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Division composition algebras through their derivation algebras

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Abstract

Composition algebras are algebras with a nondegenerate quadratic multiplicative form. In this paper we construct five families of division composition algebras and prove that any division composition algebra with nonabelian derivation algebra belongs to one of these families. As a by-product we obtain new examples of real division algebras.

Keywords: Composition algebras; Division algebras; Derivation algebra

Introduction

In the following we will always use the word algebra to be synonymous with finite-dimensional algebra, and we will assume that the fields have characteristic $\neq 2, 3$.

A division algebra is an algebra where the left and right multiplication operators L_x, R_x by any nonzero element x are bijective. Early examples of these algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. In fact, a theorem by Frobenius [27]

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ensures that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only associative real division algebras. When the associativity is substituted by the *alternative* law:

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2,$$

or, equivalently, if any two elements generate an associative subalgebra [43]; then $\mathbb{O}$ comes into play. In [5] Benkart, Britten and Osborn extended this research to real division algebras verifying the *flexible* identity $(xy)x = x(yx)$ by including the generalized real division pseudo-octonion algebras among others.

One cornerstone of the study of finite-dimensional real division algebras is the very well-known theorem by Bott and Milnor [1] and Kervaire [30] which reduces the possible dimensions of these algebras to 1, 2, 4 or 8. Apart from this result, there have been other attempts to understand these algebras which demonstrate their complexity. In [2,3] Benkart and Osborn focused on the derivation algebra of these algebras. The philosophy behind this approach is that studying the algebra of derivations is the same as studying the group of automorphisms, and this group measures the "symmetries" of the algebra. The main results in [2,3] are:

Theorem 1. *Let A be a finite-dimensional real division algebra and* $\operatorname{Der} A$ *its derivation algebra, then*:

(i) *If* $\dim A = 1$ *or* 2 *then* $\operatorname{Der} A = 0$.
(ii) *If* $\dim A = 4$ *then either* $\operatorname{Der} A = \mathrm{su}(2)$ *or* $\dim \operatorname{Der} A \leqslant 1$.
(iii) *If* $\dim A = 8$ *then* $\operatorname{Der} A$ *is one of the following Lie algebras*:
  (a) *The compact algebra G_2.*
  (b) $\mathrm{su}(3)$.
  (c) $\mathrm{su}(2) \oplus \mathrm{su}(2)$.
  (d) $\mathrm{su}(2) \oplus Z$*, where Z denotes an abelian ideal of dimension* 0 *or* 1.
  (e) *An abelian algebra H of dimension* 0, 1 *or* 2.

The following result shows the possible decompositions of A as a module for its derivation algebra.

Theorem 2. *Let A be a finite-dimensional real division algebra, then*:

(i) *If* $\operatorname{Der} A$ *is the compact Lie algebra G_2 then A is the direct sum of two irreducible modules of dimensions* 1 *and* 7.
(ii) *If* $\operatorname{Der} A = \mathrm{su}(3)$ *then A is either irreducible or the direct sum of three irreducible modules of dimensions* 1, 1 *and* 6.
(iii) *If* $\operatorname{Der} A = \mathrm{su}(2) \oplus \mathrm{su}(2)$ *then A is the direct sum of three irreducible modules of dimensions* 1, 3 *and* 4. *Here the three-dimensional module is irreducible for one copy of* $\mathrm{su}(2)$ *but trivial for the other copy. The one and four-dimensional modules are irreducible for both copies of* $\mathrm{su}(2)$.
(iv) *If* $\operatorname{Der} A = \mathrm{su}(2) \oplus Z$ *where Z is an abelian ideal of dimension* $\leqslant 1$ *then the possible dimensions of the direct summands of any decomposition of A as a sum of irreducible* $\mathrm{su}(2)$*-modules are*:
  (a) 1 *and* 3. *Here* $\operatorname{Der} A = \mathrm{su}(2)$.
  (b) 1, 1, 3 *and* 3.

(c) $1, 3$ *and* 4.
(d) $1, 1, 1, 1$ *and* 4.
(e) 3 *and* 5. *Here* $\operatorname{Der} A = \mathrm{su}(2)$.

An investigation using automorphisms was conducted in [9]. Although Benkart and Osborn gave many examples of finite-dimensional real division algebras, it remained unclear whether all the possibilities can be fulfilled. In particular, they did not provide examples of algebras with decompositions as in parts (b), (c) and (e). In [40] Rochdi finds a real division algebra with su(2) as its derivation algebra and decomposition $1 + 1 + 3 + 3$.

In the previous list we can locate the quaternions $\mathbb{H}$ as an algebra with derivations su(2) and decomposition $1+3$. The octonion algebra $\mathbb{O}$ has derivation algebra G_2 and decomposition $1+7$. These algebras are particular examples of *composition algebras* [32].

Definition 3. An algebra A over a field F is called a composition algebra if there exists a quadratic form $n : A \to F$ such that:

(i) $n(xy) = n(x)n(y)$ for all $x, y \in A$.
(ii) The bilinear form $(x, y) = \frac{1}{2}(n(x + y) - n(x) - n(y))$ is nondegenerate.

The first well-known composition algebras were those with a unit element, also termed *Hurwitz algebras*. These algebras are: the base field F, $F \oplus F$, a quadratic extension $K(\mu)$ of F, a generalized quaternion algebra $Q(\mu, \beta)$ or a generalized octonion algebra $C(\mu, \beta, \gamma)$ [43].

The interest on nonunital composition algebras arose from the work on the SU(3) particle physics. In [33] Okubo defined a new product on the vector space $\mathrm{sl}(3, F)$ of 3 by 3 trace zero matrices over a field F of characteristic not 2 or 3 by the following formula:

$$x * y = \mu xy + (1 - \mu)yx - \frac{1}{3}\operatorname{trace}(xy)\operatorname{Id}, \tag{1}$$

where xy stands for the usual matrix product, μ is a solution of the equation $3\mu(1-\mu) = 1$ and Id is the 3 by 3 identity matrix. The new algebra $\mathrm{P}_8(F) = (\mathrm{sl}(3, F), *)$ is called the *pseudo-octonion algebra over* F. $\mathrm{P}_8(F)$ is a composition algebra with the quadratic form $n(x) = \frac{1}{6}\operatorname{trace}(x^2)$. The forms of $\mathrm{P}_8(\bar{F})$, where $\bar{F}$ denotes the algebraic closure of F, are called *Okubo algebras* [17]. Since the forms of composition algebras are again composition algebras [25], Okubo algebras are also composition algebras.

Nonunital finite-dimensional composition algebras are quite related with Hurwitz algebras. In [29] Kaplansky proved that given any finite-dimensional composition algebra A and an element $a \in A$ with $n(a) \neq 0$, then the left and right multiplication operators by $u = a^2/n(a)$, L_u and R_u, are isometries, and the new algebra (A, a) with product:

$$x \circ y = R_u^{-1}(x)L_u^{-1}(y)$$

is a Hurwitz algebra with unit u^2 and the same norm (quadratic form) as A. So, the product xy of any composition algebra A can be constructed from a Hurwitz algebra $(A, \circ)$ just by

$$xy = \phi_1(x) \circ \phi_2(y), \tag{2}$$

where ϕ_i, $i = 1, 2$, are adequate isometries. Equation (2) says that any composition algebra is an *isotope* of a Hurwitz algebra.

In [24] it is proved that given an Okubo algebra A and e a nonzero idempotent of A then L_e and R_e can be written as

$$L_e = \tau J \quad \text{and} \quad R_e = \tau^{-1} J, \tag{3}$$

where J denotes the standard involution in (A, e) and τ is an automorphism of degree 3 of (A, e). Moreover, the eigenspace of eigenvalue 1 of τ, which is a subalgebra of (A, e), must be a quaternion algebra.

