \in I$ and $b \in \mathrm{N}'_{\mathrm{alt}}(A)$. Then, by Part (iv), it follows that $[L_a, L_b]$, $[R_a, R_b] \in T_I$, too. Finally, $D_{a,c}(b) \in I$ by (v), so $[D_{a,c}, L_b] = L_{D_{a,c}(b)}$, $[D_{a,c}, R_b] = R_{D_{a,c}(b)} \in T_I$. Therefore T_I is an ideal of $T'(A)$. By the semisimplicity of $T'(A)$ we must have $[T_I, T_I] = T_I$. In particular, $I = T_I(1) = [T_I, T_I](1) = [I, I]$ and therefore the only abelian ideal of $\mathrm{N}'_{\mathrm{alt}}(A)$ is 0, so $\mathrm{N}'_{\mathrm{alt}}(A)$ is semisimple. ∎

In a Malcev algebra M the subspace generated by the Jacobians is an ideal of M denoted by $J(M,M,M)$. The subspace $N(M) = \{x \in M : J(x,M,M) = 0\}$ is also an ideal and is called the *J-nucleus* of M. It is well-known that $N(M)J(M,M,M) = 0$. In fact, any finite dimensional semisimple Malcev algebra M over a perfect field of characteristic not two can be decomposed as $M = N(M) \oplus J(M,M,M)$ with $N(M)$ a semisimple Lie algebra and $J(M,M,M)$ the direct sum of simple non-Lie Malcev algebras. If the field has characteristic 0 then $N(M)$ is the direct sum of simple Lie algebras, by [16, Theorem 5.3].

PROPOSITION 4.7. *Let $\mathrm{N}'_{\mathrm{alt}}(A) = \bigoplus_i N'_i$ be the decomposition of $\mathrm{N}'_{\mathrm{alt}}(A)$ as the direct sum of ideals that are simple Malcev algebras, and let $A_i = \mathrm{alg}\langle N'_i, 1\rangle$. Then, A_i is a simple unital algebra generated by $\mathrm{N}_{\mathrm{alt}}(A_i) = F1 + N'_i$, and $A \cong \bigotimes_i A_i$.*

Proof. Given $a \in N'_i$ and $b \in N'_j$ with $i \neq j$, then by Lemma 4.2(v) we have $D_{a,b} = 3[L_a, R_b]$ and $D_{a,b}(\mathrm{N}_{\mathrm{alt}}(A)) = 0$. Since A is generated by $\mathrm{N}_{\mathrm{alt}}(A)$, it follows that $[L_a, R_b] = 0$. By Lemma 4.2(iv), we also get $[L_a, L_b] = [R_a, R_b] = 0$. By Lemma 4.4, the left and right multiplication operators by elements of A_i commute with those by elements of A_j. Therefore, we have an epimorphism $\varphi: \bigotimes_i A_i \to A$ given by the multiplication of the factors. Consider the ideals $T_{N'_i}$ of $T'(A)$ as in the proof of Proposition 4.6. Since $T'(A)$ is semisimple so is $T_{N'_i}$. The subalgebra A_i is a $T_{N'_i}$-module, and by Weyl's theorem it is completely reducible. In fact, by Lemma 4.4 any submodule is an ideal and the converse. Thus A_i is the direct sum of simple (unital) ideals. Fix A'_i to be one of these simple ideals. Clearly $\prod_i A'_i \subseteq A$ is a $T(A)$-submodule. By irreducibility it follows that $A = \prod_j A'_j$. Any other simple ideal A''_i in the decomposition of A_i verifies $A''_i = AA''_i = (\prod_{j \neq i} A'_j)A'_i A''_i = 0$. So, $A_i = A'_i$ is a central simple algebra as well as $\bigotimes_i A_i$, and consequently φ is an isomorphism. Finally, we observe that $\sum_i \mathrm{N}_{\mathrm{alt}}(A_i) \subseteq \mathrm{N}_{\mathrm{alt}}(A) \subseteq F1 + \sum_i N'_i$ implies $\mathrm{N}_{\mathrm{alt}}(A_i) = F1 + N'_i$. ∎

This proposition allows us to distinguish two cases, algebras in which $\mathrm{N}'_{\mathrm{alt}}$ is a simple Lie algebra and algebras in which $\mathrm{N}'_{\mathrm{alt}}$ is a simple non-Lie Malcev algebra.

