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An envelope for Bol algebras

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Abstract

We prove that Bol algebras arise as primitive elements of certain bialgebras which generalize the usual universal enveloping algebras of Lie and Malcev algebras. The Bol algebra is located inside the generalized left alternative nucleus of the envelope, and its binary and ternary products are naturally recovered from the product of the envelope.

Keywords: Bol algebras; Universal enveloping algebras; Loops and nonassociative algebras

1. Introduction

A vector space V equipped with a trilinear operation $[a, b, c]$ is called a *Lie triple system* if

$$[a, a, b] = 0,$$
$$[a, b, c] + [b, c, a] + [c, a, b] = 0,$$
$$\big[x, y, [a, b, c]\big] = \big[[x, y, a], b, c\big] + \big[a, [x, y, b], c\big] + \big[a, b, [x, y, c]\big]$$

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for all $x, y, a, b, c \in V$. A (*left*) *Bol algebra* $(V, [\ ,\ ,\ ], [\ ,\ ])$ is a Lie triple system $(V, [\ ,\ ,\ ])$ with an additional bilinear skew-symmetric operation $[a, b]$ satisfying

$$\big[a, b, [c, d]\big] = \big[[a, b, c], d\big] + \big[c, [a, b, d]\big] + \big[c, d, [a, b]\big] + \big[[a, b], [c, d]\big]. \tag{1}$$

Bol algebras were introduced in differential geometry to study smooth Bol loops [4,5].

A set $\mathcal{Q}$ with a binary operation $x \cdot y$ is called a *right loop* if for any $y \in \mathcal{Q}$ the right multiplication operator $R_y : x \mapsto x \cdot y$ is bijective, and there exists an element $\varepsilon \in \mathcal{Q}$, the *left neutral*, such that $\varepsilon \cdot y = y$ for any $y \in \mathcal{Q}$. It is also customary to write x/y instead of $R_y^{-1}(x)$. A *left loop* is a set $\mathcal{Q}$ with a binary operation $x \cdot y$ such that for any $x \in \mathcal{Q}$ the left multiplication operator $L_x : y \mapsto x \cdot y$ is bijective, and there exists an element $\varepsilon \in \mathcal{Q}$, the *right neutral*, such that $x \cdot \varepsilon = x$ for any $x \in \mathcal{Q}$. The element $L_x^{-1}(y)$ is usually denoted by $x \backslash y$. In case that $\langle \mathcal{Q}, \cdot, \varepsilon \rangle$ is simultaneously a left and right loop then it is called a *loop* with *identity element* ε. A *left smooth loop* $\mathcal{M}$ is a left loop equipped with a structure of smooth manifold so that the maps $(x, y) \mapsto x \cdot y$ and $(x, y) \mapsto x \backslash y$ are smooth (see [4] for a local version). The most well-known examples of loops and smooth loops are groups and Lie groups, respectively, though many other families of loops have come into scene over the years, Moufang and Bol loops among others.

A *right Bol loop* $\langle \mathcal{Q}, \cdot, \varepsilon \rangle$ is a right loop that satisfies the *right Bol property*

$$x \cdot \big[(a \cdot y) \cdot a\big] = \big[(x \cdot a) \cdot y\big] \cdot a$$

for all $a, x, y \in \mathcal{Q}$. Similarly, a *left Bol loop* satisfies the identity

$$a \cdot \big[x \cdot (a \cdot y)\big] = \big[a \cdot (x \cdot a)\big] \cdot y.$$

In any loop the following identities are equivalent:

$((ax)a)y = a(x(ay))$ *left Moufang identity*,
$((xa)y)a = x(a(ya))$ *right Moufang identity*,
$(ax)(ya) = (a(xy))a$ *middle Moufang identity*.

A loop is called a *Moufang loop* if satisfies any of them.

The classical correspondence between Lie groups and Lie algebras has been successfully extended to smooth loops. Firstly, it was achieved for Moufang loops [1]. The tangent space at the identity inherits a skew-symmetric product $[\ ,\ ]$ from the product of the so called *fundamental* vector fields over the loop, and with this product the tangent space becomes a *Malcev algebra*, i.e.,

$$[x, x] = 0 \quad \text{and} \quad \big[[x, y], [x, z]\big] = \big[\big[[x, y], z\big], x\big] + \big[\big[[y, z], x\big], x\big] + \big[\big[[z, x], x\big], y\big]$$

for any x, y, z. Later, the correspondence was studied for (local) left Bol loops with two-sided neutral. The picture here is slightly more complicated. On the one hand, the fundamental vector fields form a Lie triple system that naturally induces a structure of Lie triple system on the tangent space at the identity. On the other hand, the product of vector fields induces a binary product on this tangent space so that it becomes a Bol algebra. In contrast to Lie groups and Moufang loops, both products, binary and ternary, are needed to locally recover and classify the Bol loop [5]. For general loops the correspondence is established in terms of a family of multilinear operations called *hyperalgebra* [6].

So far, the geometrical origins of Lie, Malcev and Bol algebras are well-understood. However, algebraic settings for these algebras are less known. The Poincaré–Birkhoff–Witt theorem says that any Lie algebra L is a subalgebra of some unital associative algebra considered with the commutator product $[x, y] = xy - yx$. The universal enveloping algebra $U(L)$ is the universal object with respect to this property. This theorem was extended in [3] to Malcev algebras. Given an arbitrary algebra A, the *generalized alternative nucleus* of A is defined as $\mathrm{N_{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a)\ \forall x, y \in A\}$, where (x, y, z) denotes the associator of x, y and z. $\mathrm{N_{alt}}(A)$ is always a Malcev algebra with the commutator product $[x, y] = xy - yx$. Moreover, given a Malcev algebra M, then there exists a pair $(U(M), \iota)$, where $U(M)$ is a unital algebra and $\iota : M \hookrightarrow \mathrm{N_{alt}}(U(M)) \subseteq U(M)$ is a monomorphism of Malcev algebras, with the following universal property:

> Given a unital algebra A and $\iota' : M \to \mathrm{N_{alt}}(A) \subseteq A$ a homomorphism of Malcev algebras then there exists a unique homomorphism $\varphi : U(M) \to A$ of unital algebras such that $\iota' = \varphi \circ \iota$.