Another interesting family of composition algebras are the *standard composition algebras*. These algebras are constructed from a Hurwitz algebra $(A, \circ)$ by altering the product $\circ$ on A by one of the following new products:

$$\text{(I)} \quad x \circ y, \qquad \text{(II)} \quad \bar{x} \circ y, \qquad \text{(III)} \quad x \circ \bar{y}, \qquad \text{(IV)} \quad \bar{x} \circ \bar{y},$$

where $x \mapsto \bar{x}$ denotes the standard involution of $(A, \circ)$. Depending on which product we choose the standard composition algebra is said to be of type (I), (II), (III) or (IV), respectively. The quadratic form of these algebras is the same as that of $(A, \circ)$. Standard algebras of type (IV) are also called *para-Hurwitz algebras* [36,37].

Many papers have been concerned with the classification of composition algebras. In [42] the classification of four-dimensional composition algebras was carried out. Power associative composition algebras have been classified in [8,34]. A composition algebra is called *symmetric* if the bilinear form is associative, that is:

$$(xy, z) = (x, yz)$$

for all x, y and z, or equivalently [31]

$$x(yx) = n(x)y = (xy)x \tag{4}$$

for all x and y. In [10,11,24,36,37] it is shown that the only symmetric composition algebras are the Okubo algebras and the forms of para-Hurwitz algebras, which are disjoint families. The classification of flexible composition algebras can be found in [18,19,21,35]. In [26] the authors proved that any composition algebra of degree two is either an Okubo algebra or a form of a standard algebra. In [23,26] they deal with the classification of third power associative composition algebras showing that these algebras are flexible. Composition and division algebras satisfying $y((xz)x) = ((yx)z)x$ (right Moufang), $(xy)(zx) = (x(yz))x$ (middle Moufang) and $(x(yx))z = x(y(xz))$ (left Moufang) are classified in [6,7]. Okubo algebras over fields of characteristic three is the topic of [13]. An elegant approach to derivations and automorphisms of Hurwitz algebras is developed in [14]. Symmetric composition algebras are also very much related with the Principle of Triality [15,22] and the exceptional Lie algebras [16].

Similarly to what happens to real division algebras, not many things are known for arbitrary composition algebras. Inspired by the work of Benkart and Osborn, in [38] we started developing a similar program to study division composition algebras, over fields of characteristic not 2 or 3, through their derivation algebras. Our main result there was an analogue of Theorem 1:

Theorem 4. *Let A be a division composition algebra over a field of characteristic not* 2 *or* 3*, and* $\operatorname{Der} A$ *its derivation algebra then*:

(i) *If* $\dim A = 1$ *or* 2 *then* $\operatorname{Der} A = 0$.
(ii) *If* $\dim A = 4$ *then either* $\operatorname{Der} A$ *is a Lie algebra of type* A_1 *or* $\dim \operatorname{Der} A \leqslant 1$. *Moreover,* $\operatorname{Der} A = A_1$ *if and only if A is standard.*
(iii) *If* $\dim A = 8$ *then* $\operatorname{Der} A$ *is one of the following Lie algebras*:
   (a) G_2. *This happens if and only if A is standard.*
   (b) A_2.
   (c) $A_1 \oplus A_1$.
   (d) $A_1 \oplus Z$*, where Z denotes an abelian ideal of dimension* 0 *or* 1.
   (e) *An abelian algebra H of dimension* 0, 1 *or* 2.

In this paper we fully develop this program. Firstly, we prove the following results which reflect the surprising analogy between the families of real division algebras and division composition algebras:

Theorem 5. *Let A be as in the previous theorem, then*:

(i) *If* $\operatorname{Der} A = G_2$ *then A is the direct sum of two irreducible modules of dimensions* 1 *and* 7.
(ii) *If* $\operatorname{Der} A = A_2$ *then A is either irreducible or the direct sum of three irreducible modules of dimensions* 1, 1 *and* 6. *In case A is irreducible then A is an Okubo algebra.*
(iii) *If* $\operatorname{Der} A = A_1 \oplus A_1$ *then A is the direct sum of three irreducible modules of dimensions* 1, 3 *and* 4. *Here the three-dimensional module is irreducible for a copy of A_1 but trivial for the other. The one and four-dimensional modules are irreducible for both copies of A_1.*
(iv) *If* $\operatorname{Der} A = A_1 \oplus Z$*, where Z is an abelian ideal of dimension* $\leqslant 1$*, then the possible dimensions of the direct summands of any decomposition of A as a sum of irreducible A_1-modules are*:
   (a) 1 *and* 3. *Here* $\operatorname{Der} A = A_1$.
   (b) 1, 1, 3 *and* 3. *Here* $\operatorname{Der} A = A_1$.
   (c) 1, 3 *and* 4.
   (d) 1, 1, 1, 1 *and* 4.
   (e) 3 *and* 5. *Here* $\operatorname{Der} A = A_1$.

As it is apparent by comparing Theorems 2 and 5, one case is missing. This was the motivation to prove the following result which complements the work of Benkart and Osborn:

Proposition 6. *There are no real division algebras with derivation algebra isomorphic to* $\mathrm{su}(2) \oplus Z$*, where Z is an abelian ideal of dimension* 1*, and decomposition into irreducible* $\mathrm{su}(2)$*-modules given by* $1+1+3+3$.

Proof. We will proceed by contradiction. Let A be a real division algebra with derivation algebra $\mathrm{su}(2) \oplus Z$, where Z is an abelian ideal of dimension 1. Let us assume that $\mathrm{su}(2)$ decomposes A as $A = A_0 \oplus W$ where $A_0 = \{x \in A \mid (\mathrm{su}(2))x = 0\}$ and $W = W_1 \oplus W_2$ for some irreducible $\mathrm{su}(2)$-modules W_1 and W_2 of dimension 3. Fix $h \in Z$, for any $d \in \mathrm{su}(2)$ we have that $d(h(A_0)) = h(d(A_0)) = 0$ so $h(A_0) \subseteq A_0$. Moreover, since A_0 is a two-dimensional division subalgebra, Theorem 1 says that $d(A_0) = 0$. Now observe that $\ker h = A_0 \oplus (W \cap \ker h)$ and that $\ker h$ is also a su(2)-module; so, if $\ker h \cap W \neq 0$ then $\ker h$ contains an irreducible su(2)-module of dimension 3. In particular, the dimension of $\ker h$ is $\geqslant 5$ but it is also a division subalgebra of A, hence $\ker h = A$ and $h = 0$, which it is not possible. This shows that $\ker h = A_0$.

Let $\mathrm{sl}(2) = \mathrm{sl}(2, \mathbb{C})$ be $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{su}(2)$, $\bar{W} = \mathbb{C} \otimes_{\mathbb{R}} W$ and $\bar{S}(\alpha)$ the eigenspace of h (on $\mathbb{C} \otimes_{\mathbb{R}} A$) associated to the eigenvalue α. Since su(2) has no trivial submodules in W, the sl(2)-submodules $\mathbb{C} \otimes_{\mathbb{R}} W_i$ $(i = 1, 2)$ are irreducible. Thus, any proper sl(2)-submodule of $\bar{W}$ has dimension 3. But we have some distinguished submodules in $\bar{W}$, namely the subspaces $\bar{W}_\alpha = \bar{W} \cap \bar{S}(\alpha)$ (notice that h is in the center of $\operatorname{Der} A$ and h is semisimple by [2]); so, either $\bar{W} = \bar{W}_\alpha$ or $\bar{W} = \bar{W}_\alpha \oplus \bar{W}_\beta$ with $0 \neq \alpha, \beta \in \mathbb{C}$.

