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An envelope for Malcev algebras

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Abstract

We prove that for every Malcev algebra M there exist an algebra $U(M)$ and a monomorphism $\iota : M \to U(M)^-$ of M into the commutator algebra $U(M)^-$ such that the image of M lies into the alternative center of $U(M)$, and $U(M)$ is a universal object with respect to such homomorphisms. The algebra $U(M)$, in general, is not alternative, but it has a basis of Poincaré–Birkhoff–Witt type over M and inherits some good properties of universal enveloping algebras of Lie algebras. In particular, the elements of M can be characterized as the primitive elements of the algebra $U(M)$ with respect to the diagonal homomorphism $\Delta : U(M) \to U(M) \otimes U(M)$. An extension of Ado–Iwasawa theorem to Malcev algebras is also proved.

1. Introduction

An anticommutative algebra $(M, [\,,\,])$ is said to be a Malcev algebra if it satisfies the identity $[J(x, y, z), x] = J(x, y, [x, z])$, where $J(x, y, z) = [[x, y], z] - [[x, z], y] - [x, [y, z]]$ is the jacobian of x, y, z [4,6,8]. Since for any Lie algebra the jacobian of any three elements vanishes, Lie algebras fall into the variety of Malcev algebras. Among the non-Lie Malcev algebras, the traceless elements of an octonion algebra with the product given by the commutator is one of the most important examples [4,5,9].

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Let us denote by A^- the algebra obtained from an algebra A when the product xy is replaced by $[x, y] = xy - yx$. Starting with an associative algebra A one obtains a Lie algebra A^-, and conversely, the celebrated Poincaré–Birkhoff–Witt Theorem [1] establishes that any Lie algebra is isomorphic to a subalgebra of A^- for some associative algebra A. A weaker condition than the associativity for an algebra is to be alternative. An algebra A is called alternative if it satisfies the identities $x(xy) = x^2y$ and $(yx)x = yx^2$ [13]. When starting with an alternative algebra A one obtains a Malcev algebra A^-. However, at this time it remains an open problem whether any Malcev algebra is isomorphic to a subalgebra of A^- for some alternative algebra A [2,10,12].

There is a more general way of constructing Malcev algebras. Given an arbitrary algebra A, the generalized alternative nucleus of A is defined as

$$\mathrm{N_{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a)\ \forall x, y \in A\},$$

where $(x, y, z) = (xy)z - x(yz)$ is the usual associator [7]. This nucleus may not be a subalgebra of A, but it is closed under the commutator product $[x, y] = xy - yx$, so it is a subalgebra of A^-. In fact $(\mathrm{N_{alt}}(A), [\,,\,])$ is a Malcev algebra. If A is an alternative algebra then $\mathrm{N_{alt}}(A) = A$, and we recover the construction of Malcev algebras from alternative algebras.

It seems a natural question to ask whether any Malcev algebra is isomorphic to a subalgebra of $\mathrm{N_{alt}}(A)$ for some algebra A. The goal of this paper is to provide an affirmative answer to this question.

More specifically, we prove that for every Malcev algebra M there exist an algebra $U(M)$ and a monomorphism $\iota : M \to U(M)^-$ of M into the commutator algebra $U(M)^-$ such that the image of M lies into the alternative nucleus of $U(M)$, and $U(M)$ is a universal object with respect to such homomorphisms. The algebra $U(M)$, in general, is not alternative, but it has a basis of Poincaré–Birkhoff–Witt type over M and inherits some good properties of universal enveloping algebras of Lie algebras. In particular, the elements of M can be characterized as the primitive elements of the algebra $U(M)$ with respect to the diagonal homomorphism $\Delta : U(M) \to U(M) \otimes U(M)$. An extension of Ado–Iwasawa theorem to Malcev algebras is also proved.

2. The universal enveloping algebra

Let $(M, [\,,\,])$ be a Malcev algebra over a commutative and associative ring ϕ with $\frac{1}{2}, \frac{1}{3} \in \phi$ which is a free module over ϕ. Let $\phi\{M\}$ be the unital free non-associative algebra on a basis of M and $I(M)$ the ideal of $\phi\{M\}$ generated by the set

$$\bigl\{ab - ba - [a, b],\ (a, x, y) + (x, a, y),\ (x, a, y) + (x, y, a) \mid \\ a, b \in M \text{ and } x, y \in \phi\{M\}\bigr\}.$$

The natural object to consider in our context is $(U(M), \iota)$, where $U(M) = \phi\{M\}/I(M)$ and

$$\iota : M \to \mathrm{N}_{\mathrm{alt}}\big(U(M)\big) \subseteq U(M),$$
$$a \mapsto \iota(a) = \bar{a} = a + I(M).$$

It is clear that for any Malcev homomorphism $\varphi : M \to \mathrm{N}_{\mathrm{alt}}(A)$, with A a unital algebra, there exists a homomorphism $\bar{\varphi} : U(M) \to A$ such that $\varphi(1) = 1$ and $\varphi = \bar{\varphi} \circ \iota$. It is also clear that M is isomorphic to a subalgebra of $\mathrm{N}_{\mathrm{alt}}(A)$ for some A if and only if the map ι is injective.

Let $\{a_i \mid i \in \Lambda\}$ be a basis of M, $\leqslant$ an order in Λ and $\Omega = \{(i_1, \dots, i_n) \mid i_1, \dots, i_n \in \Lambda, n \in \mathbb{N}$ and $i_1 \leqslant \cdots \leqslant i_n\}$. If $I = (i_1, \dots, i_n) \in \Omega$ then we will usually write $\bar{a}_I$ instead of $\bar{a}_{i_1}(\bar{a}_{i_2}(\cdots(\bar{a}_{i_{n-1}}\bar{a}_{i_n})\cdots))$. We understand that if $n = 0$ then $I = \emptyset$ and $\bar{a}_I = 1$. The size n of I will be denoted by $|I|$ while I' stands for $(i_2, \dots, i_n)$ if $|I| \geqslant 1$. With this notation our main theorem can be formulated as follows:

Theorem 2.1. *The set $\{\bar{a}_I \mid I \in \Omega\}$ is a basis of $U(M)$.*

We first prove that the monomials $\{\bar{a}_I \mid I \in \Omega\}$ span $U(M)$.

Proposition 2.2. $U(M) = \mathrm{span}\langle \bar{a}_I \mid I \in \Omega\rangle$.