The aim of this paper is to construct an envelope for Bol algebras by extending the techniques developed in [3]. To start with, we define the *generalized left alternative nucleus* of an algebra A as

$$\mathrm{LN_{alt}}(A) = \bigl\{a \in A \bigm| (a, x, y) = -(x, a, y)\ \forall x, y \in A\bigr\}.$$

$\mathrm{LN_{alt}}(A)$ is a Lie triple system with the triple product

$$[a, b, c] = a(bc) - b(ac) - c[a, b].$$

In fact, any subspace V of $\mathrm{LN_{alt}}(A)$ closed under the triple product $[\ ,\ ,\ ]$ and the commutator product $[\ ,\ ]$ is a Bol algebra with these operations (see Section 2).

The main result in this paper is

Main result. *Let $(V, [\ ,\ ,\ ], [\ ,\ ])$ be a Bol algebra, then there exist a unital algebra $U(V)$ and a linear injective map $\iota : V \hookrightarrow \mathrm{LN_{alt}}(U(V))$, $a \mapsto a$, such that*

$$\iota\bigl([a, b]\bigr) = ab - ba \quad \text{and} \quad \iota\bigl([a, b, c]\bigr) = a(bc) - b(ac) - c[a, b],$$

and the following universal property holds:

> *For any unital algebra A and any linear map $\iota' : V \to \mathrm{LN_{alt}}(A)$, $a \mapsto a'$, with $\iota'([a, b]) = a'b' - b'a'$ and $\iota'([a, b, c]) = a'(b'c') - b'(a'c') - c'[a', b']$ there exists a homomorphism $\varphi : U(V) \to A$ of unital algebras satisfying $\iota' = \varphi \circ \iota$.*

The algebra $U(V)$ will be called the *universal enveloping algebra* of the Bol algebra $(V, [\ ,\ ,\ ], [\ ,\ ])$. As in the classical case, this algebra admits a Poincaré–Birkhoff–Witt type basis. Though, in general, $\mathrm{LN}_{\mathrm{alt}}(U(V))$ may be larger than V, $U(V)$ can be endowed with a natural structure of bialgebra so that if the characteristic of the ground field F is zero then V is recovered as the set of primitive elements of $U(V)$.

Throughout this paper F will denote a field of characteristic $\neq 2$.

2. Lie enveloping algebras of Bol algebras

Given an algebra A and $d_1, d_2, d_3 \in \mathrm{End}_F(A)$, (d_1, d_2, d_3) is called a ternary derivation of A if $d_1(xy) = d_2(x)y + xd_3(y)$ for any $x, y \in A$ [2]. In case that A is a unital algebra then the relations

$$d_1 = d_2 + R_{d_3(1)} \quad \text{and} \quad d_1 = d_3 + L_{d_2(1)} \tag{2}$$

hold, where L_a and R_a stand for the left and right multiplication operators by a. We will denote $L_a + R_a$ by T_a for short. Observe that $a \in \mathrm{LN}_{\mathrm{alt}}(A)$ if and only if $(L_a, T_a, -L_a)$ is a ternary derivation of A.

Lemma 1. *Let A be a unital algebra and $d, d' \in \mathrm{End}_F(A)$. Then, $(d, d', -d)$ is a ternary derivation of A if and only if $d = L_a$ and $d' = T_a$ for some $a \in \mathrm{LN}_{\mathrm{alt}}(A)$.*

Proof. Assume that $(d, d', -d)$ is a ternary derivation. By (2), $d = L_a$ and $d' = T_a$ with $a = 1/2d'(1)$. Since $(d, d', -d)$ is a ternary derivation, then $a \in \mathrm{LN}_{\mathrm{alt}}(A)$. □

Recall the notation $[a, b, c] = a(bc) - b(ac) - c[a, b]$.

Proposition 2. *Let A be an algebra and $a, b, c \in \mathrm{LN}_{\mathrm{alt}}(A)$. Then*

(i) $[L_a, L_b] = [T_a, T_b] + R_{[a,b]}$,
(ii) $[[L_a, L_b], L_c] = L_{[a,b,c]}$,
iii) $[[T_a, T_b], T_c] = T_{[a,b,c]}$,
(iv) $(\mathrm{LN}_{\mathrm{alt}}(A), [\ ,\ ,\ ])$ *is a Lie triple system.*

Proof. Let $A^{\#}$ be the unitization of A. Since $a, b, c \in \mathrm{LN}_{\mathrm{alt}}(A^{\#})$ and the ternary derivations form a Lie algebra, then $([L_a, L_b], [T_a, T_b], [L_a, L_b])$ is a ternary derivation of $A^{\#}$, which by (2) shows that $[L_a, L_b] = [T_a, T_b] + R_{[a,b]}$. Moreover, $([[L_a, L_b], L_c], [[T_a, T_b], T_c], -[[L_a, L_b], L_c])$ is also a ternary derivation, so Lemma 1 implies that $[[L_a, L_b], L_c] = L_e$ and $[[T_a, T_b], T_c] = T_e$ for some $e \in \mathrm{LN}_{\mathrm{alt}}(A^{\#})$. Evaluating these operators on 1 leads to $[[L_a, L_b], L_c] = L_{[a,b,c]}$ and $[[T_a, T_b], T_c] = T_{[a,b,c]}$. Part (iv) follows from (ii). □

Proposition 3. *Let A be an algebra and V a subspace of $\mathrm{LN}_{\mathrm{alt}}(A)$ closed under $[\ ,\ ,\ ]$ and $[\ ,\ ]$. Then $(V, [\ ,\ ,\ ], [\ ,\ ])$ is a Bol algebra.*

Proof. Given $a, b, c, e \in \mathrm{LN}_{\mathrm{alt}}(A)$, in $A^{\#}$

$$\big[[L_a, L_b], [L_c, L_e]\big](1) = \begin{cases} [L_{[a,b,c]}, L_e](1) + [L_c, L_{[a,b,e]}](1) \\ \quad = \big[[a,b,c], e\big] + \big[c, [a,b,e]\big], \\ \big[[T_a, T_b] + R_{[a,b]}, [T_c, T_e] + R_{[c,e]}\big](1) \\ \quad \overset{\langle 1\rangle}{=} \big[a, b, [c,e]\big] + [c,e][a,b] - \big[c, e, [a,b]\big] - [a,b][c,e] \end{cases}$$

holds, where $\langle 1\rangle$ follows from the identities $[T_x, T_y](1) = 0$ and $[T_x, T_y](z) = [x, y, z]$. So, $[a, b, [c, e]] = [[a, b, c], e] + [c, [a, b, e]] + [c, e, [a, b]] + [[a, b], [c, e]]$. □