PROPOSITION 4.8. *If $\mathrm{N}'_{\mathrm{alt}}(A)$ is a simple Lie algebra, then A is a simple associative algebra.*

Proof. Since $J(a,c,b) = 6(a,c,b)$ for any $a,b,c \in \mathrm{N}_{\mathrm{alt}}(A)$ [27, 37], the hypothesis implies that $(a,b,c) = 0$ and thus $D_{a,b}(c) = \mathrm{ad}_{[a,b]}(c) = [[a,b],c]$ by Lemma 4.2. Since $\mathrm{N}'_{\mathrm{alt}}(A)$ is a simple Lie algebra, any derivation of $\mathrm{N}'_{\mathrm{alt}}(A)$ has the form $\mathrm{ad}_{[a,b]}$; thus, we obtain an epimorphism from the derivations of A onto the derivations of $\mathrm{N}'_{\mathrm{alt}}(A)$ that is in fact an isomorphism because A is generated by $\mathrm{N}_{\mathrm{alt}}(A)$. Given $a \in \mathrm{N}'_{\mathrm{alt}}(A)$, we denote by D_a the unique derivation of A that restricts to ad_a over $\mathrm{N}'_{\mathrm{alt}}(A)$. It is not difficult to check that $\mathrm{span}_F\langle D_a - \mathrm{ad}_a : a \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$ is an ideal of $T'(A)$ that kills $\mathrm{N}_{\mathrm{alt}}(A)$. Since $T'(A)$ is semisimple then the subspace killed by an ideal is a submodule of A and, by irreducibility, it must be all of A. Therefore, $\mathrm{ad}_a = D_a$ is a derivation. But $(L_a - R_a, T_a + R_a, -L_a - T_a), (L_a - R_a, L_a - R_a, L_a - R_a) \in \mathrm{Tder}(R)$ implies $(0, 3R_a, -3L_a) \in \mathrm{Tder}(A)$ which can be written as $(x, a, y) = 0$. Thus $(a, x, y) = (x, y, a) = -(x, a, y) = 0$ and $a \in \mathrm{N}(A)$. Since A is generated by $\mathrm{N}'_{\mathrm{alt}}(A)$ and $\mathrm{N}(A)$ is a subalgebra, this finishes the proof. ∎

Now we will assume that $\mathrm{N}'_{\mathrm{alt}}(A)$ is a simple non-Lie Malcev algebra, that is, $\mathrm{N}'_{\mathrm{alt}}(A) = C'$ where $C = \mathrm{Zor}(F)$ denotes the split octonion algebra, which is the only octonion algebra up to isomorphism over an algebraically closed field.

PROPOSITION 4.9. *We have that*

(i) $\mathrm{Der}(A) = \mathrm{span}_F\langle D_{a,b} : a, b \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$ *is a simple Lie algebra of type* G_2.

(ii) $\mathrm{span}_F\langle D_{a,b},\ \mathrm{ad}_a : a, b \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$ *is a simple Lie algebra of type* B_3.

(iii) $T'(A)$ *is a simple Lie algebra of type* D_4.

(iv) *The maps* $\zeta, \eta\colon T'(A) \to T'(A)$ *given by* $\zeta\colon L_a \mapsto T_a$, $R_a \mapsto -R_a$, $D_{a,b} \mapsto D_{a,b}$, *and* $\eta\colon L_a \mapsto -L_a$, $R_a \mapsto T_a$, $D_{a,b} \mapsto D_{a,b}$ *can be identified with the automorphisms corresponding to the permutations* (13) *and* (14) *of the Dynkin diagram of* D_4.