The eigenvalues of any derivation of A are purely imaginary [2] and, as a consequence, the trace of any such derivation is zero. In case that $\bar{W} = \bar{W}_\alpha$ the trace of h is 6α, so $\alpha = 0$ which is a contradiction. In case that $\bar{W} = \bar{W}_\alpha \oplus \bar{W}_\beta$ we obtain that $3(\alpha + \beta) = 0$ or, equivalently, $\alpha = -\beta$. Now $W^2 \subseteq (\bar{W}_\alpha \oplus \bar{W}_{-\alpha})^2 \cap A \subseteq (\bar{S}(2\alpha) \oplus \bar{S}(-2\alpha) \oplus \bar{S}(0)) \cap A$. However, $\bar{S}(\pm 2\alpha) = 0$ so $W^2 \subseteq \bar{S}(0) \cap A = A_0$ but this is again a contradiction because, since the left and right multiplication operators by nonzero elements are bijective, this would imply $6 = \dim W \leqslant \dim W^2 < \dim A_0 = 2$. $\square$

Contrary to what was done for real division algebras, we will not give some examples of division composition algebras with nonabelian derivation algebra, but rather will provide the whole description of these algebras. To this end, in Section 1 we construct five families $\mathcal{T}$, $\mathcal{W}$, $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{O}$ of division composition algebras, compute their derivation algebras and their decompositions as modules for these Lie algebras. The results show that all possibilities in Theorem 5 are fulfilled. Later on, in Section 3, we manage to fit any algebra in Theorem 5 into one of our families, which completes the description. Explicitly, if we denote any irreducible module by its dimension, and we will do so very often, then we have that

$$\text{if} \quad A = \begin{cases} 1+7 \text{ or } 1+3+4, \\ 1+1+6 \text{ or } 1+1+1+1+4, \\ 1+1+3+3, \\ 3+5, \\ 8, \end{cases} \qquad \text{then} \quad A \in \begin{cases} \mathcal{T}, \\ \mathcal{Q}, \\ \mathcal{W}, \\ \mathcal{S}, \\ \mathcal{O}, \end{cases} \quad \text{respectively.}$$

Furthermore, as a by-product of our research we obtain other interesting results. Looking at the constructions in Section 1 we have:

Theorem 7. *If there are nonzero scalars $\beta, \gamma \in F$ such that the octonion algebra $C(-1, \beta, \gamma)$ is a division algebra then there also exist examples of (composition) division algebras over F for all possible decompositions in Theorem 5.*

Since $\mathbb{R}$ is an example of such a field ($\mathbb{O} = C(-1, -1, -1)$), this result complements the result in [3]. The following characterization of Okubo algebras will be straightforward from Theorems 4, 5 and Corollary 13

Theorem 8. *A division composition algebra is an Okubo algebra if and only if it is an irreducible module for its derivation algebra.*

The distinguished role played by Hurwitz and Okubo algebras in the family of composition algebras is emphasized by the following theorem:

Theorem 9. *Let A be a division composition algebra, then there exist isometries ϕ_1 and ϕ_2 such that A with the new product $*$ defined by $x * y = \phi_1(x)\phi_2(y)$ is either a Hurwitz or an Okubo algebra and* $\mathrm{Der}\, A = \{d \in \mathrm{Der}\,(A, *) \mid d\phi_1 = \phi_1 d$ *and* $d\phi_2 = \phi_2 d\}$.

The reader can skip the details in the construction of the families and return later to them.

This paper is structured as follows:

1. Examples of division algebras.
   - 1.1 General results about composition algebras.
   - 1.2 Examples of dimension 1, 2 and 4.
   - 1.3 Examples of dimension 8.
     - Algebras $(C, \tau_u, \tau_v, \delta_1, \delta_2)$ and the family $\mathcal{T}$.
     - Algebras $(C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2)$ and the family $\mathcal{W}$.
     - Algebras $(C, \phi_{Q,1}, \phi_{Q,2})$ and the family $\mathcal{Q}$.
     - Algebras $(P, S, \epsilon_1, \epsilon_2)$ and the families $\mathcal{O}$ and $\mathcal{S}$.
2. Module structure of A.
3. Description of division composition algebras.

1. Examples of division algebras

Throughout this section we will construct five different families of eight-dimensional division composition algebras $\mathcal{T}$, W, Q, O and $\mathcal{S}$, and we will compute their derivation algebras. As we will see later, any eight-dimensional division composition algebra with nonabelian derivation algebra belongs to one of these five families. We will also give examples in dimension 1, 2 and 4.

1.1. General results about composition algebras

Given a vector space S over F and $\bar{F}$ the algebraic closure of F, $\bar{S}$ will denote $\bar{F} \otimes_F S$. We will need some results in our arguments. From [25]:

Lemma 10. *Let A be a composition algebra and $x \in A$. The following are equivalent*:

(i) *L_x is bijective,*
(ii) *$n(x) \neq 0$,*
(iii) *R_x is bijective.*

So, whenever we talk about division composition algebras we will keep in mind that the norm of any nonzero element is nonzero.

Proposition 11. *Let A be a composition algebra and $d \in \mathrm{Der}\, A$, then*

$$(dx, y) + (x, dy) = 0.$$

In fact, given $d \in \mathrm{Der}\, A$ and $\bar{A}_\alpha = \{x \in \bar{A} \mid dx = \alpha x\}$ we have [38]:

Table 1
Split octonions

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

Proposition 12. *Let A be an eight-dimensional division composition algebra. Any nonzero derivation of A acts semisimply on $\bar{A}$, and $\bar{A}$ decomposes as a sum of eigenspaces in one of the following ways*:

(i) $\bar{A} = \bar{A}_0 \oplus \bar{A}_{\pm\alpha} \oplus \bar{A}_{\pm\beta} \oplus \bar{A}_{\pm(\alpha+\beta)}$, *with* $\alpha, \beta \in \bar{F}$ *with* $\dim \bar{A}_0 = 2$.
(ii) $\bar{A} = \bar{A}_0 \oplus \bar{A}_{\pm\alpha}$ *with* $\alpha \in \bar{F}$ *and* $\dim \bar{A}_\alpha = \dim \bar{A}_{-\alpha} = 2$, *and* $\dim \bar{A}_0 = 4$.
(iii) $\bar{A} = \bar{A}_0 \oplus \bar{A}_{\pm\alpha} \oplus \bar{A}_{\pm2\alpha}$ *with* $\alpha \in \bar{F}$ *and* $\dim \bar{A}_0 = 2 = \dim \bar{A}_\alpha = \dim \bar{A}_{-\alpha}$, *and* $\dim \bar{A}_{2\alpha} = \dim \bar{A}_{-2\alpha} = 1$.

Corollary 13. *Given $H \subseteq \operatorname{Der} A$ with H abelian, then* $\dim\{x \in A \mid Hx = 0\} \geqslant 2$.

From Lemma 13 in [38]:

Lemma 14. *With the same notation as in Proposition* 12, *the decomposition*

$$\bar{A} = \bar{A}_0 \oplus (\bar{A}_\alpha \oplus \bar{A}_{-2\alpha}) \oplus (\bar{A}_{-\alpha} \oplus \bar{A}_{2\alpha})$$

is a nontrivial $\mathbb{Z}_3$-gradation of $\bar{A}$.

These gradations are related with *canonical bases* of octonions. Over algebraically closed fields the only Cayley–Dickson algebra is the split one, the Cayley–Dickson algebra with zero divisors. This algebra possesses a basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ where the product is given by Table 1.

Any basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of a split octonion algebra C where the product is given Table 1 is called *canonical basis*. Canonical bases arise from the Peirce decomposition of C. Given two orthogonal idempotents e_1, e_2 of C [41] set $C_{ij} = \{x \in C \mid e_1x = ix \text{ and } xe_1 = jx\}$, with $i, j = 0, 1$. It is well known that $C_{11} = Fe_1$, $C_{00} = Fe_2$ as well as that C_{10} and C_{01} are dual with respect to the bilinear form $(,)$, and $\dim C_{10} = \dim C_{01} = 3$. In fact, using formulas in [43] it is easy to prove:

Lemma 15. *Given x_1, x_2 and x_3 in C_{10} with $(x_3, x_1x_2) = 1/2$ then*

$$\{e_1, e_2, x_1, x_2, x_3, x_2x_3, x_3x_1, x_1x_2\} \tag{5}$$

is a canonical basis of C.

Note 1. Given $x_1, x_2 \in C_{10}$ linearly independent there always exists $x_3 \in C_{10}$ such that (5) is a canonical basis.