Let us recall some definitions and constructions on Bol algebras. Given a Bol algebra $(V, [\ ,\ ,\ ], [\ ,\ ])$ over F, a *pseudodifferentiation* of $(V, [\ ,\ ,\ ], [\ ,\ ])$ is a linear map $D : V \to V$ for which there exists $z \in V$ (the *companion* of D) with

$$D\big([c,d]\big) = \big[D(c), d\big] + \big[c, D(d)\big] + [c, d, z] + \big[z, [c,d]\big]$$

for all $c, d \in V$. The companion is not necessarily unique and it depends on D. The set of all companions of D is denoted by $\mathrm{com}(D)$. With the notation $D_{a,b} : c \mapsto [a, b, c]$, condition (1) is equivalent to saying that $D_{a,b}$ is a pseudodifferentiation with companion $[a, b]$ for any $a, b \in V$. The pseudodifferentiations of V form a Lie algebra, denoted by $\mathrm{pder}\, V$, under the natural product $[D, D'] = DD' - D'D$. The Lie subalgebra $\mathrm{ipder}\, V$ generated by $\{D_{a,b} \mid a, b \in V\}$ is called the (*Lie*) *algebra of inner pseudodifferentiations* of V. The *enlarged algebra* $\mathrm{Pder}\, V$ *of pseudodifferentiations* of V is defined as

$$\mathrm{Pder}\, V = \big\{(D, z) \,\big|\, D \in \mathrm{pder}\, V,\ z \in \mathrm{com}(D)\big\},$$

and it becomes a Lie algebra with the product

$$\big[(D, z), (D', z')\big] = \big([D, D'], D(z') - D'(z) - [z, z']\big).$$

The *enlarged algebra of inner pseudodifferentiations* is defined as

$$\mathrm{IPder}\, V = \big\{(D, z) \in \mathrm{Pder}\, V \,\big|\, D \in \mathrm{ipder}\, V,\ z \in \mathrm{com}(D)\big\}.$$

Given a subalgebra K of $\mathrm{Pder}\, V$ with $\mathrm{IPder}\, V \subseteq K$ and a copy $\bar V = \{\bar a \mid a \in V\}$ of V, the product on K is extended to a product on $K \times \bar V$ by

$$\big[(D, z), \bar b\big] = \overline{D(b)}, \qquad [\bar a, \bar b] = \big(D_{a,b}, [a, b]\big)$$

and skew-symmetry to obtain a $\mathbb{Z}_2$-graded Lie algebra $\mathrm{Env}(V, K)$. The subspace

$$L = \big\langle (D, z) - \bar z \in \mathrm{Env}(V, K) \,\big|\, (D, z) \in K \big\rangle$$

is a Lie subalgebra and $\mathrm{Env}(V, K) = L \oplus \bar V$. Furthermore,

$$[\bar a, \bar b] = \big(\big(D_{a,b}, [a, b]\big) - \overline{[a, b]}\big) + \overline{[a, b]}$$

implies that $\overline{[a,b]} = \mathrm{pr}([\bar{a},\bar{b}])$ where pr( ) denotes the projection on $\bar{V}$ parallel to L. Therefore, the binary and triple products on V are recovered as

$$\overline{[a,b]} = \mathrm{pr}\big([\bar{a},\bar{b}]\big) \quad \text{and} \quad \overline{[a,b,c]} = \big[[\bar{a},\bar{b}],\bar{c}\big].$$

The Lie algebra $\mathrm{Env}(V,K)$ is called a *Lie enveloping algebra* of the Bol algebra V.

Looking at $\mathrm{Env}(V,K)$ as $L \oplus \bar{V}$, we may define $\gamma_{a,b} = (D_{a,b},[a,b]) - \overline{[a,b]}$ the projection on L of $[\bar{a},\bar{b}]$. These maps satisfy the relation

$$[\gamma_{a,b},\gamma_{c,d}] = \gamma_{[a,b,c],d} + \gamma_{c,[a,b,d]} + \gamma_{[a,b],[c,d]}. \tag{3}$$

Modeled on these enveloping algebras $\mathrm{Env}(V,K)$ we define another Lie algebra. Let $\{\tau_a \mid a \in V\}$ be a copy of V, and $\mathrm{E}(V)$ the Lie algebra generated by $\{\tau_a \mid a \in V\}$ with relations

$$\big[[\tau_a,\tau_b],\tau_c\big] = \tau_{[a,b,c]}, \tag{4}$$

$$[\delta_{a,b},\delta_{c,d}] = \delta_{[a,b,c],d} + \delta_{c,[a,b,d]} + \delta_{[a,b],[c,d]}, \tag{5}$$

where $\delta_{a,b}$ stands for $[\tau_a,\tau_b] - \tau_{[a,b]}$. By abuse of notation, we continue to write τ_a for the image of τ_a in $\mathrm{E}(V)$. This notation will be fully justified after proving Corollary 5.

Proposition 4. *There exists an automorphism θ of the Lie algebra $\mathrm{E}(V)$ such that $\theta^2 = \mathrm{id}$ and $\theta(\tau_a) = -\tau_a$ for all $a \in V$.*