Proof. Any derivation of A induces a derivation of $\mathrm{N}'_{\mathrm{alt}}(A)$, and since A is generated by $\mathrm{N}_{\mathrm{alt}}(A)$ then any two derivations that agree on $\mathrm{N}'_{\mathrm{alt}}(A)$ must be equal. It is known that $\mathrm{Der}(\mathrm{N}'_{\mathrm{alt}}(A)) = \langle D_{a,b}|_{\mathrm{N}'_{\mathrm{alt}}(A)} : a, b \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$; therefore, this yields the first part of (i). Consider a standard basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of C [11] and $f = e_1 - e_2$. Relative to the subalgebra $H = \mathrm{span}_F\langle D_{u_1,v_1}, D_{u_2,v_2}, L_f, R_f\rangle$, $T'(A)$ decomposes as the direct sum of root spaces in the way given in Table I (note that the elements in the right column are not 0 by evaluating them in appropriate elements of the standard basis).

Therefore, H is a *Cartan subalgebra* of $T'(A)$ and the root system corresponds to a simple Lie algebra of type D_4. The automorphism ζ leaves H invariant and permutes the root spaces corresponding to α_1 and α_3, but fixes those corresponding to α_2 and α_4. Thus, ζ can be thought of as the automorphism (13) of the Dynkin diagram of D_4. Similarly, η corresponds to (14).

The subalgebra $\mathrm{span}_F\langle D_{a,b}, \mathrm{ad}_a : a, b \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$ is the algebra fixed by the automorphism $\zeta\eta\zeta$ which corresponds to the automorphism (34) of the Dynkin diagram. This algebra is known to be a simple Lie algebra of type

TABLE I

Root	Element that spans the root space
α_1	$L_{u_3} - R_{u_3} - D_{e_1, u_3}$
α_2	D_{u_2, v_3}
α_3	$L_{u_3} + 2R_{u_3} - D_{e_1, u_3}$
α_4	$2L_{u_3} + R_{u_3} + D_{e_1, u_3}$
$\alpha_1 + \alpha_2$	$L_{u_2} - R_{u_2} - D_{e_1, u_2}$
$\alpha_2 + \alpha_3$	$L_{u_2} + 2R_{u_2} - D_{e_1, u_2}$
$\alpha_2 + \alpha_4$	$2L_{u_2} + R_{u_2} + D_{e_1, u_2}$
$\alpha_1 + \alpha_2 + \alpha_3$	$2L_{v_1} + R_{v_1} + D_{e_2, v_1}$
$\alpha_1 + \alpha_2 + \alpha_4$	$L_{v_1} + 2R_{v_1} - D_{e_2, v_1}$
$\alpha_2 + \alpha_3 + \alpha_4$	$L_{v_1} - R_{v_1} - D_{e_2, v_1}$
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	D_{v_1, u_3}
$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	D_{v_1, u_2}
Negative roots	Change α_i by $-\alpha_i$, e_1 by e_2 and u_i by v_i in the previous rows

B_3. Finally, $\mathrm{Der}(A) = \mathrm{span}_F\langle D_{a,b} : a, b \in \mathrm{N}'_{\mathrm{alt}}(A)\rangle$ is the algebra fixed by the automorphism $\eta\zeta$, which corresponds to (134) as an automorphism of the Dynkin diagram, therefore it is a Lie algebra of type G_2. ∎

Let $\{\alpha_1, \dots, \alpha_4\}$ be a basis of a root system of D_4 as in the above table and $\lambda_1, \dots, \lambda_4$ be the corresponding fundamental weights. If λ is a dominant weight, we will denote by $V(\lambda)$ the irreducible module of highest weight λ.

PROPOSITION 4.10. *The Lie algebra A is isomorphic as a D_4-module to $V(n\lambda_1)$ for some n.*

Proof. Since $F1$ is a trivial submodule for B_3, the branching rules for the inclusion $B_3 \subseteq D_4$ [14, Theorem 8.1.4] imply that there exists an n such that $A \cong V(n\lambda_1)$. ∎

Since $C \cong V(\lambda_1)$, this proposition allows us to identify A with the submodule of $C \otimes_F \cdots \otimes_F C$ generated by $v_0 \otimes \cdots \otimes v_0$ with v_0 the highest weight of C. This submodule obviously lies in $\mathrm{Sym}^n(C)$ and it is killed by the contraction $\mathrm{Sym}^n(C) \to \mathrm{Sym}^{n-2}(C)$, $x \otimes \cdots \otimes x \mapsto n(x)x \otimes \cdots \otimes x$. In fact, this is the kernel of this contraction [13, Example 19.21].