These bases become useful when studying $\mathbb{Z}_3$-gradations of the split octonion algebra [12,25]:

Proposition 16. *Let C be a Cayley–Dickson algebra and $C = C^{(0)} \oplus C^{(1)} \oplus C^{(2)}$ a $\mathbb{Z}_3$-gradation which is not trivial ($C^{(0)} \neq 0$). Then C is split, $e \in C^{(0)}$ and there exists a canonical basis such that either*

(i) $C^{(0)} = \mathrm{span}\{e_1, e_2\}$, $C^{(1)} = \mathrm{span}\{u_1, u_2, u_3\}$ *and* $C^{(2)} = \mathrm{span}\{v_1, v_2, v_3\}$, *or*
(ii) $C^{(0)} = \mathrm{span}\{e_1, e_2, u_1, v_1\}$, $C^{(1)} = \mathrm{span}\{u_2, v_3\}$ *and* $C^{(3)} = \mathrm{span}\{u_3, v_2\}$.

Let us say a few words about the derivations and automorphisms of Hurwitz algebras (see [28]). Any automorphism σ of C with $\sigma(e_i) = e_i$, $i = 1, 2$ must preserve C_{10} and C_{01}, and $\sigma|_{C_{01}} = (\sigma|_{C_{10}})^*$, the adjoint of $\sigma|_{C_{10}}$ with respect to $(,)$. Moreover, $\det \sigma|_{C_{10}} = 1$, that is, $\sigma|_{C_{10}} \in \mathrm{SL}(C_{10})$. In fact, there is a group isomorphism

$$\bigl\{\sigma \in \mathrm{Aut}(C) \mid \sigma(e_i) = e_i,\ i = 1, 2\bigr\} \cong \mathrm{SL}(C_{10}) \tag{6}$$

given by $\sigma \mapsto \sigma|_{C_{10}}$ [28].

Now, let C be a Cayley–Dickson algebra and Q a quaternion subalgebra, so $C = Q \perp v_0 Q$ where v_0 is an arbitrary element of $Q^{\perp}$ with $n(v_0) \neq 0$ that we fix. Since Q is a central simple associative algebra, any derivation $d \in \mathrm{Der}\, Q$ is inner, that is, $d = ad_a : x \mapsto ax - xa$ for some $a \in Q$ and $a \perp e$ where e denotes the unit element. Given any element $b \in Q \cap e^{\perp}$ we can extend d to a derivation $d_{a,b}$ of C by the following formulas:

$$\begin{aligned} d_{a,b}(x) &= [a, x], \\ d_{a,b}(v_0 x) &= v_0\bigl(xb + [a, x]\bigr) \quad \forall x \in Q. \end{aligned}$$

In fact, $\mathrm{Der}\,(C, Q) := \{h \in \mathrm{Der}\, C \mid h(Q) \subseteq Q\} = \mathrm{span}\{d_{a,b} \mid a, b \in Q \cap e^{\perp}\}$. It is also easy to check that $\mathrm{Der}\,(C, Q)$ is the direct sum of two ideals, each of them being a simple Lie algebra of type A_1, namely

$$\mathrm{Der}\,(C, Q) = \mathrm{span}\bigl\{d_{a,a} \mid a \in Q \cap e^{\perp}\bigr\} \oplus \mathrm{span}\bigl\{d_{0,b} \mid b \in Q \cap e^{\perp}\bigr\}. \tag{7}$$

If C is also a division algebra then the first summand decomposes C as $1 + 3 + 4$, and the second as $1 + 1 + 1 + 1 + 4$.

The analogue of the second summand in the group of automorphisms is the subgroup $\mathrm{Aut}(C, Q) := \{\tau \in \mathrm{Aut}(C) \mid \tau(x) = x,\ \forall x \in Q\}$. These automorphisms can be constructed from elements $c \in Q$ with $n(c) = 1$ as follows:

$$\begin{aligned} \tau_c(x) &= x, \\ \tau_c(v_0 x) &= v_0(xc) \quad \forall x \in Q. \end{aligned} \tag{8}$$

In fact, $\mathrm{Aut}(C, Q) = \{\tau_c \mid c \in Q,\ n(c) = 1\}$.

In regard to the derivations which fix a subalgebra $K = \mathrm{span}\{e, f\}$, $(e \perp f)$ of dimension 2, they can be described in terms of an hermitian form on $K^\perp$. Following [20], $K^\perp$ is a K-vector space of dimension 3 and the map $\sigma : K^\perp \times K^\perp \mapsto K$ defined by

$$\begin{aligned}\sigma(x, y) &= -\text{orthogonal projection of } xy \text{ on } K\\ &= (x, y)e + (x, fy)\frac{f}{n(f)}\end{aligned} \tag{9}$$

is an hermitian form with respect to the nontrivial F-automorphism $x \mapsto \bar{x}$ of K, that is,

$$\begin{aligned}&a\sigma(x, y) = \sigma(ax, y) = \sigma(x, \bar{a}y) \quad \text{and}\\ &\sigma(x, y) = \overline{\sigma(y, x)}, \quad a \in K,\ x, y \in K^\perp.\end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned}\mathrm{Der}\,(C, K) &:= \{d \in \mathrm{Der}\,C \mid d(K) = 0\}\\ &\cong \{d \in \mathrm{End}_K(K^\perp) \mid \sigma(dx, y) + \sigma(x, dy) = 0\} = A_2,\end{aligned}$$

where the isomorphism is given by restriction, and C decomposes, in the division case, as $1 + 1 + 6$ as a $\mathrm{Der}\,(C, K)$-module.

1.2. Examples of dimension 1, 2 and 4

By Theorem 4 the derivation algebra of any division composition algebra of dimension $\leqslant 2$ is trivial. Examples of these division composition algebras are the base field F, any quadratic field extension $F(\mu)$ of F, or any isotope of these.

Four-dimensional composition algebras have been classified in [42]. We have:

Theorem 17. *The product of any composition algebra of dimension four is given by one of the following formulas*:

$$\begin{array}{llll}\text{(i)} & ax(ab)^{-1}yb, & \text{(ii)} & axb\bar{y}(ba)^{-1},\\ \text{(iii)} & (ba)^{-1}\bar{x}ayb, & \text{(iv)} & \bar{a}\bar{x}(ba)^{-1}\bar{y}\bar{b},\end{array} \tag{11}$$

where xy denotes the product of a quaternion algebra, $x \mapsto \bar{x}$ its standard involution and a, b are fixed elements with $n(a) \neq 0 \neq n(b)$.

Notice that if $a, b \in Fe$, where e denotes the unit element of the quaternion algebra, we recover the standard composition algebras.

We establish the following result without proof:

Proposition 18. *Let A be a four-dimensional division composition algebra whose product is given by one of the formulas* (11). *Then*

(i) $\mathrm{Der}\,A = A_1$ *if and only if $a, b \in Fe$. In this case A is standard and decomposes as* $1 + 3$.
(ii) $\mathrm{Der}\,A = Fd$ *with d semisimple if and only if* $\dim \mathrm{alg}\{a, b\} = 2$.
(iii) $\mathrm{Der}\,A = 0$ *otherwise.*

It is clear that from any division quaternion algebra we can get examples of new four-dimensional division composition algebras with derivations as in Theorem 4 just by choosing suitable a and b.

1.3. Examples of dimension 8

Algebras $(C, \tau_u, \tau_v, \delta_1, \delta_2)$ *and the family* $\mathcal{T}$

Given C a division octonion algebra, Q a quaternion subalgebra, $u, v \in Q$ with $n(u) = n(v) = 1$ and $\delta_1, \delta_2 = 0, 1$ we can define a new product on C by

$$x * y = J^{\delta_1}\tau_u(x) J^{\delta_2}\tau_v(y), \tag{12}$$

where τ_u and τ_v are automorphisms as in (8), and J denotes the standard involution $x \mapsto J(x) = \bar{x}$ of C. The new algebra obtained in this way will be denoted by $(C, \tau_u, \tau_v, \delta_1, \delta_2)$, and $\mathcal{T}$ will stand for the whole family of these algebras.