Proof. Consider $U = \mathrm{span}\langle \bar{a}_I \mid I \in \Omega\rangle$ and $U_n = \mathrm{span}\langle \bar{a}_I \mid I \in \Omega$ and $|I| \leqslant n\rangle$. Since $\iota(M) \subseteq U$ and $U(M)$ is generated by $\iota(M)$, it suffices to prove that U is a subalgebra of $U(M)$. Suppose that we have proved that $\bar{a}U_{n-1} \subseteq U_n$ and $[U_{n-1}, \bar{a}] \subseteq U_{n-1}$; then given $a \in M$ and $I \in \Omega$ with $|I| = n$,

$$\begin{aligned}
[\bar{a}_I, \bar{a}] &= [\bar{a}_{i_1}\bar{a}_{I'}, \bar{a}] = [\bar{a}_{i_1}, \bar{a}]\bar{a}_{I'} + \bar{a}_{i_1}[\bar{a}_{I'}, \bar{a}] + 3(\bar{a}_{i_1}, \bar{a}_{I'}, \bar{a}) \\
&= [\bar{a}_{i_1}, \bar{a}]\bar{a}_{I'} + \bar{a}_{i_1}[\bar{a}_{I'}, \bar{a}] + \frac{1}{2}\Big(\big[[\bar{a}_{i_1}, \bar{a}_{I'}], \bar{a}\big] - \big[[\bar{a}_{i_1}, \bar{a}], \bar{a}_{I'}\big] \\
&\quad - \big[\bar{a}_{i_1}, [\bar{a}_{I'}, \bar{a}]\big]\Big) \in U_n,
\end{aligned}$$

where we have used the identities

$$[xy, z] - [x, z]y - x[y, z] = (x, y, z) - (x, z, y) + (z, x, y)$$

and

$$\begin{aligned}
\big[[x, y], z\big] - \big[[x, z], y\big] - \big[x, [y, z]\big] &= (x, y, z) - (x, z, y) + (z, x, y) \\
&\quad - (y, x, z) + (y, z, x) - (z, y, x),
\end{aligned}$$

valid in any algebra. Therefore $[U_n, \bar{a}] \subseteq U_n$. On the other hand,

$$\begin{aligned}
\bar{a}_{i_0}\bar{a}_I &= \bar{a}_{i_0}(\bar{a}_{i_1}\bar{a}_{I'}) = (\bar{a}_{i_0}\bar{a}_{i_1})\bar{a}_{I'} - (\bar{a}_{i_0}, \bar{a}_{i_1}, \bar{a}_{I'}) \\
&= (\bar{a}_{i_1}\bar{a}_{i_0})\bar{a}_{I'} + [\bar{a}_{i_0}, \bar{a}_{i_1}]\bar{a}_{I'} + (\bar{a}_{i_0}, \bar{a}_{I'}, \bar{a}_{i_1})
\end{aligned}$$

$$\begin{aligned}
&\equiv \bar a_{i_1}(\bar a_{i_0}\bar a_{I'}) + 2(\bar a_{i_0}, \bar a_{I'}, \bar a_{i_1})\\
&= \bar a_{i_1}(\bar a_{i_0}\bar a_{I'}) + \frac{1}{3}\big(\big[[\bar a_{i_0}, \bar a_{I'}], \bar a_{i_1}\big] - \big[[\bar a_{i_0}, \bar a_{i_1}], \bar a_{I'}\big] - \big[\bar a_{i_0}, [\bar a_{I'}, \bar a_{i_1}]\big]\big)\\
&\equiv \bar a_{i_1}(\bar a_{i_0}\bar a_{I'}) \mod U_{|I|}
\end{aligned}$$

allows us to move $\bar a_{i_0}$ around, and place it in a position so that $i_1 \leqslant \cdots \leqslant i_0 \leqslant \cdots \leqslant i_n$. Therefore, $\bar a U_n \subseteq U_{n+1}$ and we conclude the induction. Since $U_n \bar a \subseteq \bar a U_n + [\bar a, U_n] \subseteq U_{n+1}$ we also obtain that $U_n \bar a \subseteq U_{n+1}$. In particular, $\bar a U + U\bar a \subseteq U$.

Suppose we have proved that $\bar a_I U \subseteq U$ for any $I \in \Omega$ with $|I| < n$. Given $I \in \Omega$ with $|I| = n$ and $x \in U$, then

$$\begin{aligned}
\bar a_I x = (\bar a_{i_1}\bar a_{I'})x &= \bar a_{i_1}(\bar a_{I'}x) + (\bar a_{i_1}, \bar a_{I'}, x) = \bar a_{i_1}(\bar a_{I'}x) + (\bar a_{I'}, x, \bar a_{i_1})\\
&= \bar a_{i_1}(\bar a_{I'}x) + (\bar a_{I'}x)\bar a_{i_1} - \bar a_{I'}(x\bar a_{i_1}) \in \bar a_{i_1}U + U\bar a_{i_1} - \bar a_{I'}U \subseteq U.
\end{aligned}$$

Thus, U is a subalgebra and $U(M) = U$. □

Corollary 2.3. *If M is a Lie algebra then $U(M)$ is isomorphic to the universal enveloping algebra of M.*

Proof. Let U be the universal enveloping algebra of M. Since M is a Lie algebra then by [7] $U(M)$ is an associative algebra, and by the universal property of U we obtain a homomorphism $U \to U(M)$ with $a \mapsto \bar a$, $a \in M$. Conversely, since U is associative, $M \subseteq U = \mathrm{N_{alt}}(U)$ and we also obtain a homomorphism $U(M) \to U$, $\bar a \mapsto a$, $a \in M$ that is the inverse of the previous one. □

Definition 2.4. $(U(M), \iota)$ will be called the universal enveloping algebra of the Malcev algebra M.