Finally, in order to prove Theorem 4.1 we have to determine the product on A. This product is not a D_4-homomorphism of $V(n\lambda_1) \otimes_F V(n\lambda_1) \to V(n\lambda_1)$ since that would imply that D_4 acts as derivations, which is not true. Given an automorphism ξ of D_4 and V a module, we denote by V_ξ the vector space V but with a new action given by $d \circ x = \xi(d)x$ for all

$d \in D_4$ and $x \in V$. Then (2) implies that

$$\begin{aligned} V(n\lambda_1)_\zeta \otimes_F V(n\lambda_1)_\eta &\to V(n\lambda_1) \\ x \otimes y &\mapsto xy \end{aligned}$$

is a D_4-homomorphism. Since $V(n\lambda_1)_\eta \cong V(n\lambda_3)$ and $V(n\lambda_1)_\eta \cong V(n\lambda_4)$, the product is a D_4-homomorphism from $V(n\lambda_3) \otimes_F V(n\lambda_4)$ onto $V(n\lambda_1)$. However, since

$$\dim\big(\mathrm{Hom}_{D_4}(V(n\lambda_3) \otimes_F V(n\lambda_4), V(n\lambda_1))\big) = 1$$

[25], then we only have a possibility that is fulfilled by the induced product of $\mathrm{Sym}^n(C)$. This proves Theorem 4.1.

Remark 4.11. The commutative nucleus $K(T_n(C))$ is killed by ad_a for any a, so it is killed by the action of B_3. Since the decomposition of $V(n\lambda_1)$ as a B_3-module is multiplicity free this implies that $K(T_n(C)) = F$.

The generalized alternative nucleus $\mathrm{N}_{\mathrm{alt}}(T_n(C))$ is the direct sum of simple Malcev ideals. One of these ideals is F and the other is C'. Since any other ideal would be killed by the action of B_3 we have that $\mathrm{N}_{\mathrm{alt}}(T_n(C)) = C$. In particular, $N(T_n(C)) = F$.

If A is as in Theorem 4.1 and $A = A_1 \otimes_F \cdots \otimes_F A_m$ is the decomposition as a tensor product given by the theorem, then Proposition 3.3 implies that

$$\mathrm{N}_{\mathrm{alt}}(A) = \mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F F \otimes_F \cdots \otimes_F + \cdots + F \otimes_F \cdots \otimes_F F \otimes_F \mathrm{N}_{\mathrm{alt}}(A_m).$$

As in Corollary 3.10, this implies that the decomposition is unique up to order and isomorphism of the factors.

5. CONNECTIONS WITH STRUCTURABLE ALGEBRAS

In [6] Allison classified the finite dimensional central simple structurable algebras over fields of characteristic zero. Later, Smirnov [33, 34] showed that there was a gap in the list provided by Allison, and one has to include in the previous list the algebra of symmetric octonion tensors [7], a 35-dimensional algebra which in our notation corresponds to $T_2(C)$.

We want to analyze the connection between structurable algebras and algebras generated by its generalized alternative nucleus. Recall from [6] that any structurable algebra $(A, ^-)$ is skew-alternative; that is, the skew-symmetric elements for the involution lie in the generalized alternative nucleus. In his work, Allison first reduces the classification of finite dimensional central simple (as algebras with involution) structurable algebras to the case in which the algebra is central simple, so up to a scalar extension one may assume that the field is algebraically closed. After that, he splits the proof into two cases, depending on whether or not the algebra is generated by the skew-symmetric elements. If the algebra is not generated by the skew-symmetric elements (Case 1), then one obtains either a central simple Jordan algebra with the identity as involution, an algebra constructed from a nondegenerate Hermitian form on a module over a unital central simple associative algebra with involution, or an algebra with involution constructed from an admissible triple. The second case (Case 2), where the Kantor–Smirnov structurable algebra is missed, deals with algebras generated by the skew-symmetric elements. In this case Allison and Smirnov obtain that the only possibilities are either a central simple associative algebra with involution, the tensor product of an octonion algebra with a composition algebra, or the Kantor–Smirnov structurable algebra.