Proof. Let us use the temporary notation $\tau'_a = -\tau_a$ and $\delta'_{a,b} = [\tau'_a,\tau'_b] - \tau'_{[a,b]}$. In $\mathrm{E}(V)$

$$\begin{aligned}
[\delta'_{a,b},\delta'_{c,d}] &= [\delta_{a,b},\delta_{c,d}] + 2[\delta_{a,b},\tau_{[c,d]}] + 2[\tau_{[a,b]},\delta_{c,d}] + 4[\tau_{[a,b]},\tau_{[c,d]}] \\
&\overset{\langle 1\rangle}{=} \delta_{[a,b,c],d} + \delta_{c,[a,b,d]} + \delta_{[a,b],[c,d]} + 2\tau_{[a,b,[c,d]]} - 2[\tau_{[a,b]},\tau_{[c,d]}] \\
&\quad - 2\tau_{[c,d,[a,b]]} + 2[\tau_{[c,d]},\tau_{[a,b]}] + 4[\tau_{[a,b]},\tau_{[c,d]}] \\
&= \delta_{[a,b,c],d} + \delta_{c,[a,b,d]} + \delta_{[a,b],[c,d]} + 2\tau_{[a,b,[c,d]]} - 2\tau_{[c,d,[a,b]]} \\
&\overset{\langle 2\rangle}{=} \delta_{[a,b,c],d} + \delta_{c,[a,b,d]} + \delta_{[a,b],[c,d]} \\
&\quad + 2\tau_{[[a,b,c],d]} + 2\tau_{[c,[a,b,d]]} + 2\tau_{[[a,b],[c,d]]} \\
&= \delta'_{[a,b,c],d} + \delta'_{c,[a,b,d]} + \delta'_{[a,b],[c,d]}
\end{aligned}$$

holds, where $\langle 1\rangle$ follows from the defining relations of $\mathrm{E}(V)$ and $\langle 2\rangle$ from (1). This shows that (5) is satisfied if we change δ by δ'. Similarly, (4) holds if τ is replaced by τ'. Therefore, there exists a Lie algebra endomorphism θ of $\mathrm{E}(V)$ such that $\theta(\tau_a) = -\tau_a$ $\forall a \in V$. Since θ^2 fixes the generators of $\mathrm{E}(V)$, then $\theta^2 = \mathrm{id}$. □

Corollary 5. *Let $E_+ = \langle [\tau_a,\tau_b] \mid a,b \in V\rangle$ and $E_- = \langle \tau_a \mid a \in V\rangle \subseteq \mathrm{E}(V)$. Then $\mathrm{E}(V) = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-gradation and $E_- \cong V$ as vector spaces.*

Proof. The $\mathbb{Z}_2$-gradation in the statement is the one induced by the automorphism θ of order two, so we only need to show that $E_- \cong V$ as vector spaces. By (3), for any enveloping algebra $\mathrm{Env}(V, K)$ there exists a homomorphism from $\mathrm{E}(V)$ to $\mathrm{Env}(V, K)$ sending τ_a to $\bar{a}$. Since $a \mapsto \bar{a}$ is injective, so $a \mapsto \tau_a$ is. $\square$

3. An envelope for Bol algebras

Let $(V, [\ ,\ ,\ ], [\ ,\ ])$ be a Bol algebra and $E = \mathrm{E}(V)$ the Lie algebra defined in Section 2. Consider the left ideals of the universal enveloping algebra $U(E)$ of E defined by

$$K_L = U(E)\langle \delta_{a,b} \mid a, b \in V\rangle \quad \text{and} \quad K_T = U(E)\big\langle [\tau_a, \tau_b] \mid a, b \in V\big\rangle.$$

These left ideals provide two E-modules

$$U(E)/K_L \quad \text{and} \quad U(E)/K_T.$$

The elements in $U(E)/K_L$ will be denoted by $\bar{x}$ where $x \in U(E)$, while the elements in $U(E)/K_T$ will be denoted by $[x]$. A third module $(U(E)/K_L)_\theta$ appears when we twist the action of E on $U(E)/K_L$ by the automorphism θ. As vector spaces $(U(E)/K_L)_\theta$ and $U(E)/K_L$ are the same, but the action of E on $(U(E)/K_L)_\theta$ is defined by $d \circ \bar{x} = \overline{\theta(d)x}$.

Let us introduce some temporary notation:

- $\{\tau_{a_i} \mid i \in \Lambda\}$ denotes an ordered basis of E_-.
- $\tau_\emptyset = 1$, $\tau_I = \tau_{a_{i_1}} \cdots \tau_{a_{i_n}}$ if $I = (i_1, \ldots, i_n)$.
- $I' = (i_2, \ldots, i_n)$ and $|I| = n$ if $I = (i_1, \ldots, i_n)$.
- If $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_m)$, then $I * J = (k_1, \ldots, k_{n+m})$ with $\{k_1, \ldots, k_{n+m}\} = \{i_1, \ldots, i_n, j_1, \ldots, j_m\}$ and $k_1 \leqslant \cdots \leqslant k_{n+m}$.

By the classical Poincaré–Birkhoff–Witt theorem we know that

$$\big\{[\tau_I] \;\big|\; I = (i_1, \ldots, i_n) \text{ with } i_1 \leqslant \cdots \leqslant i_n \text{ and } n \geqslant 0\big\}$$

is a basis of $U(E)/K_T$. The subspaces

$$\big(U(E)/K_T\big)_n = \big\langle [\tau_I] \;\big|\; |I| \leqslant n\big\rangle$$

form a filtration of $U(E)/K_T$. In the same way, $U(E)/K_L$ has a basis $\{\bar{\tau}_I \mid, I = (i_1, \ldots, i_n)$ with $i_1 \leqslant \cdots \leqslant i_n$ and $n \geqslant 0\}$ and a filtration given by the subspaces $(U(E)/K_L)_n = \langle \bar{\tau}_I \mid |I| \leqslant n\rangle$. It is easy to prove that

$$\begin{aligned} &\tau_{a_{i_0}}[\tau_I] \equiv [\tau_{I*(i_0)}] \quad \operatorname{mod}\big(U(E)/K_T\big)_{|I|} \quad \text{and} \\ &\tau_{a_{i_0}}\bar{\tau}_I \equiv \bar{\tau}_{I*(i_0)} \quad \operatorname{mod}\big(U(E)/K_L\big)_{|I|}. \end{aligned} \tag{6}$$

The following theorem is similar to [3, Proposition 3.1].