Corollary 2.5. *For every Malcev algebra M, the algebra $U(M)$ has no zero divisors. Moreover, if M is finite-dimensional then $U(M)$ is left and right noetherian.*

Proof. It is easy to see that the sequence of subspaces U_n from the proof of Proposition 2.2 defines an ascending filtration on $U(M)$, that is, $U(M) = \bigcup_n U_n$ and $U_i U_j \subseteq U_{i+j}$. Moreover, it follows easily from the proof of proposition that the corresponding graded algebra $gr\, U(M)$ is associative and commutative. By Theorem 2.1, the algebra $gr\, U(M)$ is isomorphic to the polynomial algebra $\phi[a_1, \ldots, a_n, \ldots]$. Now, the corollary is proved just as in case of Lie algebras (see [1]). □

3. Proof of Theorem 2.1

As for Lie algebras, we only need a minimum of information about Malcev algebras to prove Theorem 2.1. In our case this information is the relationship between Malcev algebras and Lie triple systems. From a Malcev algebra $(M, [\,,\,])$ one can obtain a Lie triple system $(M, [\,,\,,\,])$ by

$$[a,b,c] = \frac{1}{3}R(a,b)(c) = \frac{1}{3}\big(2\big[[a,b],c\big] - \big[[b,c],a\big] - \big[[c,a],b\big]\big)$$

(we have added the scalar $1/3$ for convenience) [5]. The operators $\mathrm{ad}_a : b \mapsto [a,b]$ are derivations of this triple system, so the theory of Lie triple systems provides us with a $\mathbb{Z}_2$-graded Lie algebra $L(M, [\,,\,,\,]) = L(M) \oplus M$, where $L(M)$ is the Lie algebra generated by the operators $\{\mathrm{ad}_a \mid a \in M\}$, with the product given by the rules

$$\begin{aligned}
&L(M) \text{ is a Lie subalgebra of } L(M, [\,,\,,\,]),\\
&[\varphi, a] = \varphi(a),\\
&[a,b] = \frac{1}{3}R(a,b),
\end{aligned}$$

$\varphi \in L(M)$, $a, b \in M$, and skewsymmetry. While this connection between Malcev algebras and Lie algebras has valuable consequences in the theory of Malcev algebras, $L(M, [\,,\,,\,])$ will be too small for our purposes.

In any algebra A the left and right multiplication operators by elements of $\mathrm{N_{alt}}(A)$ satisfy the relations

$$\begin{aligned}
[L_a, R_b] &= [R_a, L_b],\\
[L_a, L_b] &= L_{[a,b]} - 2[L_a, R_b],\\
[R_a, R_b] &= -R_{[a,b]} - 2[L_a, R_b]
\end{aligned}$$

(see [7] for details, though these relations were implicit in the proof of Proposition 2.2). Roughly speaking, these relations tell us that the Lie algebra generated by these operators depends on the Malcev algebra $\mathrm{N_{alt}}(A)$ rather than on the particular A. Therefore, we proceed to define some kind of universal version of this algebra that will be helpful in the following. Let $\mathcal{L}(M)$ be the Lie algebra generated by $\{\lambda_a, \rho_a \mid a \in M\}$ with relations

$$\begin{aligned}
&\lambda_{\alpha a+\beta b} = \alpha\lambda_a + \beta\lambda_b, && \rho_{\alpha a+\beta b} = \alpha\rho_a + \beta\rho_b,\\
&[\lambda_a, \lambda_b] = \lambda_{[a,b]} - 2[\lambda_a, \rho_b], && [\rho_a, \rho_b] = -\rho_{[a,b]} - 2[\lambda_a, \rho_b],\\
&[\lambda_a, \rho_b] = [\rho_a, \lambda_b], &&
\end{aligned} \tag{1}$$

$a, b \in M$, $\alpha, \beta \in \phi$.

Proposition 3.1. *There exists an epimorphism of Lie algebras*

$$\mathcal{L}(M) \to L(M, [\ ,\ ])$$

such that $\lambda_a \mapsto \frac{1}{2}(\mathrm{ad}_a + a)$ *and* $\rho_a \mapsto \frac{1}{2}(-\mathrm{ad}_a + a)$.

For short we define in $\mathcal{L}(M)$ the elements $\mathrm{ad}_a = \lambda_a - \rho_a$, $T_a = \lambda_a + \rho_a$ and $D_{a,b} = [\lambda_a, \lambda_b] + [\rho_a, \rho_b] + [\lambda_a, \rho_b] = \mathrm{ad}_{[a,b]} - 3[\lambda_a, \rho_b]$.

Proposition 3.2. *The algebra $\mathcal{L}(M)$ is a $\mathbb{Z}_2$-graded Lie algebra $\mathcal{L}(M) = \mathcal{L}_+ \oplus \mathcal{L}_-$ with $\mathcal{L}_+ = \mathrm{span}\langle \mathrm{ad}_a, D_{a,b} \mid a, b \in M\rangle$ and $\mathcal{L}_- = \mathrm{span}\langle T_a \mid a \in M\rangle$. Moreover, the mapping $T_a \mapsto a$ gives a linear isomorphism from $\mathcal{L}_-$ onto M.*

Proof. From relations (1) we have $[T_a, T_b] = [\lambda_a, \lambda_b] + [\rho_a, \rho_b] + 2[\lambda_a, \rho_b] = \mathrm{ad}_{[a,b]} - 2[\lambda_a, \rho_b]$. Thus,

$$3[T_a, T_b] = \mathrm{ad}_{[a,b]} + 2D_{a,b}. \tag{2}$$

Similarly,

$$[\mathrm{ad}_a, T_b] = [\lambda_a, \lambda_b] - [\rho_a, \rho_b] = \lambda_{[a,b]} + \rho_{[a,b]} = T_{[a,b]}. \tag{3}$$

Before computing $[D_{a,b}, T_c]$ we observe that

$$\begin{aligned}[\mathrm{ad}_a, \mathrm{ad}_b] &= [\lambda_a, \lambda_b] + [\rho_a, \rho_b] - 2[\lambda_a, \rho_b] = \mathrm{ad}_{[a,b]} - 6[\lambda_a, \rho_b] \\ &= -\mathrm{ad}_{[a,b]} + 2D_{a,b},\end{aligned} \tag{4}$$

so, $2D_{a,b} = \mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b]$. Now we use this relation toghether with the Jacobi identity and (3) to obtain $2[D_{a,b}, T_c] = [\mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b], T_c] = T_{[[a,b],c]+[[a,c],b]+[a,[b,c]]}$. If we set $D_{a,b}(c) = \frac{1}{2}([[a,b],c] + [[a,c],b] + [a,[b,c]])$ then

$$[D_{a,b}, T_c] = T_{D_{a,b}(c)}. \tag{5}$$

Equalities (2), (3), (4) and the Jacobi identity imply that

$$\begin{aligned}2[D_{a,b}, \mathrm{ad}_c] &= \bigl[3[T_a, T_b] - \mathrm{ad}_{[a,b]}, \mathrm{ad}_c\bigr] \\ &= 3[T_{[a,c]}, T_b] + 3[T_a, T_{[b,c]}] + \mathrm{ad}_{[[a,b],c]} - 2D_{[a,b],c} \\ &= 2\mathrm{ad}_{D_{a,b}(c)} + 2(D_{a,[b,c]} + D_{b,[c,a]} + D_{c,[a,b]}).\end{aligned} \tag{6}$$