Since the skew-symmetric elements lie in the generalized alternative nucleus, then Case 2 falls naturally into our context. So we can use Theorem 4.1 to give a new proof of this case. We will assume that A is a finite dimensional central simple structurable algebra, over an algebraically closed field of characteristic zero, which is generated by the skew-symmetric elements. The key point is Lemma 14 in Allison's paper which establishes that A is spanned by $\{s, s^2 : s \text{ is skew-symmetric}\}$. Let us write $A = A_1 \otimes_F \cdots \otimes_F A_m$ as given by Theorem 4.1. Since for any skew-symmetric element s,

$$s, s^2 \in \sum_{i,j} F \otimes_F \cdots \otimes_F F \otimes_F A_i \otimes_F F \otimes_F \cdots \otimes_F F \otimes_F A_j \otimes_F F \otimes_F \cdots \otimes_F F,$$

then $m \leq 2$. If $m = 1$, then A is either associative or isomorphic to $T_n(C)$ with $n \geq 2$. Since $\mathrm{N}_{\mathrm{alt}}(T_n(C)) \cong C$ then, in the latter case, Lemma 14 also implies that $\dim T_n(C) = \dim V(n\lambda_1) \leq 35$, so $n \leq 2$, and we obtain the octonions and the Kantor–Smirnov structurable algebra. Finally, if $m = 2$ then $A = A_1 \otimes_F A_2$ and Lemma 14 implies that $A = A_1 \otimes_F F + F \otimes_F A_2 + \mathrm{N}_{\mathrm{alt}}(A_1) \otimes_F \mathrm{N}_{\mathrm{alt}}(A_2)$. In particular, A_1 and A_2 are alternative, so A is either the tensor product of two octonion algebras or the tensor product of an octonion algebra and an associative algebra. In the second case, the involution of A preserves the associative nucleus and its centralizer so it preserves each factor in the tensor product. If S_i denotes the skew-symmetric elements of A_i, then Lemma 14 implies that $A = A_1 \otimes_F F + F \otimes_F A_2 + S_1 \otimes_F S_2$, so the set of symmetric elements of A_i must be F and A_i are quadratic algebras. Therefore the associative factor must be isomorphic to the two-by-two matrices, which is isomorphic to the quaternions.

6. INVARIANT BILINEAR FORMS

A symmetric bilinear form $(\ ,\ ): A \times A \to F$ on an F-algebra A is said to be *associative* if $(xy, z) = (x, yz)$ for any $x, y, z \in A$. If the algebra has an involution $x \mapsto \bar{x}$ with $(x, y) = (\bar{x}, \bar{y})$, the new bilinear form $\langle x, y\rangle = (\bar{x}, y)$ is symmetric and verifies $\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle$ and $\langle xy, z\rangle = \langle y, \bar{x}z\rangle$; that is, $\langle\ ,\ \rangle$ is invariant. In [33], Schafer proves that, up to scalar multiples, there is only one invariant symmetric bilinear form on a finite dimensional central simple structurable algebra over a field of characteristic zero. That invariant form was constructed by Allison in [6]. In this section we construct an associative symmetric bilinear form on any algebra generated by its generalized alternative nucleus.

PROPOSITION 6.1. *Let A be an algebra generated by its generalized alternative nucleus; then the symmetric bilinear form*

$$(x, y) = \operatorname{trace}(L_x L_y)$$

is associative. If A is unital, then $(x, y) = \operatorname{trace}(L_{xy})$.