A straightforward computation shows that a derivation $d_{a,b} \in \mathrm{Der}(C, Q)$ commutes with an automorphism $\tau_c \in \mathrm{Aut}(C, Q)$ if and only if $b - a$ commutes with c. Since the only elements in Q commuting with c are those in $\mathrm{alg}\{c\} = \mathrm{span}\{e, c_0\}$, where $c_0 = c - (e, c)e$ is orthogonal to e, if $c \notin Fe$ or Q if $c = \pm e$, then we have that $\{d_{a,b} \in \mathrm{Der}(C, Q) \mid [d_{a,b}, \tau_c] = 0\}$ is

$$\begin{cases} \mathrm{Der}(C, Q) = A_1 \oplus A_1 & \text{if } c = \pm e, \\ \text{or} & \\ \mathrm{span}\{d_{a,a} \mid a \in Q,\ a \perp e\} \oplus F d_{0,c_0} = A_1 \oplus F d_{0,c_0} & \text{if } c \neq \pm e. \end{cases} \tag{13}$$

Proposition 19. *Given* $(A, *) = (C, \tau_u, \tau_v, \delta_1, \delta_2)$ *as in* (12) *and* $L = \mathrm{Der}\, A$ *then*:

(i) *If* $u, v = e$ *then* A *is standard,* $L = G_2$ *and* $A = 1 + 7$.
(ii) *If* $u, v = \pm e$ *but one of them is* $\neq e$ *then* $L = A_1 \oplus A_1$ *and* $A = 1 + 3 + 4$ *as an* L*-module. This decomposition also works for one copy of* A_1 *but* $A = 1 + 1 + 1 + 1 + 4$ *for the other.*
(iii) $u^2 + u + e = 0$, $v = u^2$ *and* $\delta_1 = \delta_2 = 1$ *if and only if* A *is an Okubo algebra. Here* $L = A_2$ *and* A *is an irreducible* L*-module.*
(iv) *If* $\dim \mathrm{alg}\{u, v\} = 2$ *but* A *is not an Okubo algebra then* $L = A_1 \oplus Z$ *with* $\dim Z = 1$ *and* $A = 1 + 3 + 4$ *as an* A_1*-module.*
(v) *If* $\dim \mathrm{alg}\{u, v\} = 4$ *then* $L = A_1$ *and* $A = 1 + 3 + 4$.

Proof. Let us start with part (iii). Assume that A is an Okubo algebra. Since e is an idempotent, by (3), the left and right multiplication operators by e in A, namely $J^{\delta_1}\tau_u$, $J^{\delta_2}\tau_v$, should be antiautomorphisms of degree 3 of C, one the inverse of the other. So $\delta_1 = \delta_2 = 1$, $\tau_u^3 = 1$ and $\tau_v = \tau_u^{-1}$, that is, $u^3 = e$ and $v = u^{-1} = u^2$. The identity $u^3 - e = 0$ factors as $(u - e)(u^2 + u + e) = 0$ and, since C is a division algebra, either $u = e$ or $u^2 + u + e = 0$. In the first case $\tau_u = \mathrm{Id} = \tau_v$ and A is a para-Hurwitz algebra, which contradicts our assumption of A being an Okubo algebra.

Let us assume now that $u^2 + u + e = 0$, $v = u^{-1}$, $\delta_1 = 1 = \delta_2$. In order to prove that A is an Okubo algebra, it suffices to check that the subalgebra fixed by τ_u has dimension 4. This subalgebra contains Q, so its dimension is $\geqslant 4$. Moreover, since C is a division composition algebra, any subalgebra is a composition algebra too and so, its dimension must be either 1, 2, 4 or 8. In particular the dimension of the subalgebra fixed by τ_u must be 4 or 8. Since $\tau_u \neq \mathrm{Id}$ then this dimension is 4 and A is an Okubo algebra.

Part (i) is obvious.

(ii) In this case (13) shows that $\mathrm{Der}(C,Q) = A_1 \oplus A_1 \subseteq L$. Since we have proved in part (iii) that this is not an Okubo algebra, by Theorem 33 A contains a trivial L-module. However, the only subspace killed by $\mathrm{Der}(C,Q)$ is Fe, so $Fe = \{x \in A \mid Lx = 0\}$. In particular, any derivation commutes with the left and right multiplication operators by e, thus $L = \{d \in \mathrm{Der}\, C \mid [d,\tau_u] = 0 = [d,\tau_v]\}$. This forces that any derivation leaves invariant the eigenspace of eigenvalue 1 of τ_u and τ_v, that in this case is Q. Therefore $\mathrm{Der}(C,Q) \subseteq \mathrm{Der}\, A \subseteq \mathrm{Der}(C,Q)$, which proves the statement in this case.

(iv) We can write $\mathrm{alg}\{u,v\}$ as $\mathrm{alg}\{c_0\}$ where $c_0 \in e^{\perp}$. By (13) we have that $\mathrm{span}\{d_{a,a} \mid a \in Q \cap e^{\perp}\} \oplus F d_{0,c_0} = A_1 \oplus Z$ is contained in L. As before, $Fe = \{x \in A \mid Lx = 0\}$ and $L = \{d \in \mathrm{Der}\, C \mid [d,\tau_u] = 0 = [d,\tau_v]\} = A_1 \oplus Z$.

(v) Similar. □

Algebras $(C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2)$ and the family $\mathcal{W}$

Let C be a division octonion algebra and $K = \mathrm{span}\{e, f\}$, where $f \perp e$, a two-dimensional subalgebra. Given W a three-dimensional subspace of $K^{\perp}$ such that $K^{\perp} = W \perp fW$, we have

Lemma 20. *Any skew-symmetric transformation d of W relative to $(,)$ extends to a derivation in $\mathrm{Der}(C,K)$ given by:*

$$d|_K = 0, \qquad d|_W = d \quad \textit{and} \quad d(fx) = fd(x) \quad \forall x \in W.$$

Proof. It is enough to show that d is K-linear and skew for the hermitian form $\sigma(,)$ in (9). On one hand, $d(f(fx)) = d(f^2x) = -n(f)\,dx = f^2 d(x) = f\,d(fx)$ says that d is K-linear. On the other hand, for any $x, y \in W$, $(d(x)y, f) = -(dx, fy) = 0$ so $\sigma(dx, y) = -(d(x)y, e) = (dx, y) = -(x, dy) = -\sigma(x, dy)$. Now, since d is K-linear and σ is hermitian,

$$\sigma\big(d(ax), by\big) = a\bar{b}\sigma(dx, y) = -a\bar{b}\sigma(x, dy) = -\sigma\big(ax, d(by)\big),$$

for all $a, b \in K$ and $x, y \in W$. Therefore d is σ-skew. □

Under this extension, $\mathrm{skew}(W, n)$ (the skew-symmetric transformations relative to n) embeds in $\mathrm{Der}(C,K)$ as a subalgebra isomorphic to A_1 and decomposes C as $1 + 1 + 3 + 3$.

Given C, K, W, f as above, $\phi_{K,1}$ and $\phi_{K,2}$ two isometries of K, and $\epsilon_1, \epsilon_2 = \pm 1$ with ϵ_1 or $\epsilon_2 \neq 1$, we can define two isometries ϕ_1, ϕ_2 on C by:

$$\begin{aligned} &\phi_1|_K = \phi_{K,1}, \qquad \phi_1|_W = \mathrm{Id}, \qquad \phi_1|_{fK} = \epsilon_1\,\mathrm{Id},\\ &\phi_2|_K = \phi_{K,2}, \qquad \phi_2|_W = \mathrm{Id}, \qquad \phi_2|_{fK} = \epsilon_2\,\mathrm{Id}, \end{aligned}$$

and a new product on C given by

$$x * y = \phi_1(x)\phi_2(y).$$

We denote this new division composition algebra by $(C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2)$ and the family of all these algebras by $\mathcal{W}$. It is clear that $\mathrm{skew}(W, n) \cong A_1 \subseteq \mathrm{Der}(C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2)$ and $A_1 = \{d \in \mathrm{Der}(C,K) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$.