In any Malcev algebra the map $c \mapsto D_{a,b}(c)$ is a derivation [8]. In fact, $2D_{a,b}(c) = 2[[a,b],c] - J(a,b,c)$ implies that

$$
\begin{aligned}
&2\big(D_{a,b}(c)+D_{b,c}(a)+D_{c,a}(b)\big)\\
&\quad=2\big[[a,b],c\big]-J(a,b,c)+2\big[[b,c],a\big]-J(b,c,a)+2\big[[c,a],b\big]-J(c,a,b)\\
&\quad=2J(a,b,c)-3J(a,b,c)=-J(a,b,c) \qquad (7)
\end{aligned}
$$

by skewsymmetry of the jacobian [8]. Another relation satisfied by the derivations $D_{a,b}$ in any Malcev algebra is $D_{a,[b,c]}+D_{b,[c,a]}+D_{c,[a,b]}=0$ [8]. We now show that this is also true in $\mathcal{L}(M)$. Since $\mathcal{L}(M)$ is a Lie algebra then $J(\mathrm{ad}_a,\mathrm{ad}_b,\mathrm{ad}_c)=0$. On the other hand, we can compute $J(\mathrm{ad}_a,\mathrm{ad}_b,\mathrm{ad}_c)$ by using (4) and (6):

$$
\begin{aligned}
&J(\mathrm{ad}_a,\mathrm{ad}_b,\mathrm{ad}_c)\\
&\quad=[-\mathrm{ad}_{[a,b]}+2D_{a,b},\mathrm{ad}_c]+[-\mathrm{ad}_{[b,c]}+2D_{b,c},\mathrm{ad}_a]+[-\mathrm{ad}_{[c,a]}+2D_{c,a},\mathrm{ad}_b]\\
&\quad=\mathrm{ad}_{J(a,b,c)}+2\mathrm{ad}_{D_{a,b}(c)+D_{b,c}(a)+D_{c,a}(b)}+8(D_{a,[b,c]}+D_{b,[c,a]}+D_{c,[a,b]})\\
&\quad=8(D_{a,[b,c]}+D_{b,[c,a]}+D_{c,[a,b]})
\end{aligned}
$$

where the last equality follows from (7). Therefore,

$$D_{a,[b,c]}+D_{b,[c,a]}+D_{c,[a,b]}=0$$

and

$$[D_{a,b},\mathrm{ad}_c]=\mathrm{ad}_{D_{a,b}(c)}. \qquad (8)$$

Finally, we use (4) and (8) to obtain $[D_{a,b},D_{c,d}]=D_{D_{a,b}(c),d}+D_{c,D_{a,b}(d)}$. This proves the first part of the proposition. The isomorphism between $\mathcal{L}_-$ and M follows from the previous proposition. □

Let $S(M)$ be the usual symmetric tensor algebra on M that we can identify with $S(\mathcal{L}_-)$. The $\mathbb{Z}_2$-gradation on $\mathcal{L}(M)$ allows us to define easily a structure of $\mathcal{L}(M)$-module on $S(M)$. Since the method works in general, we start with a $\mathbb{Z}_2$-graded Lie algebra $\mathcal{L}=\mathcal{L}_+\oplus\mathcal{L}_-$, its universal enveloping algebra $U(\mathcal{L})$, the left ideal K of $U(\mathcal{L})$ generated by $\mathcal{L}_+$, i.e., $K=U(\mathcal{L})\mathcal{L}_+$, and the $\mathcal{L}$-module $U(\mathcal{L})/K$. By the Poincaré–Birkhoff–Witt Theorem, given a basis $\{x_i \mid i\in\Lambda_-\}$ of $\mathcal{L}_-$ and an order $\leqslant$ on Λ_-, then $\{x_{i_1}\cdots x_{i_n}+K \mid i_1\leqslant\cdots\leqslant i_n \text{ and } n\in\mathbb{N}\}$ is a basis of $U(\mathcal{L})/K$ (if $n=0$ then $x_{i_1}\cdots x_{i_n}=1$ by convention). Therefore, we have a linear isomorphism $\theta: S(\mathcal{L}_-)\to U(\mathcal{L})/K$ defined on the basis $\{x_{i_1}\cdots x_{i_n} \mid i_1\leqslant\cdots\leqslant i_n \text{ and } n\in\mathbb{N}\}$ of $S(\mathcal{L}_-)$ by $x_{i_1}\cdots x_{i_n}\mapsto x_{i_1}\cdots x_{i_n}+K$. With this isomorphism, we can pull back the $\mathcal{L}$-module structure of $U(\mathcal{L})/K$ to $S(\mathcal{L}_-)$ by defining $\lambda\circ x=\theta^{-1}(\lambda\theta(x))$, $\lambda\in\mathcal{L}$ and $x\in S(\mathcal{L}_-)$. Let $S(\mathcal{L}_-)=\bigoplus_{i=0}^{\infty}S(\mathcal{L}_-)^i$ be the usual gradation on $S(\mathcal{L}_-)$, then $S(\mathcal{L}_-)=\bigcup_{n=0}^{\infty}S(\mathcal{L}_-)_n$ with $S(\mathcal{L}_-)_n=\bigoplus_{i=0}^{n}S(\mathcal{L}_-)^i$ becomes a filtration of $S(L_-)$. We set $S(\mathcal{L}_-)_{-1}=0$.

Lemma 3.3.