Proof. We prove that $(xy, z) = (y, zx)$ by induction on the degree of x on $\mathrm{N}_{\mathrm{alt}}(A)$. If $x = a \in \mathrm{N}_{\mathrm{alt}}(A)$ then

$$\begin{aligned}
\operatorname{trace}(L_{ay} L_z) &= \operatorname{trace}(L_a L_y L_z + [R_a, L_y] L_z)\\
&= \operatorname{trace}(L_y L_z L_a + [R_a, L_y] L_z)\\
&= \operatorname{trace}(L_y L_{za} - L_y[L_z, R_a] + [R_a, L_y] L_z)\\
&= \operatorname{trace}(L_y L_{za} + [R_a, L_y L_z])\\
&= \operatorname{trace}(L_y L_{za}),
\end{aligned}$$

where we have used Lemma 4.2. Thus $(ay, z) = (y, za)$ and we get the first step in the induction. Now suppose that $x = ax_0$ or $x = x_0 a$ with $a \in \mathrm{N}_{\mathrm{alt}}(A)$ and that $(x_0 y, z) = (y, zx_0)$ for any y, z. In the first case it follows that

$$\begin{aligned}
(xy, z) &= ((ax_0)y, z) = (a(x_0 y), z) + ((a, x_0, y)z)\\
&= (y, (za)x_0) + ((a, x_0, y)z)\\
&= (y, zx) + (y, (z, a, x_0)) + ((a, x_0, y)z)\\
&= (y, zx) - (y, (a, z, x_0)) + ((x_0, y, a)z)\\
&= (y, zx) - (y, [R_{x_0}, L_a](z)) + ([R_a, L_{x_0}](y), z)\\
&= (y, zx) - ([R_a, L_{x_0}](y), z) + ([R_a, L_{x_0}](y), z) = (y, zx).
\end{aligned}$$

The second case is analogous. This completes the induction. If A is unital, then $(x, y) = (xy, 1) = \mathrm{trace}(L_{xy})$. ∎

We denote the radical of this bilinear form by Rad. Since the form is associative, Rad is an ideal. We remark that if this form is nondegenerate then it is, up to scalar multiples, the only nondegenerate associative bilinear form on A [9].

COROLLARY 6.2. *If* $(\ ,\ )$ *is nondegenerate then* A *is the direct sum of algebras as in Theorem* 4.1.

Proof. Let I be an ideal with $I^2 = 0$ and $x \in I$. Given $y \in A$, $x(y(x(yA))) \subseteq xI = 0$; thus $L_x L_y$ is nilpotent and $(x, y) = 0$. Since Rad $= 0$, we obtain $I = 0$. By [31, Theorem 2.6], A is the direct sum of ideals A_i that are simple unital algebras. Moreover, $\mathrm{N}_{\mathrm{alt}}(A) = \mathrm{N}_{\mathrm{alt}}(\bigoplus_i A_i) = \bigoplus_i \mathrm{N}_{\mathrm{alt}}(A_i)$ implies that $A_i = \mathrm{alg}\langle \mathrm{N}_{\mathrm{alt}}(A_i)\rangle$. ∎

Remark 6.3. Let A be generated by $\mathrm{N}_{\mathrm{alt}}(A)$ and suppose that the associative bilinear form $(x, y) = \mathrm{trace}(L_x L_y)$ is nondegenerate. If A is unital, the corollary yields the classification of A. In general, we consider the unital algebra $A^{\#} = A \oplus F1$, which contains A as an ideal. Since the bilinear forms on A and $A^{\#}$ agree, it follows that $A^{\#} = A \oplus Fe$ with Fe the orthogonal complement of A, which is an ideal. Now $e = \alpha 1 + x$ with $x \in A$, and $\alpha \neq 0$ implies that $0 \neq e^2 \in Fe$, and we can assume that e is an idempotent. Therefore $\mathrm{trace}(L_e L_e) = 1$ and the bilinear form on $A^{\#}$ is nondegenerate. By the corollary, $A^{\#}$ is the direct sum of simple unital ideals, but A is an ideal and thus it is the sum of some of these ideals. This implies that A must be unital if the bilinear form is nondegenerate.

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