Proposition 21. *Any algebra A in $\mathcal{W}$ either decomposes as $1+1+3+3$ and $\operatorname{Der} A = A_1$ or belongs to $\mathcal{W}\cap\mathcal{T}$ and decomposes as $1+3+4$.*

Proof. Let $(A,*) = (C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2) \in \mathcal{W}$. By Propositions 34 and 35, the possible decompositions of $(A,*)$ as a module for its derivation algebra L are:

$$\text{(i)}\quad 1+7, \qquad \text{(ii)}\quad 8, \qquad \text{(iii)}\quad 1+1+6,$$
$$\text{(iv)}\quad 1+3+4, \quad \text{or} \quad \text{(v)}\quad 1+1+3+3.$$

Choose $a, b \in K$ such that $\phi_1(a) = e$ and $\phi_2(b) = e$, so the left multiplication operator by a is ϕ_2, and the right multiplication operator by b is ϕ_1. Now let us assume that A decomposes as $1+1+3+3$ as an L-module, that is, the fifth possibility. Since $A_1 \subseteq L$ and K is the subspace killed by A_1 we have that K is the sum of the two trivial L-modules of A. In particular, any derivation kills a and b, thus it commutes with ϕ_1 and ϕ_2. Therefore, $L = \{d \in \operatorname{Der}(C,K) \mid [d,\phi_1] = 0 = [d,\phi_2]\} = A_1$, which proves the statement. In case (iv), by Proposition 38, $(A,*)$ also belongs to $\mathcal{T}$. So we will be done if we prove that the other possibilities do not occur.

Let us start with (i). Here, by Proposition 36, $(A,*)$ is standard, so there exists $e_0 \in K$ (e_0 is killed by $L \supset A_1$) and $\delta_1, \delta_2 = 0, 1$ such that for any $x \perp e_0$ we have $e_0 * x = (-1)^{\delta_1} x$ and $x * e_0 = (-1)^{\delta_2} x$. However,

$$a * x = \phi_1(a)\phi_2(x) = x = (-1)^{\delta_1} e_0 * x \quad \forall x \in W,$$
$$a * y = \epsilon_2 y = \epsilon_1 (-1)^{\delta_1} e_0 * y \quad \forall y \in fW.$$

Therefore, $(-1)^{\delta_1} e_0 = a = \epsilon_2 (-1)^{\delta_1} e_0$, that is, $\epsilon_2 = 1$. Analogous computations with b lead to $\epsilon_1 = 1$, which is a contradiction.

(ii) By Theorem 8, $(A,*)$ is an Okubo algebra. Take $x \in W$ and $y \in fW$, then by (4)

$$n(x)a = x * (a * x) = x * x = xx = -n(x)e,$$
$$n(y)a = y * (a * y) = \epsilon_2 y * y = -\epsilon_1 n(y)e.$$

Which shows that in this case $\epsilon_1 = 1$. However, if we work with b we obtain $\epsilon_2 = 1$, which gives the contradiction.

(iii) As above, in this case L kills a and b, and commutes with ϕ_1 and ϕ_2. As a consequence, L preserves the eigenspace of eigenvalue -1 of ϕ_1 and ϕ_2 in $K^\perp$. So this eigenspace turns out to be a three-dimensional L-submodule, which is not possible. □

Since there are no algebras in $\mathcal{T}$ with decomposition $1+1+3+3$ it follows from the previous proposition that $\mathcal{W}\cap\mathcal{T}$ is the set of algebras in $\mathcal{W}$ decomposing as $1+3+4$. We would like to describe $\mathcal{W}\cap\mathcal{T}$ in terms of K, W, ϕ_1, ϕ_2. To accomplish that, we should bear in mind that any isometry of K can be written as $J^\delta(ax)$ for some $a \in K$ with $n(a) = 1$ and $\delta = 0, 1$ [39]. Also, using (9) and (10), it is not difficult to check that $W \subseteq K^\perp$ is orthogonal to fW if and only if $W = \operatorname{span}\{w_1, w_2, c(w_1 w_2)\}$ for some σ-orthogonal elements w_1, w_2 and some $c \in K$.

The following examples belong to $\mathcal{W}\cap\mathcal{T}$:

Example 22. Given $0 \neq c \in K = \mathrm{span}\{e, f\}$ $(e \perp f)$ a two-dimensional subalgebra of C, consider $W = \mathrm{span}\{w_1, w_2, c(w_1w_2)\} \subset K^{\perp}$ where w_1 and w_2 are σ-orthogonal relative to the hermitian form σ in (9) and $W' = fW$. Define

$$\phi_i(x) = J^{\delta_i}(a_i x), \qquad \phi_i|_W = \mathrm{Id}, \qquad \phi_i|_{W'} = -\mathrm{Id}, \quad x \in K,\ i = 1, 2,$$

where $a_1 = -(-1)^{\delta_1} J^{\delta_1}(\frac{cc}{n(c)})$, $a_2 = -(-1)^{\delta_2} J^{\delta_2}(\frac{\bar{c}\bar{c}}{n(c)})$ and $\delta_i = 0, 1$. The algebra $\mathrm{Der}(A, *)$ with $x * y = \phi_1(x)\phi_2(y)$ is $A_1 \oplus A_1$ and decomposes A as $1 + 3 + 4$.

(*Sketch of proof:* Consider $e_0 = (-1)^{\delta_1+\delta_2} e$ and $Q = Fe_0 \oplus cW$. It is straightforward to check that Q is a subalgebra of $(A, *)$ and that the coordinate matrices of L_{e_0} and R_{e_0} in a basis taken from the decomposition $Fe_0 \oplus cW \oplus Ff \oplus cW'$ agree with those of $\tilde{J}^{\delta_2+1}\tilde{\tau}_{-e_0}$ and $\tilde{J}^{\delta_1+1}\tilde{\tau}_{-e_0}$ where the tilde denotes the corresponding operator on $\tilde{C} = ((A, *), e_0)$ (recall the notation from the introduction). Proposition 19 concludes the proof.)

Example 23. Let c, K, W and W' as in Example 22. Define

$$\phi_i(x) = J^{\delta_i}(a_i x), \qquad \phi_i|_W = \mathrm{Id}, \qquad \phi_1|_{W'} = -\mathrm{Id} = -\phi_2|_{W'}, \quad x \in K,\ i = 1, 2,$$

where $a_1 = -(-1)^{\delta_1} \frac{\bar{c}\bar{c}}{n(c)}$, $a_2 = (-1)^{\delta_2} a_1 J^{\delta_2}(a_1^{-1})$ and $\delta_i = 1, 2$. The algebra $\mathrm{Der}(A, *)$ with $x * y = \phi_1(x)\phi_2(y)$ is $A_1 \oplus A_1$ and decomposes A as $1 + 3 + 4$. (*Sketch of proof:* Take $e_0 = (-1)^{\delta_2} a_1^{-1}$ and $Q = Fe_0 \oplus \bar{c}W$, which is a subalgebra of $(A, *)$. The coordinate matrices of L_{e_0} and R_{e_0} in a basis from the decomposition $Fe_0 \oplus \bar{c}W \oplus Ffe_0 \oplus \bar{c}W'$ agree with those of $\tilde{J}^{\delta_2}$ and $\tilde{J}^{\delta_1+1}\tilde{\tau}_{-e}$ where the tilde denotes the corresponding operators over $\tilde{C} = ((A, *), e_0)$.)

Example 24. Since an algebra and its opposite share the same derivation algebra, opposite algebras of those in Example 23 also belong to $\mathcal{W} \cap \mathcal{T}$.

Proposition 25. *The only algebras in* $\mathcal{W}$ *which do not decompose as* $1 + 1 + 3 + 3$ *are those in Examples* 22, 23 *and* 24.

Proof. Consider an algebra $(A, *) = (C, \phi_{K,1}, \phi_{K,2}, W, \epsilon_1, \epsilon_2)$ in $\mathcal{W}$ which decomposes as $1 + 3 + 4$ and write $\phi_{K,i}(x)$ as $J^{\delta_i}(a_i x)$ for any $x \in K$ and $i = 1, 2$, and $W' = fW$.

First we assume that $\phi_i|_{W'} = -\mathrm{Id}$. The three-dimensional submodule M in the decomposition $1 + 3 + 4$ must be irreducible for A_1. Since $KW = K^{\perp}$ we have $cw \in M$ for some $c \in K$ and $w \in W$. Hence, $M \cap cW \neq 0$ and by irreducibility $M = cW$. Take e_0 the idempotent spanning the trivial submodule 1 in $1 + 3 + 4$. By imposing that L_{e_0} and R_{e_0} acts on cW as a possible switch of sign, that e_0 is idempotent, that $Fe_0 \oplus cW$ must be a subalgebra of $(A, *)$ and computing the coordinate matrices of L_{e_0} and R_{e_0} in a basis coming from the decomposition $Fe_0 \oplus cW \oplus Ff \oplus cW'$, it follows that $(A, *)$ is as in Example 22.