Theorem 6. *There exists a homomorphism of E-modules*

$$*: U(E)/K_T \otimes \big(U(E)/K_L\big)_\theta \to U(E)/K_L$$

defined recursively by

$$[1] * \bar{x} = \bar{x} \quad \textit{and} \quad [\tau_I] * \bar{x} = \tau_{a_{i_1}}\big([\tau_{I'}] * \bar{x}\big) - [\tau_{I'}] * \overline{\theta(\tau_{a_{i_1}})x}.$$

Proof. We will prove that

$$d\big([\tau_I] * \bar{x}\big) - \big(d[\tau_I]\big) * \bar{x} - [\tau_I] * \big(\theta(d)\bar{x}\big) = 0$$

by induction on $|I|$. For $|I| = 0$ and $d = d_0 + \tau_a$ with $d_0 \in E_+$ we have

$$\begin{aligned}
&d\big([1] * \bar{x}\big) - \big(d[1]\big) * \bar{x} - [1] * \big(\theta(d)\bar{x}\big)\\
&\quad = \big(d - \theta(d)\big)\bar{x} - [\tau_a] * \bar{x} = 2\tau_a\bar{x} - [\tau_a] * \bar{x}\\
&\quad = 2\tau_a\bar{x} - \tau_a\big([1] * \bar{x}\big) + [1] * \theta(\tau_a)\bar{x} = 2\tau_a\bar{x} - \tau_a\bar{x} - \tau_a\bar{x} = 0.
\end{aligned}$$

In the general case we observe that

$$\begin{aligned}
&d\big([\tau_I] * \bar{x}\big) - \big(d[\tau_I]\big) * \bar{x} - [\tau_I] * \big(\theta(d)\bar{x}\big)\\
&\quad = d\big(\tau_{a_{i_1}}\big([\tau_{I'}] * \bar{x}\big) - [\tau_{I'}] * \theta(\tau_{a_{i_1}})\bar{x}\big) - \big(d[\tau_I]\big) * \bar{x} - \tau_{a_{i_1}}\big([\tau_{I'}] * \theta(d)\bar{x}\big)\\
&\qquad + [\tau_{I'}] * \big(\theta(\tau_{a_{i_1}})\theta(d)\bar{x}\big)\\
&\quad \overset{\langle 1\rangle}{=} d\tau_{a_{i_1}}\big([\tau_{I'}] * \bar{x}\big) - \big(d[\tau_{I'}]\big) * \theta(\tau_{a_{i_1}})\bar{x} - [\tau_{I'}] * \big(\theta(d)\theta(\tau_{a_{i_1}})\bar{x}\big)\\
&\qquad - \big(d[\tau_I]\big) * \bar{x} - \tau_{a_{i_1}}\big([\tau_{I'}] * \theta(d)\bar{x}\big) + [\tau_{I'}] * \big(\theta(\tau_{a_{i_1}})\theta(d)\bar{x}\big)\\
&\quad \overset{\langle 2\rangle}{=} [d, \tau_{a_{i_1}}]\big([\tau_{I'}] * \bar{x}\big) + \tau_{a_{i_1}}\big(\big(d[\tau_{I'}]\big) * \bar{x} + [\tau_{I'}] * \theta(d)\bar{x}\big) - \big(d[\tau_{I'}]\big) * \theta(\tau_{a_{i_1}})\bar{x}\\
&\qquad - [\tau_{I'}] * \big[\theta(d), \theta(\tau_{a_{i_1}})\big]\bar{x} - \big(d[\tau_I]\big) * \bar{x} - \tau_{a_{i_1}}\big([\tau_{I'}] * \theta(d)\bar{x}\big)\\
&\quad \overset{\langle 3\rangle}{=} \big([d, \tau_{a_{i_1}}][\tau_{I'}]\big) * \bar{x} + \tau_{a_{i_1}}\big(\big(d[\tau_{I'}]\big) * \bar{x}\big) - \big(d[\tau_{I'}]\big) * \theta(\tau_{a_{i_1}})\bar{x} - \big(d[\tau_I]\big) * \bar{x}\\
&\quad \overset{\langle 4\rangle}{=} \tau_{a_{i_1}}\big(\big(d[\tau_{I'}]\big) * \bar{x}\big) - \big(\tau_{a_{i_1}}d[\tau_{I'}]\big) * \bar{x} - \big(d[\tau_{I'}]\big) * \theta(\tau_{a_{i_1}})\bar{x},
\end{aligned}$$

where $\langle 1\rangle$ follows by using induction on $d([\bar{\tau}_{I'}] * \theta(\tau_{a_{i_1}})\bar{x})$, $\langle 2\rangle$ by subtracting $\tau_{a_{i_1}} d([\bar{\tau}_{I'}] * \bar{x})$ and adding $\tau_{a_{i_1}}((d[\tau_{I'}]) * \bar{x} + [\tau_{I'}] * \theta(d)\bar{x})$ (they both are equal by induction); $\langle 3\rangle$ follows by induction on $[d, \tau_{a_{i_1}}]([\tau_{I'}] * \bar{x})$, some simplifications and the fact that θ is an automorphism, and $\langle 4\rangle$ by simplification. So, it suffices to prove the induction step with $d = \tau_{a_{i_0}}$ for some $\tau_{a_{i_0}}$. If $i_0 \leqslant i_1$ then it follows from the very definition of $*$. Therefore, we may assume that $i_0 > i_1$. By (6), $\tau_{a_{i_0}}[\tau_{I'}] = [\tau_{I' * (i_0)}] + r$ with $r \in (U(E)/K_T)_{|I'|}$, and by the previous computations and the hypothesis of induction we have

$$\begin{aligned}
&\tau_{a_{i_0}}\big([\tau_I]*\bar{x}\big)-\big(\tau_{a_{i_0}}[\tau_I]\big)*\bar{x}-[\tau_I]*\theta(\tau_{a_{i_0}})\bar{x}\\
&\quad=\tau_{a_{i_1}}\big(\big(\tau_{a_{i_0}}[\tau_{I'}]\big)*\bar{x}\big)-\big(\tau_{a_{i_1}}\tau_{a_{i_0}}[\tau_{I'}]\big)*\bar{x}-\big(\tau_{a_{i_0}}[\tau_{I'}]\big)*\theta(\tau_{a_{i_1}})\bar{x}\\
&\quad=\tau_{a_{i_1}}\big([\tau_{I'*(i_0)}]*\bar{x}\big)-\big(\tau_{a_{i_1}}[\tau_{I'*(i_0)}]\big)*\bar{x}-[\tau_{I'*(i_0)}]*\theta(\tau_{a_{i_1}})\bar{x}
\end{aligned}$$

which by definition vanishes. □

While this homomorphism $*$ induces a product on $U(E)/K_L$, by identifying $[\tau_I]$ with $\bar{\tau}_I$, the algebra thus obtained has no unit element in general (observe that, for instance, $[\tau_a]*\bar{1}=2\bar{\tau}_a$). A natural way to overcome this defect is to use an isotope.