(i) *$\mathcal{L}_+ \circ S(\mathcal{L}_-)_n \subseteq S(\mathcal{L}_-)_n$ and $\mathcal{L}_- \circ S(\mathcal{L}_-)_n \subseteq S(\mathcal{L}_-)_{n+1}$,*
(ii) *if $i_1 \leqslant \cdots \leqslant i_{n+1}$ then $x_{i_k} \circ (x_{i_1} \cdots \hat{x}_{i_k} \cdots x_{i_{n+1}}) \equiv x_{i_1} \cdots x_{i_k} \cdots x_{i_{n+1}} \bmod S(\mathcal{L}_-)_{n-1}$, where $\hat{x}_{i_k}$ denotes that this factor is omitted.*

Proof. We prove (i) by induction. The case $n = 0$ is trivial. Now observe that

$$\begin{aligned}
&\theta\big(x_{i_k} \circ (x_{i_1} \cdots \hat{x}_k \cdots x_{i_{n+1}})\big) \\
&\quad = x_{i_k} x_{i_1} \cdots \hat{x}_k \cdots x_{i_{n+1}} + K \\
&\quad = ([x_{i_k}, x_{i_1}] x_{i_2} \cdots \hat{x}_{i_k} \cdots x_{i_{n+1}} + K) + (x_{i_1} x_{i_k} x_{i_2} \cdots \hat{x}_{i_k} \cdot s x_{i_{n+1}} + K).
\end{aligned}$$

Since $[x_{i_k}, x_{i_1}] \in \mathcal{L}_+$, the first summand lies in $\theta(S(\mathcal{L}_-)_{n-1})$ by the hypothesis of induction, so modulo $\theta(S(\mathcal{L}_-)_{n-1})$ we can move x_{i_k} around and place it in the right order. Thus, $x_{i_k} \circ (x_{i_1} \cdots \hat{x}_{i_k} \cdots x_{i_{n+1}}) \equiv x_{i_1} \cdots x_{i_k} \cdots x_{i_{n+1}} \bmod S(\mathcal{L}_-)_{n-1}$ and $\mathcal{L}_- \circ S(\mathcal{L}_-)_n \subseteq S(\mathcal{L}_-)_{n+1}$. Given $\lambda_+ \in \mathcal{L}_+$,

$$\begin{aligned}
\theta(\lambda_+ \circ x_{i_1} \cdots x_{i_n}) &= \lambda_+ x_{i_1} \cdots x_{i_n} + K = [\lambda_+, x_{i_1} \cdots x_{i_n}] + K \\
&= \sum x_{i_1} \cdots [\lambda_+, x_{i_j}] \cdots x_{i_n} + K.
\end{aligned}$$

Since $[\lambda_+, x_{i_j}] \in \mathcal{L}_-$, by the hypothesis of induction we get that each summand lies in $\theta(S(\mathcal{L}_-)_n)$, therefore $\mathcal{L}_+ \circ S(\mathcal{L}_-)_n \subseteq S(\mathcal{L}_-)_n$ as desired. Part (ii) has been proved along the way. $\square$

Observe that the elements $\lambda'_a = T_a, \rho'_a = -\rho_a$ and $\lambda''_a = -\lambda_a, \rho''_a = T_a$ satisfy the relations defining $\mathcal{L}(M)$, so in $\mathcal{L}(M)$ we have endomorphisms ζ, η with

$$\begin{aligned}
\zeta(\lambda_a) &= T_a, & \eta(\lambda_a) &= -\lambda_a, \\
\zeta(\rho_a) &= -\rho_a, & \eta(\rho_a) &= T_a,
\end{aligned}$$

that turn out to be automorphisms since $\zeta^2 = \mathrm{id} = \eta^2$. These automorphisms are a generalization of the automorphisms involved in the Principle of Local Triality in the case of D_4. In general, they may not be inherited by $L(M, [\ ,\ ])$ since the kernel of the epimorphism in Proposition 3.1 may not be invariant, so we cannot expect to obtain such a principle for an arbitrary $L(M, [\ ,\ ])$. The automorphism $\zeta\eta\zeta$ sends λ_a to $-\rho_a$, and ρ_a to $-\lambda_a$ and it is the responsible for the $\mathbb{Z}_2$-gradation on $\mathcal{L}(M)$. The structure of $\mathcal{L}(M)$-module of $S(M)$ can be twisted from any automorphism ξ of $\mathcal{L}(M)$ by $\xi(\lambda) \circ x$ to get another module $S(M)_\xi$. So, from ζ and η we obtain two extra modules $S(M)_\zeta$ and $S(M)_\eta$.

Proposition 3.4. *If there exists a homomorphism*

$$* : S(M)_\zeta \otimes S(M)_\eta \to S(M)$$

of $\mathcal{L}(M)$-modules satisfying

(i) $a * x = 2\lambda_a \circ x$ *and* $x * a = 2\rho_a \circ x$ *for any* $a \in M$, $x \in S(M)$,
(ii) $1 * x = x = x * 1$ *for any* $x \in S(M)$,

then Theorem 2.1 *holds.*

Proof. We can think of $*$ as a product on $S(M)$. If that is the case then

$$\begin{aligned} a * (x * y) &= 2\lambda_a \circ (x * y) = 2\big(\zeta(\lambda_a) \circ x\big) * y + 2x * \big(\eta(\lambda_a) \circ y\big) \\ &= 2(T_a \circ x) * y - 2x * (\lambda_a \circ y) \\ &= (a * x) * y + (x * a) * y - x * (a * y) \end{aligned}$$

and

$$\begin{aligned} (x * y) * a &= 2\rho_a \circ (x * y) = 2\big(\zeta(\rho_a) \circ x\big) * y + 2x * \big(\eta(\rho_a) \circ y\big) \\ &= -2(\rho_a \circ x) * y + 2x * (T_a \circ y) \\ &= x * (y * a) - (x * a) * y + x * (a * y) \end{aligned}$$

imply that $M \subseteq \mathrm{N_{alt}}((S(M), *))$.

Observe that

$$\begin{aligned} a_{i_1} * (a_{i_2} \cdots a_{i_n}) &= 2\lambda_{a_{i_1}} \circ (a_{i_2} \cdots a_{i_n}) = T_{a_{i_1}} \circ (a_{i_2} \cdots a_{i_n}) + \mathrm{ad}_{a_{i_1}} \circ (a_{i_2} \cdots a_{i_n}) \\ &= a_{i_1} \cdots a_{i_n} + \mathrm{ad}_{a_{i_1}} \circ (a_{i_2} \cdots a_{i_n}) \equiv a_{i_1} \cdots a_{i_n} \mod S(M)_{n-1}. \end{aligned}$$

By iterating this argument we obtain that

$$a_{i_1} * (a_{i_2} * (\cdots (a_{i_{n-1}} * a_{i_n}) \cdots)) \equiv a_{i_1} \cdots a_{i_n} \mod S(M)_{n-1}.$$

Therefore, we have that the set

$$\big\{a_{i_1} * \big(a_{i_2} * (\cdots (a_{i_{n-1}} * a_{i_n}) \cdots)\big) \mid (i_1, \ldots, i_n) \in \Omega\big\} \tag{9}$$

is a basis of $S(M)$. Now, the homomorphism $U(M) \to (S(M), *)$ from the universal property of $U(M)$ maps a linear generator set $\{\bar{a}_I \mid I \in \Omega\}$ of $U(M)$ onto a basis (9) of $S(M)$, therefore it is an isomorphism. □