Now let us assume that $\phi_1|_{W'} = -\mathrm{Id}$ and $\phi_2|_{W'} = \mathrm{Id}$. Take $c \in K$ such that $M = \bar{c}W$. Since $\phi_1(a_1^{-1}) = e$ we have that $a_1^{-1} * x = x$ $\forall x \in K^{\perp}$. Applying $d \in L$ in this equation we get $d(a_1^{-1}) * x = (\mathrm{Id} - \phi_2)(dx) \in K$ $\forall x \in K^{\perp}$. By dimensions it must be $d(a_1^{-1}) = 0$. By imposing that, up to sign, a_1^{-1} must be idempotent, that $Fa_1^{-1} \oplus \bar{c}W$ is a subalgebra and computing the coordinate matrices of L_{e_0} and R_{e_0} with $e_0 = (-1)^{\delta_2} a_1^{-1}$ in a basis arising from the decomposition $Fe_0 \oplus \bar{c}W \oplus Ffe_0 \oplus (e\bar{c})W$, we get that $(A, *)$ is as in Example 23.

In case $\phi_1|_{W'} = \mathrm{Id}$ and $\phi_2|_{W'} = -\mathrm{Id}$ it is enough to consider its opposite and use the previous paragraph to see that it must be as in Example 24. □

Algebras $(C, \phi_{Q,1}, \phi_{Q,2})$ *and the family* $\mathcal{Q}$

Let C be a division octonion algebra with standard involution $J : x \mapsto \bar{x}$ and Q a quaternion subalgebra. Given two isometries $\phi_{Q,1}, \phi_{Q,2}$ on Q we can extend them to be isometries ϕ_1, ϕ_2 of C by the following formulas:

$$\phi_1|_Q = \phi_{Q,1}, \qquad \phi_1|_{Q^\perp} = \mathrm{Id},$$
$$\phi_2|_Q = \phi_{Q,2}, \qquad \phi_2|_{Q^\perp} = \mathrm{Id}.$$

With these isometries we define a new algebra $(A, *) = (C, \phi_{Q,1}, \phi_{Q,2})$ with the following product

$$x * y = \phi_1(x)\phi_2(y).$$

The family of all these algebras will be denoted by $\mathcal{Q}$.

Notice that $A_1 = \{d_{0,c} \in \mathrm{Der}\, C \mid c \in Q \cap e^\perp\}$ commutes with ϕ_1 and ϕ_2, so $A_1 \subseteq L := \mathrm{Der}(C, \phi_{Q,1}, \phi_{Q,2})$. Moreover, this subalgebra decomposes A as $1 + 1 + 1 + 1 + 4$.

We would like to show that $L = \{d \in \mathrm{Der}\, C \mid [d, \phi_1] = 0 = [d, \phi_2]\}$ and then compute it. In particular, A should contain a trivial submodule, which in this case is the same as $(A, *)$ not to being an Okubo algebra (Proposition 34). Let us check this point. Assume the contrary that $(A, *)$ is an Okubo algebra. Then by (4):

$$n(x)y = x * (y * x) = x * (yx) = x\phi_2(yx) \quad \forall x, y \in Q^\perp.$$

Thus,

$$\phi_2(yx) = n(x)x^{-1}y = \bar{x}y = -xy = yx - t(yx)e \quad \forall 0 \neq x, y \in Q, \tag{14}$$

where we have used the linearization $xy + yx = -2(x, y)e = 2(xy, e)e = t(xy)e$ of $x^2 = -(x, x)e$ in $e^\perp$. Since $Q = Q^\perp Q^\perp$ [43], we obtain that

$$\phi_2(c) = c - t(c)e = -\bar{c} \quad \forall c \in Q,$$

and in a similar way that $\phi_1(c) = -\bar{c}$ $\forall c \in Q$. Now, if we look at the definition of ϕ_1 and ϕ_2, we realize that $\phi_1 = -J = \phi_2$. Therefore $(A, *)$ is a para-Hurwitz algebra, which contradicts our assumption of $(A, *)$ being an Okubo algebra.

Associated to the $\mathbb{Z}_2$-gradation $Q \oplus Q^\perp$ of $(A, *)$ we have a $\mathbb{Z}_2$-gradation of L given by $L = L_0 \oplus L_1$ where $L_0 = \{d \in L \mid d(Q) \subseteq Q\}$ and $L_1 = \{d \in L \mid d(Q) \subseteq Q^\perp\}$. Let $a, b \in Q$ be such that $\phi_1(a) = e = \phi_2(b)$, that is, $a * x = \phi_2(x)$ and $x * b = \phi_1(x)$ for all $x \in A$. Given $d \in L_1$, the existence of a trivial submodule inside A implies that there exists $x \in Q^\perp$ killed by d. Now $0 = dx = d(a * x) = da * x$ implies $da = 0$, and similarly $db = 0$. Therefore, any derivation in L_1 commutes with ϕ_1 and ϕ_2. Moreover, given $d \in L_0$ and $x \in Q^\perp$ we have $dx = d(a * x) = a * dx + da * x = dx + da * x$, so $da = 0$. Since the same is true for b, this shows that $L = \{d \in \mathrm{Der}\, C \mid [d, \phi_1] = 0 = [d, \phi_2]\}$.

In order to separate the different possible algebras L that we obtain in our construction we will use an alternative description of the isometries of Q. In [42] it is shown that any isometry of

Q can be written as

$$x \mapsto aJ^{\delta}(x)b$$

for some $a, b \in Q$ with $n(ab) = 1$ and $\delta = 0, 1$. Therefore there exist values a_i, b_i, δ_i $(i = 1, 2)$ such that

$$\phi_{Q,i} : x \mapsto a_i J^{\delta_i}(x) b_i.$$

Lemma 26. *A derivation $0 \neq d = ad_c \in \operatorname{Der} Q$, with $c \in Q \cap e^{\perp}$, commutes with $\phi_{Q,i}$ if and only if a_i and $b_i \in \operatorname{alg}\{c\}$.*

Proof. Since any derivation commutes with the standard involution we can rewrite the commutation condition as:

$$(da_i)xb_i + a_i x db_i = 0 \quad \forall x \in Q. \tag{15}$$

Using the explicit form of d we get:

$$ca_i x b_i - a_i c x b_i + a_i x c b_i - a_i x b_i c = 0.$$

Multiplying on the left by a_i^{-1} and on the right by b_i^{-1} we obtain:

$$a_i^{-1} c a_i x - cx + xc - x b_i c b_i^{-1} = 0. \tag{16}$$

In particular, with $x = e$ we have that $a_i^{-1} c a_i = b_i c b_i^{-1}$, so we can rewrite (16) as:

$$\left[a_i^{-1} c a_i - c, x\right] = 0 \quad \forall x \in Q.$$

Since the center of the algebra Q is Fe we get that $a_i^{-1} c a_i - c$ belongs to Fe. However, since it also belongs to $e^{\perp}$, it must be zero. That is, $a_i^{-1} c a_i = c$, or equivalently, $a_i \in \operatorname{alg}\{c\}$. If we recall that $a_i^{-1} c a_i = b_i c b_i^{-1}$ then we also obtain that $b_i \in \operatorname{alg}\{c\}$.