Lemma 7. *The map $\phi: U(E)/K_T \to U(E)/K_L$ given by $[x]\mapsto [x]*\bar{1}$ is a linear isomorphism.*

Proof. We first observe that $[\tau_I]*\bar{\tau}_J \equiv 2^{|I|}\bar{\tau}_{I*J} \mod (U(E)/K_L)_{|I|+|J|-1}$. In fact, by definition and induction,

$$\begin{aligned}
[\tau_I]*\bar{\tau}_J &= \tau_{a_{i_1}}\big([\tau_{I'}]*\bar{\tau}_J\big)-[\tau_{I'}]*\theta(\tau_{a_{i_1}})\bar{\tau}_J \equiv 2^{|I|-1}\tau_{a_{i_1}}\bar{\tau}_{I'*J}+[\tau_{I'}]*\bar{\tau}_{(i_1)*J}\\
&\equiv 2^{|I|-1}\bar{\tau}_{I*J}+2^{|I|-1}\bar{\tau}_{I*J}=2^{|I|}\bar{\tau}_{I*J} \mod \big(U(E)/K_L\big)_{|I|+|J|-1}.
\end{aligned}$$

As a particular case, $\phi([\tau_I])\equiv 2^{|I|}\bar{\tau}_I \mod (U(E)/K_L)_{|I|-1}$. Thus, since ϕ maps a basis of $U(E)/K_T$ onto a basis of $U(E)/K_L$, ϕ must be a linear isomorphism. □

The product that we will consider on $U(E)/K_L$ is

$$U(E)/K_L\otimes U(E)/K_L\to U(E)/K_L,\quad \bar{x}\otimes\bar{y}\mapsto \phi^{-1}(\bar{x})*\bar{y}. \tag{7}$$

Theorem 8. *The algebra $U(E)/K_L$ defined by* (7) *satisfies*

(i) $\bar{1}$ *is the unit element,*
(ii) $\bar{\tau}_a\in \mathrm{LN}_{\mathrm{alt}}(U(E)/K_L)$ *for any* $a\in V$,
(iii) $\bar{\tau}_{[a,b]}=[\bar{\tau}_a,\bar{\tau}_b]$ *and* $\bar{\tau}_{[a,b,c]}=\bar{\tau}_a(\bar{\tau}_b\bar{\tau}_c)-\bar{\tau}_b(\bar{\tau}_a\bar{\tau}_c)-\bar{\tau}_c[\bar{\tau}_a,\bar{\tau}_b]$.

Proof. Part (i) is obvious from construction. Previous to obtain parts (ii) and (iii), we note that $\bar{\tau}_a\bar{y}=\tau_a\bar{y}$. In fact, $[\tau_a]*\bar{1}=2\bar{\tau}_a$, so $\phi^{-1}(\bar{\tau}_a)=1/2[\tau_a]$ and

$$\bar{\tau}_a\bar{y}=\phi^{-1}(\bar{\tau}_a)*\bar{y}=\frac{1}{2}[\tau_a]*\bar{y}=\tau_a\bar{y}.$$

Now,

$$\begin{aligned}
\bar{\tau}_a(\bar{x}\bar{y}) &= \tau_a\big(\phi^{-1}(\bar{x})*\bar{y}\big)=\big(\tau_a\phi^{-1}(\bar{x})\big)*\bar{y}-\phi^{-1}(\bar{x})*(\tau_a\bar{y})\\
&=\big(\tau_a\phi^{-1}(\bar{x})\big)*\bar{y}-\bar{x}(\bar{\tau}_a\bar{y})=\big(\phi^{-1}\big(\phi\tau_a\phi^{-1}(\bar{x})\big)\big)*\bar{y}-\bar{x}(\bar{\tau}_a\bar{y})\\
&=\big(\phi\tau_a\phi^{-1}(\bar{x})\big)\bar{y}-\bar{x}(\bar{\tau}_a\bar{y})
\end{aligned}$$

with $\bar{y} = \bar{1}$ implies that $\phi\tau_a\phi^{-1}(\bar{x}) = \bar{\tau}_a\bar{x} + \bar{x}\bar{\tau}_a = T_{\bar{\tau}_a}(\bar{x})$, thus $(L_{\bar{\tau}_a}, T_{\bar{\tau}_a}, -L_{\bar{\tau}_a})$ is a ternary derivation of $U(E)/K_L$, which is the statement in (ii). Finally,

$$\begin{aligned}\bar{\tau}_a\bar{\tau}_b - \bar{\tau}_b\bar{\tau}_a &= \overline{\tau_a\tau_b} - \overline{\tau_b\tau_a} = \overline{\delta_{a,b} + \tau_{[a,b]}} = \bar{\tau}_{[a,b]},\\ \bar{\tau}_a(\bar{\tau}_b\bar{\tau}_c) - \bar{\tau}_b(\bar{\tau}_a\bar{\tau}_c) - \bar{\tau}_c[\bar{\tau}_a, \bar{\tau}_b] &= \overline{\tau_a\tau_b\tau_c - \tau_b\tau_a\tau_c - \tau_c\tau_a\tau_b + \tau_c\tau_b\tau_a}\\ &= \overline{\big[[\tau_a, \tau_b], \tau_c\big]} = \bar{\tau}_{[a,b,c]}\end{aligned}$$

establishes (iii). □

4. Main result

Given a Bol algebra $(V, [\ ,\ ,\ ], [\ ,\ ])$, let $F\{V\}$ be the free unital nonassociative algebra on a basis of V and

$$\begin{aligned}U(V) = F\{V\}/\operatorname{ideal}\big\langle &ab - ba - [a,b], a(bc) - b(ac) - c[a,b] - [a,b,c],\\ &(a,x,y) + (x,a,y) \;\big|\; a,b,c \in V \text{ and } x,y \in F\{V\}\big\rangle.\end{aligned}$$

The natural embedding $V \to F\{V\}$ induces a map $\iota : V \to U(V)$. By construction $(U(V), \iota)$ verifies the universal property:

> Given a unital algebra A and a linear map $\iota' : V \to \mathrm{LN}_{\mathrm{alt}}(A) \subseteq A$ $a \mapsto a'$ with $\iota'([a,b,c]) = a'(b'c') - b'(a'c') - c'[a',b']$ and $\iota'([a,b]) = [a',b']$ then there exists a homomorphism $\varphi : U(V) \to A$ of unital algebras such that $\iota' = \varphi \circ \iota$.