The definition of the product $*$ is quite straightforward. We keep the notation $a_I = a_{i_1} \cdots a_{i_n}$ where $I = (i_1, \ldots, i_n) \in \Omega$. Recall that by Lemma 3.3 the element $r_I = a_I - 2\lambda_{a_{i_1}} \circ a_{I'}$ lies in $S(M)_{|I|-1}$ (in particular, if $|I| = 1$ then $r_I = 0$). We set $1 * x = x$, and assume that we have defined $a_J * x$ for any a_J with $|J| < |I|$. Then we define

$$a_I * x = 2T_{a_{i_1}} \circ (a_{I'} * x) - 2a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x. \tag{10}$$

As an explanation for this formula, observe that if $*$ has to satisfy all the requirements in Proposition 3.4 then $a_I * x = (2\lambda_{a_{i_1}} \circ a_{I'} + r_I) * x = (2\lambda_{a_{i_1}} \circ a_{I'}) * x + r_I * x = 2T_{a_{i_1}} \circ (a_{I'} * x) - 2a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x$. Also note that with this definition $a_{i_1} * x = 2T_{a_{i_1}} \circ x - 2\rho_{a_{i_1}} \circ x = 2\lambda_{a_{i_1}} \circ x$, so $a * x = 2\lambda_a * x$.

Proposition 3.5. *For any $\lambda \in \mathcal{L}(M)$ and $x, y \in S(M)$ we have*

$$\lambda \circ (x * y) = \bigl(\zeta(\lambda) \circ x\bigr) * y + x * \bigl(\eta(\lambda) \circ y\bigr).$$

Proof. We will prove that $\lambda \circ (a_I * x) = (\zeta(\lambda) \circ a_I) * x + a_I * (\eta(\lambda) \circ x)$ by induction on $|I|$. If $|I| = 0$ then $a_I = 1$ and the formula becomes $\lambda \circ x = (\zeta(\lambda) \circ 1) * x + \eta(\lambda) \circ x$. We write λ as $\lambda = D + \lambda_a + \rho_b$, where D is a linear combination of elements D_{a_i,b_i} (these elements are fixed by ζ and η), so that $\lambda - \eta(\lambda) = \lambda_{2a-b}$ and $\zeta(\lambda) \circ 1 = a - 1/2b$. Thus, $(\zeta(\lambda) \circ 1) * x = 2\lambda_{a-1/2b} \circ x = \lambda_{2a-b} \circ x = (\lambda - \eta(\lambda)) \circ x$ as desired. For the general case, we observe that

$$\begin{aligned}
&\lambda \circ (a_I * x) - \bigl(\zeta(\lambda) \circ a_I\bigr) * x - a_I * \bigl(\eta(\lambda) \circ x\bigr) \\
&\stackrel{(1)}{=} \lambda \circ \bigl(2T_{a_{i_1}} \circ (a_{I'} * x) - 2a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x\bigr) \\
&\quad - \bigl(\zeta(\lambda) \circ (2\lambda_{a_{i_1}} \circ a_{I'} + r_I)\bigr) * x - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr) \\
&\quad + 2a_{I'} * \bigl(\rho_{a_{i_1}} \eta(\lambda) \circ x\bigr) - r_I * \bigl(\eta(\lambda) \circ x\bigr) \\
&\stackrel{(2)}{=} 2\lambda T_{a_{i_1}} \circ (a_{I'} * x) - 2\lambda \circ \bigl(a_{I'} * (\rho_{a_{i_1}} \circ x)\bigr) - 2\bigl(\zeta(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\bigr) * x \\
&\quad - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr) + 2a_{I'} * \bigl(\rho_{a_{i_1}} \eta(\lambda) \circ x\bigr) \\
&= 2[\lambda, T_{a_{i_1}}] \circ (a_{I'} * x) + 2T_{a_{i_1}} \lambda \circ (a_{I'} * x) - 2\lambda \circ \bigl(a_{I'} * (\rho_{a_{i_1}} \circ x)\bigr) \\
&\quad - 2\bigl(\zeta(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\bigr) * x - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr) + 2a_{I'} * \bigl(\rho_{a_{i_1}} \eta(\lambda) \circ x\bigr) \\
&\stackrel{(3)}{=} 2[\lambda, T_{a_{i_1}}] \circ (a_{I'} * x) + 2T_{a_{i_1}} \lambda \circ (a_{I'} * x) - 2\bigl(\zeta(\lambda) \circ a_{I'}\bigr) * (\rho_{a_{i_1}} \circ x) \\
&\quad - 2a_{I'} * \bigl(\eta(\lambda)\rho_{a_{i_1}} \circ x\bigr) - 2\bigl(\zeta(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\bigr) * x \\
&\quad - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr) + 2a_{I'} * \bigl(\rho_{a_{i_1}} \eta(\lambda) \circ x\bigr) \\
&\stackrel{(4)}{=} 2\bigl(\bigl[\zeta(\lambda), \lambda_{a_{i_1}}\bigr] \circ a_{I'}\bigr) * x + 2T_{a_{i_1}} \lambda \circ (a_{I'} * x) - 2\bigl(\zeta(\lambda) \circ a_{I'}\bigr) * (\rho_{a_{i_1}} \circ x) \\
&\quad - 2\bigl(\zeta(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\bigr) * x - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr) \\
&= -2\bigl(\lambda_{a_{i_1}} \zeta(\lambda) \circ a_{I'}\bigr) * x + 2T_{a_{i_1}} \circ \bigl(\lambda \circ (a_{I'} * x)\bigr) \\
&\quad - 2\bigl(\zeta(\lambda) \circ a_{I'}\bigr) * (\rho_{a_{i_1}} \circ x) - 2T_{a_{i_1}} \circ \bigl(a_{I'} * \bigl(\eta(\lambda) \circ x\bigr)\bigr)
\end{aligned}$$

$$\overset{(5)}{=} 2T_{a_{i_1}} \circ \big(\big(\zeta(\lambda) \circ a_{I'}\big) * x\big) - 2\big(\lambda_{a_{i_1}} \circ \big(\zeta(\lambda) \circ a_{I'}\big)\big) * x$$
$$- 2\big(\zeta(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x),$$

where equality (1) follows from the definitions, equalities (2)–(5) follow by the hypothesis of induction, and the others by manipulations. This shows that without lost of generality we can assume that $\lambda = T_{a_{i_0}}$ for some a_{i_0}. Moreover, since $a_{i_0} * a_{I'} \equiv a_{\hat{I}'} \bmod S(M)_{n-2}$ with $\hat{I}' = (i_2, \dots, i_0, \dots, i_n)$ and $i_2 \leqslant \cdots \leqslant i_0 \leqslant \cdots \leqslant i_n$, then the equality

$$T_{a_{i_0}} \circ (a_I * x) - (\lambda_{a_{i_0}} \circ a_I) * x - a_I * (\rho_{a_{i_0}} \circ x)$$
$$= T_{a_{i_1}} \circ \big((a_{i_0} * a_{I'}) * x\big) - \big(\lambda_{a_{i_1}} \circ (a_{i_0} * a_{I'})\big) * x - (a_{i_0} * a_{I'}) * (\rho_{a_{i_1}} \circ x)$$

and the hypothesis of induction allow us to assume that $i_0 \leqslant i_1$. Finally, if $i_0 \leqslant i_1$ then

$$2(\lambda_{a_{i_0}} \circ a_I) * x = (a_{(i_0,I)} - r_{(i_0,I)}) * x$$
$$= 2T_{a_{i_0}} \circ (a_I * x) - 2a_I * (\rho_{a_{i_0}} \circ x) + r_{(i_0,I)} * x - r_{(i_0,I)} * x$$
$$= 2T_{a_{i_0}} \circ (a_I * x) - 2a_I * (\rho_{a_{i_0}} \circ x),$$

which proves the proposition. □

Proposition 3.6. *For any $x \in S(M)$ and $a \in M$ we have that $1 * x = x * 1 = x$, $a * x = 2\lambda_a \circ x$ and $x * a = 2\rho_a \circ x$.*

Proof. Observe that if $\delta_a = \eta(\mathrm{ad}_a)$ then $\zeta(\delta_a) = \delta_a$, so

$$\delta_a \circ (x * 1 - x) = \big(\zeta(\delta_a) \circ x\big) * 1 + x * (\mathrm{ad}_a \circ 1) - \delta_a \circ x$$
$$= (\delta_a \circ x) * 1 - \delta_a \circ x. \tag{11}$$

Consider now $S = \mathrm{span}\langle \delta_{a_1} \cdots \delta_{a_n} \circ 1 \mid a_1, \dots, a_n \in M, n \in \mathbb{N}\rangle$. We keep the convention that if $n = 0$ then $\delta_{a_1} \cdots \delta_{a_n} \circ 1 = 1$. Observe that by Eq. (11) $x * 1 - x = 0$ for any $x \in S$. Therefore, to get the first part of the proposition it suffices to show that $S(M) = S$. Since $\delta_a \circ 1 = (-2\lambda_a - \rho_a) \circ 1 = -3/2a$ then $S(M)_1 \subseteq S$. Given a_I with $I = (i_1, \dots, i_n)$ then $\delta_{a_{i_1}} \circ a_{I'} = (-3/2T_{a_{i_1}} - 1/2\mathrm{ad}_{a_{i_1}}) \circ a_{I'} = -3/2a_I - 1/2\mathrm{ad}_{a_{i_1}} \circ a_{I'}$, thus by Lemma 3.3 and induction we have that $a_I \in S$. Therefore $S(M) = S$.

Finally, $\rho_a \circ x = \rho_a \circ (x * 1) = (\zeta(\rho_a) \circ x) * 1 + x * (\eta(\rho_a) \circ 1) = -\rho_a \circ x + x * (T_a \circ 1)$ so $x * a = 2\rho_a \circ x$. □

4. Malcev algebras as primitive elements in their enveloping algebras

Let C be an algebra with unit 1 over a field F and assume that there exists an algebra homomorphism $\delta : C \to C \otimes_F C$. An element $p \in C$ is called *primitive* with respect to δ (or simply δ-*primitive*) if $\delta(p) = 1 \otimes p + p \otimes 1$.

If L is a Lie algebra and $U(L)$ is its universal enveloping algebra, then it is easy to see that the diagonal mapping

$$\Delta : L \to L \otimes L, \qquad \Delta(l) = 1 \otimes l + l \otimes 1, \quad l \in L,$$

may be extended to an algebra homomorphism of $U(L)$ to $U(L) \otimes_F U(L)$, which we will also denote by Δ. The well-known Friedrichs criterion (see [1]) says that if F has characteristic 0 then the set of Δ-primitive elements of $U(L)$ coincides with L.

Note that in [11] the Friedrichs criterion was generalized for primitive elements in free nonassociative algebras.

We will now show that this criterion admits a generalization for Malcev algebras and their universal enveloping algebras.

Proposition 4.1. *Let M be a Malcev algebra over ϕ and $U(M)$ be its universal enveloping algebra. Then, the diagonal mapping*

$$\Delta : M \to M \otimes M, \qquad \Delta(l) = 1 \otimes l + l \otimes 1, \quad l \in M,$$

may be extended to an algebra homomorphism of $U(M)$ to $U(M) \otimes_\phi U(M)$.

Proof. Evidently, it suffices to prove that $\Delta(M) \subseteq \mathrm{N}_{\mathrm{alt}}(U(M) \otimes U(M))$. Let $m \in M$, $a, b, c, d \in U(M)$, then we have

$$\begin{aligned}
\big(\Delta(m), a \otimes b, c \otimes d\big) &= ac \otimes (m, b, d) + (m, a, c) \otimes bd \\
&= -ac \otimes (b, m, d) - (a, m, c) \otimes bd \\
&= -(a \otimes b, 1 \otimes m, c \otimes d) - (a \otimes b, m \otimes 1, c \otimes d) \\
&= -\big(a \otimes b, \Delta(m), c \otimes d\big).
\end{aligned}$$

Similarly, $(\Delta(m), a \otimes b, c \otimes d) = (a \otimes b, c \otimes d, \Delta(m))$, which proves the proposition. □

Theorem 4.2. *Let F be a field of characteristic 0 and let M be a Malcev algebra over F. Then M coincides with the set of Δ-primitive elements in the universal enveloping algebra $U(M)$.*