Theorem 9. *The map $\iota : V \to U(V)$ is injective.*

Proof. By Theorem 8 and the universal property of $(U(V), \iota)$, there exists an epimorphism

$$\varphi : U(V) \to U(E)/K_L, \quad \iota(a) \mapsto \bar{\tau}_a.$$

Since the map $a \mapsto \bar{\tau}_a$ is injective, then the same holds for ι. □

By abuse of notation, we will identify a with $\iota(a)$, and V will be thought to be contained in $U(V)$. To establish the existence of a Poincaré–Birkhoff–Witt type basis, we need to fix some more notation:

- $a_\emptyset = 1$, $a_I = a_{i_1}(a_{i_2}(\cdots(a_{i_{n-1}}a_{i_n})\cdots)) \in U(V)$ where $I = (i_1, \ldots, i_n)$,
- $U(V)_{-1} = 0$ and $U(V)_n = \langle a_I \mid |I| \leqslant n\rangle$ for all $n \geqslant 0$,
- $\operatorname{gr}(U(V)) = \bigoplus_{n=0}^{\infty} U(V)_n/U(V)_{n-1}$ is the graded algebra associated with the filtration $U(V) = \bigcup_{n=0}^{\infty} U(V)_n$,
- $[b_1, \ldots, b_n]_n = [b_1(\cdots(b_{n-1}b_n)\cdots)]_n = b_1(\cdots(b_{n-1}b_n)\cdots) + U(V)_{n-1}$ with $b_1, \ldots, b_n \in V$.

Theorem 10. *The elements* $\{a_I \mid I = (i_1, \ldots, i_n),\ i_1 \leqslant \cdots \leqslant i_n$ *and* $n \geqslant 0\}$ *form a basis of* $U(V)$.

Proof. The image of a_I by the epimorphism $\varphi : U(V) \to U(E)/K_L$ coming from the universal property is $\bar{\tau}_I$. Since the later are linearly independent, the same holds for the former. Consequently, it suffices to show that the set in the statement spans $U(V)$. The result will follow once we had proved that $\{a_I + U(V)_{n-1} \mid I = (i_1, \ldots, i_n),\ i_1 \leqslant \cdots \leqslant i_n$ and $n \geqslant 0\}$ spans $\mathrm{gr}(U(V))$. Since $[[L_a, L_b], L_c] = L_{[a,b,c]}$, then

$$\begin{aligned}
[b_1, \ldots, b_n]_n &= \bigl[L_{b_1} \cdots L_{b_n}(1)\bigr]_n \\
&= [b_1, \ldots, b_{i+1}, b_i, \ldots, b_n]_n + \bigl[L_{b_1} \cdots [L_{b_i}, L_{b_{i+1}}] \cdots L_{b_n}(1)\bigr]_n \\
&= [b_1, \ldots, b_{i+1}, b_i, \ldots, b_n]_n + \bigl[L_{b_1} \cdots \widehat{L}_{b_i} \widehat{L}_{b_{i+1}} \cdots L_{b_n}[L_{b_i}, L_{b_{i+1}}](1)\bigr]_n \\
&= [b_1, \ldots, b_{i+1}, b_i, \ldots, b_n]_n,
\end{aligned}$$

where $\widehat{L}_b$ means that the operator L_b is omitted. Therefore, $[b_1, \ldots, b_n]_n$ does not depend on the order of the elements. If by induction on n we assume that $[b_1, \ldots, b_n]_n \times [c_1, \ldots, c_m]_m = [b_1, \ldots, b_n, c_1, \ldots, c_m]_{n+m}$, then

$$\begin{aligned}
&[b_1, \ldots, b_{n+1}]_{n+1}[c_1, \ldots, c_m]_m \\
&\quad = \frac{1}{2} T_{[b_1]_1}\bigl([b_2, \ldots, b_{n+1}]_n\bigr)[c_1, \ldots, c_m]_m \\
&\quad \overset{\langle 1 \rangle}{=} \frac{1}{2}[b_1]_1\bigl([b_2, \ldots, b_{n+1}]_n[c_1, \ldots, c_m]_m\bigr) + \frac{1}{2}[b_2, \ldots, b_{n+1}]_n[b_1, c_1, \ldots, c_m]_{m+1} \\
&\quad \overset{\langle 2 \rangle}{=} [b_1, \ldots, b_{n+1}, c_1, \ldots, c_m]_{n+m+1},
\end{aligned}$$

where $\langle 1 \rangle$ is a consequence of $[b_1]_1 \in \mathrm{LN}_{\mathrm{alt}}(\mathrm{gr}(V))$ (recall the product on $\mathrm{gr}(U(V))$) and the definition of the symbols $[\ ]_n$, and $\langle 2 \rangle$ follows from the hypothesis of induction. Thus, $\mathrm{gr}(U(V))$ is associative and commutative. Since, in addition, it is generated by $\{[b]_1 \mid b \in V\}$, then the result follows. $\square$

One important feature of the universal enveloping algebras of Lie algebras is that they are Hopf algebras. The universal property of $U(V)$ allows us to define on $U(V)$ a structure of bialgebra. A straightforward computation proves that the map

$$V \to U(V) \otimes U(V), \quad a \mapsto \Delta(a) = a \otimes 1 + 1 \otimes a$$

satisfies

(1) $\Delta(a) \in \mathrm{LN}_{\mathrm{alt}}(U(V) \otimes U(V))$,
(2) $\Delta([a, b]) = [\Delta(a), \Delta(b)]$,
(3) $\Delta([a, b, c]) = \Delta(a)(\Delta(b)\Delta(c)) - \Delta(b)(\Delta(a)\Delta(c)) - \Delta(c)[\Delta(a), \Delta(b)]$.