Proof. Let us rewrite the basis (9) of $U(M)$ in the form

$$a_1^{k_1} a_2^{k_2} \cdots a_s^{k_s}, \quad k_i \geqslant 0. \tag{12}$$

Given an element

$$f = \sum_{k_1, k_2, \ldots, k_s} \alpha_{k_1, k_2, \ldots, k_s} a_1^{k_1} a_2^{k_2} \cdots a_s^{k_s}, \quad \alpha_{k_1, k_2, \ldots, k_s} \in F,$$

then

$$\Delta(f) = f \otimes 1 + 1 \otimes f + \sum_r v_r \otimes w_r$$
$$+ \sum_{j=1}^{s} a_j \otimes \left(\sum_{k_1+k_2+\cdots+k_s>1} k_j \, \alpha_{k_1,k_2,\ldots,k_s} a_1^{k_1} a_2^{k_2} \cdots a_j^{k_j-1} \cdots a_s^{k_s} \right),$$

where v_r is a nonempty word of type (12) and the length of v_r is more than 1. In case that f is primitive we must have

$$\sum_{j=1}^{s} a_j \otimes \left(\sum_{k_1+k_2+\cdots+k_s>1} k_j \, \alpha_{k_1,k_2,\ldots,k_s} a_1^{k_1} a_2^{k_2} \cdots a_j^{k_j-1} \cdots a_s^{k_s} \right) + \sum_r v_r \otimes w_r = 0.$$

Since the elements $u \otimes v$, where u, v are of type (12), form a base of $U(M) \otimes_F U(M)$, the last equality yields $w_r = 0$ for all r, and

$$\sum_{k_1+k_2+\cdots+k_s>1} k_j \, \alpha_{k_1,k_2,\ldots,k_s} a_1^{k_1} a_2^{k_2} \cdots a_j^{k_j-1} \cdots a_s^{k_s} = 0$$

for every $j = 1, \ldots, s$ with $k_j > 0$. Therefore, for every summand in f with $\sum_i k_i > 1$, we have $\alpha_{k_1,k_2,\ldots,k_s} = 0$, and so $f \in M$. $\square$

5. An extension of Ado–Iwasawa theorem to Malcev algebras

The theorem of Ado–Iwasawa says that any finite-dimensional Lie algebra has a faithful finite-dimensional representation. For Malcev algebras Filippov [3] proved that this theorem does not hold. He shows that a free nilpotent Malcev algebra of index 8 on a set of 6 generators over a unital commutative associative ring containing $1/6$ has no faithful representations.

For Lie algebras the Poincaré–Birkhoff–Witt Theorem says that any Lie algebra L is a subalgebra of A^- for some unital associative algebra A. In the case that L is finite-dimensional, then the theorem of Ado–Iwasawa says that A can be taken finite-dimensional too. In the previous sections we have shown that any Malcev algebra M is obtained as a subalgebra of $\mathrm{N_{alt}}(A)$ for some algebra A with the commutator product. In this section we will use the classical theorem of Ado–Iwasawa to prove that if M is finite-dimensional then A can be taken finite-dimensional too.

Lemma 5.1. *Let $\mathcal{L}$ be a finite-dimensional Lie algebra, σ an automorphism of $\mathcal{L}$ with $\sigma^2 = \mathrm{id}$, and $\mathcal{L}_+ = \{x \in \mathcal{L} \mid \sigma(x) = x\}$. Then, there exists a finite-dimensional module V and $e \in V$ such that*

$$\mathcal{L}_+ = \{x \in \mathcal{L} \mid xe = 0\}.$$

Proof. By the Ado–Iwasawa theorem, $\mathcal{L}$ has a finite-dimensional faithful representation W. Let $\{w_1, \ldots, w_n\}$ be a basis of W and $\{f_1, \ldots, f_n\}$ its dual. In $W \otimes W^*$ the element

$e = w_1 \otimes f_1 + \cdots + w_n \otimes f_n$ spans a trivial submodule. However, in $V = W \otimes (W^*)_\sigma$ (the action in W^* is twisted by σ) we have

$$xe = \sum_{i=1}^{n} xw_i \otimes f_i + w_i \otimes \sigma(x) f_i = \sum_{i=1}^{n} -w_i \otimes xf_i + w_i \otimes \sigma(x) f_i$$
$$= \sum_{i=1}^{n} w_i \otimes \bigl(\sigma(x) - x\bigr) f_i,$$

therefore, $xe = 0$ if and only if $(\sigma(x) - x) f_i = 0$ for all $i = 1, \ldots, n$. Since W is a faithful representation this is equivalent to saying that $x \in \mathcal{L}_+$. □

Theorem 5.2. *Let M be a finite-dimensional Malcev algebra over a field of characteristic $\neq 2, 3$. Then, there exists a unital finite-dimensional algebra A and a monomorphism of Malcev algebras $\iota : M \to \mathrm{N}_{\mathrm{alt}}(A)$.*

Proof. Let $\mathcal{L} = \mathcal{L}(M)$ and $\sigma = \zeta\eta\zeta$ be the automorphism responsible for the $\mathbb{Z}_2$-gradation $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ of $\mathcal{L}(M)$. By the previous lemma, we can choose a finite-dimensional $\mathcal{L}$-module V and $e \in V$ such that $\mathcal{L}_+ = \{x \in \mathcal{L} \mid xe = 0\}$. Looking at V as an $U(\mathcal{L})$-module, we define the left ideal of finite codimension

$$I = \bigl\{ y \in U(\mathcal{L}) \mid ye = 0 \bigr\}$$

which obviously contains $K = U(\mathcal{L})\mathcal{L}_+$. I/K is a submodule of $U(\mathcal{L})/K$ of finite codimension, so $\mathcal{I} = \theta^{-1}(I/K)$ is a submodule of $S(\mathcal{L}_-)$ of finite codimension too. Identifying $S(\mathcal{L}_-)$ with $U(M)$ this means that $\mathcal{I} \subseteq U(M)$ is invariant by left and right multiplications by elements of M, so it is an ideal of $U(M)$ (see Lemma 4.4 in [7]) of finite codimension. Moreover, since $K \subseteq I$ and $I \cap \mathcal{L}_- = 0$ then

$$\theta(\mathcal{I} \cap M) = I \cap (\mathcal{L}_- + K)/K = (I \cap \mathcal{L}_-) + K/K = K/K = 0,$$

which proves that $\mathcal{I} \cap M = 0$. Finally, consider the unital finite-dimensional algebra $A = U(M)/\mathcal{I}$. Since $M \cap \mathcal{I} = 0$, then the map $\iota : M \to \mathrm{N}_{\mathrm{alt}}(A)$ induced by $\iota : M \to U(M)$ is a monomorphism, as desired. □

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