Therefore, it induces a homomorphism of unital algebras

$$\Delta : U(V) \to U(V) \otimes U(V).$$

Under this map V goes to

$$\operatorname{Prim}\bigl(U(V), \Delta\bigr) = \bigl\{x \in U(V) \bigm| \Delta(x) = x \otimes 1 + 1 \otimes x\bigr\},$$

the set of primitive elements. Similarly, the map $V \to F$ given by $a \mapsto 0$ provides a homomorphism of unital algebras, the counit,

$$\varepsilon : U(V) \to F$$

which kills V.

Theorem 11. *$(U(V), \cdot, 1, \Delta, \varepsilon)$ is a bialgebra and, over fields of characteristic zero,* $\operatorname{Prim}(U(V), \Delta) = V$.

Proof. Use Friedrich's criterion as in [3]. □

5. Right Bol algebras

In [4] a finite-dimensional vector space V over $\mathbb{R}$ (the definitions also work over arbitrary fields) with a trilinear operation $(\eta; \xi, \zeta)$ is called a Lie triple system if for all $\xi, \eta, \zeta, \nu, \tau \in V$,

$$(\eta; \xi, \xi) = 0,$$
$$(\xi; \eta, \zeta) + (\eta; \zeta, \xi) + (\zeta; \xi, \eta) = 0,$$
$$\bigl((\xi; \nu, \tau); \eta, \zeta\bigr) + \bigl(\xi; (\eta; \nu, \tau), \zeta\bigr) + \bigl(\xi; \eta, (\zeta; \nu, \tau)\bigr) = \bigl((\xi; \eta, \zeta); \nu, \tau\bigr).$$

A *right Bol algebra* is defined as a Lie triple system with an additional bilinear skew-symmetric operation $\xi \cdot \eta$ such that

$$\bigl((\tau \cdot \zeta); \xi, \eta\bigr) = (\tau; \xi, \eta) \cdot \zeta + \tau \cdot (\zeta; \xi, \eta) + \bigl((\xi \cdot \eta); \tau, \zeta\bigr) + (\tau \cdot \zeta) \cdot (\xi \cdot \eta).$$

From a right Bol algebra V we can obtain a left Bol algebra V^{opp} by considering V with the operations

$$[a, b] = -a \cdot b \quad \text{and} \quad [a, b, c] = -(c; a, b). \tag{8}$$

We define the *generalized right alternative nucleus* of an algebra A as

$$\mathrm{RN}_{\mathrm{alt}}(A) = \bigl\{a \in A \bigm| (x, y, a) = -(x, a, y)\ \forall x, y \in A\bigr\}.$$

If A^{opp} denotes the opposite algebra of A, then $\mathrm{RN}_{\mathrm{alt}}(A) = \mathrm{LN}_{\mathrm{alt}}(A^{\mathrm{opp}})$.

Given a right Bol algebra $(V, (\ ;\ ,\ ), \cdot)$, we define the *universal enveloping algebra* $U(V)$ of V as $U(V) = U(V^{\mathrm{opp}})^{\mathrm{opp}}$. From the properties of $U(V^{\mathrm{opp}})$ it is clear that there exists a linear injective map

$$\iota : V \to \mathrm{RN}_{\mathrm{alt}}\big(U(V)\big), \quad a \mapsto a$$

such that

$$\iota(a \cdot b) = ab - ba \quad \text{and} \quad \iota\big((a; b, c)\big) = (ab)c - (ac)b - [b, c]a$$

and that $U(V)$ satisfies the corresponding universal property. Similar results to Theorems 10 and 11 are easily obtained.

6. Connections with Malcev algebras

As mentioned in the Introduction, for any Malcev algebra $(M, [\ ,\ ])$ over a field of characteristic $\neq 2, 3$ there exist a unital algebra $U(M)$ and a monomorphism of Malcev algebras

$$\iota : M \hookrightarrow \mathrm{N}_{\mathrm{alt}}\big(U(M)\big).$$

Malcev algebras are examples of Bol algebras. The binary product $[a, b]$ together with the ternary product

$$[a, b, c] = \big[[a, b], c\big] - \frac{1}{3} J(a, b, c),$$

where $J(a, b, c) = [[a, b], c] + [[b, c], a] + [[c, a], b]$, make M a left Bol algebra [4]. Therefore, we may consider the universal enveloping algebra, that we will temporary denote by $U_B(M)$ to avoid confusion, of this left Bol algebra.

Proposition 12. *Under the above assumptions, $U_B(M) \cong U(M)$.*

Proof. The proof is based on the universal property of $U_B(M)$ and the existence of Poincaré–Birkhoff–Witt type bases. On the one hand, $\iota : M \to U(M)$ maps M into $\mathrm{N}_{\mathrm{alt}}(U(M)) \subseteq \mathrm{LN}_{\mathrm{alt}}(U(M))$, and $\iota([a, b]) = ab - ba$. On the other hand,

$$\begin{aligned} a(bc) - b(ac) - c[a, b] &= \big[[L_a, L_b], L_c\big](1) \overset{\langle 1 \rangle}{=} \big[L_{[a,b]} - 2[L_a, R_b], L_c\big](1) \\ &= \big[[a, b], c\big] - 2(a, b, c) \overset{\langle 2 \rangle}{=} \big[[a, b], c\big] - \frac{1}{3} J(a, b, c) \\ &= \iota\big([a, b, c]\big), \end{aligned}$$

where in $\langle 1\rangle$ we have used that $[L_a, L_b] = L_{[a,b]} - 2[L_a, R_b]$ holds for any $a, b \in \mathrm{N}_{\mathrm{alt}}(U(M))$ [2], and $\langle 2\rangle$ follows from the identity

$$\big[[a,b],c\big] + \big[[b,c],a\big] + \big[[c,a],b\big]$$
$$= (a,b,c) - (a,c,b) + (c,a,b) - (b,a,c) + (b,c,a) - (c,b,a)$$

valid in any algebra [7]. Thus, by the universal property of $U_B(M)$, there exists a homomorphism of unital algebras $\varphi : U_B(M) \to U(M)$ with $\varphi(a) = a$ for any $a \in M$. This homomorphism maps the Poincaré–Birkhoff–Witt basis of $U_B(M)$ onto the Poincaré–Birkhoff–Witt basis of $U(M)$, so it is an isomorphism. $\square$

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