In order to prove the sufficient condition it is enough to note that if a_i, b_i belong to $\operatorname{alg}\{c\}$ then $da_i = 0 = db_i$ and (15) follows. □

As before, let $Q \oplus Q^{\perp}$ be the $\mathbb{Z}_2$-gradation of A and $L = L_0 \oplus L_1$ the induced gradation on L:

Lemma 27. *For any $d \in L_1$ we have $da_i = db_i = 0$, $i = 1, 2$. Moreover, if $L_1 \neq 0$ then $b_i = (-1)^{\delta_i} a_i / n(a_i)$, $i = 1, 2$.*

Proof. Since d commutes with ϕ_i we have for any $x \in Q$

$$J^{\delta_i} dx = d(a_i x b_i). \tag{17}$$

With $x = e$ we get $d(a_i b_i) = 0$ and with $x = b_i$ we get $(-1)^{\delta_i} db_i = J^{\delta_i} db_i = d(a_i b_i b_i) = (a_i b_i) db_i$, so either $a_i b_i = (-1)^{\delta_i} e$ or $db_i = 0$. The same reasoning with $x = a_i$ leads to that either $a_ib_i = (-1)^{\delta_i}e$ or $da_i = 0$. In order to prove the first part of the lemma, let us assume that $a_ib_i = (-1)^{\delta_i}e$, that is, $b_i = (-1)^{\delta_i}a_i^{-1}$. Now (17) reads as

$$dx = d\big(a_ixa_i^{-1}\big) \quad \forall x \in Q.$$

If we take x in Q orthogonal to $\operatorname{alg}\{a_i\} = \operatorname{span}\{e, a_i\}$ then $a_ix = x\bar{a}_i$, so

$$dx = d\big(a_ixa_i^{-1}\big) = d\left(a_ix\frac{\bar{a}_i}{n(a_i)}\right) = d\left(\frac{a_i^2}{n(a_i)}x\right),$$

or equivalently,

$$d\left(\left(\frac{a_i^2}{n(a_i)} - e\right)x\right) = 0.$$

Notice that if $a_i^2/n(a_i) - e$ is nonzero then it maps $\operatorname{alg}\{a_i\}^\perp$ into itself, and, by the last equation, $d(\operatorname{alg}\{a_i\}^\perp) = 0$. However, $\operatorname{alg}\{a_i\}^\perp$ generates all Q, so $dQ = 0$. Since d is skew and $dQ^\perp \subseteq Q$ then $d = 0$. Therefore, we get that $a_i^2 = n(a_i)e$. Since $a_i^2 = t(a_i)a_i - n(a_i)e$ this means that $a_i \in Fe$ and so, $da_i = 0 = db_i$.

If $L_1 \neq 0$ then we have $x \in Q$ and $d \in L_1$ with $dx \neq 0$. By (17) $(-1)^{\delta_i}dx = a_i((dx)b_i) = (\bar{b}_ia_i)dx$ so $\bar{b}_ia_i = (-1)^{\delta_i}e$ and $b_i = (-1)^{\delta_i}a_i/n(a_i)$. $\square$

We will discuss L in terms of a_i and b_i.

Proposition 28. *Let L be* $\operatorname{Der}(C, \phi_{Q,1}, \phi_{Q,2})$ *and a_i, b_i, δ_i as above. Then*:

(i) *If $a_i, b_i \in Fe$ and $a_ib_i = (-1)^{\delta_i}e$, $i = 1, 2$ then $L = G_2$ and $A = 1 + 7$.*
(ii) *If $a_i, b_i \in Fe$ $i = 1, 2$ but $a_1b_1 \neq (-1)^{\delta_1}e$ or $a_2b_2 \neq (-1)^{\delta_2}e$ then $L = A_1 \oplus A_1$. In this case $A = 1 + 3 + 4$ as an L-module and also as an A_1-module for one copy of A_1; for the other copy of A_1, A splits as $1 + 1 + 1 + 1 + 4$.*
(iii) *If $\dim \operatorname{alg}\{a_i, b_i \mid i = 1, 2\} = 2$ and $b_i = (-1)^{\delta_i}a_i/n(a_i)$, $i = 1, 2$ then $L = A_2$ and $A = 1 + 1 + 6$.*
(iv) *If $\dim \operatorname{alg}\{a_i, b_i \mid i = 1, 2\} = 2$ but either $b_1 \neq (-1)^{\delta_1}a_1/n(a_1)$ or $b_2 \neq (-1)^{\delta_2}a_2/n(a_2)$ then $L = A_1 \oplus Z$ with $\dim Z = 1$ and $A = 1 + 1 + 1 + 1 + 4$ as an A_1-module.*
(v) *If $\dim \operatorname{alg}\{a_i, b_i \mid i = 1, 2\} = 4$ then $L = A_1$ and A splits as $1 + 1 + 1 + 1 + 4$.*

Proof. (i) The conditions on a_i, b_i, δ_i are equivalent to $\phi_1, \phi_2 \in \{\mathrm{Id}, -J\}$, so in this case $(A, *) = (C, \phi_{Q,1}, \phi_{Q,2})$ is standard, $L = G_2$ and $A = 1 + 7$.

(ii) From the hypothesis we have that either ϕ_1 or ϕ_2 acts on $Q \cap e^\perp$ as $-\mathrm{Id}$. Since L commutes with ϕ_1 and ϕ_2 it follows that $Q \cap e^\perp$ is a submodule. In fact, Q is a submodule because of Fe is also a submodule. This says that $L = \{d \in \operatorname{Der}(C, Q) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$. From Lemma 26 we can conclude that $L = \operatorname{Der}(C, Q) = A_1 \oplus A_1$, and by (7) we get the statement in this case.

(iii) Here $K = \operatorname{alg}\{a_i, b_i \mid i = 1, 2\}$ in $(A, *)$ is a trivial module of dimension 2, and it is also a subalgebra of C, so $L = \{d \in \operatorname{Der}(C, K) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$. The conditions on a_i and b_i say that $\phi_1|_{K^\perp} = \mathrm{Id} = \phi_2|_{K^\perp}$, thus $L = \operatorname{Der}(C, K) = A_2$ and A decomposes as $1 + 1 + 6$.

(iv) By the previous lemma we have that $L_1 = 0$ so $L = \{d \in \mathrm{Der}(C, Q) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$. Now Lemma 26 says that $L = A_1 \oplus Z(L)$ where $\dim Z(L) = 1$. Notice that $Q = 1 + 1 + 2$ as a L-module, although it is trivial as an A_1-module.

(v) Here we have that Q is killed by L, so $L = \{d \in \mathrm{Der}(C, Q) \mid dQ = 0\} = A_1$. $\square$

Algebras $(P, S, \epsilon_1, \epsilon_2)$ and the families $\mathcal{O}$ and $\mathcal{S}$

Let P be an Okubo algebra with product xy. By (1), $\mathrm{Der}\, P = \{ad_x : y \mapsto xy - yx \mid x \in P\} = A_2$ and P is irreducible as a module for this Lie algebra. We will denote by $\mathcal{O}$ the family of division Okubo algebras over F. For a deeper insight into Okubo algebras we refer to [10,11]. Our concerns are more related not with general Okubo algebras but with division Okubo algebras with idempotents, so in the following P stands for a division Okubo algebra with a nonzero idempotent e (such algebras exist whenever we have a division octonion algebra $C(-1, \beta, \gamma)$ [24]) and product denoted by xy. Recall from (3) that

$$xy = \tau(\bar{x}) \circ \tau^{-1}(\bar{y}), \tag{18}$$

where $(P, \circ)$ is an octonion algebra with unit e, standard involution $x \mapsto \bar{x}$, and τ is an automorphism of order 3 such that the set of fixed points by τ is a quaternion subalgebra Q of $(P, \circ)$.

Let us focus for a while on the Lie algebra $(P, [,])$, where $[x, y] = xy - yx$ is the usual commutator. Consider $x_0 \in Q^\perp$, then $\tau(x_0) = x_0 \circ u$ for some $u \in Q$. Since $\tau^3(x_0) = x_0$ and $\tau(x_0) \neq x_0$ then $u^3 = e$ and $u \neq e$, so $u^2 + u + e = 0$. Take $a \in Q$ orthogonal to $\mathrm{alg}\{u\}$, and $S = \mathrm{span}\{a, x_0, a \circ x_0\}$.

Lemma 29.

(i) *S is a subalgebra of $(P, [,])$ of type A_1.*
(ii) *$P = S \oplus S^\perp$ is the decomposition of P as a direct sum of irreducible S-submodules.*

Proof. (i) Using (18) we have that

$$[a, x_0] = \tau(\bar{a}) \circ \tau^{-1}(\bar{x_0}) - \tau(\bar{x_0}) \circ \tau^{-1}(\bar{a}) = a \circ \tau^{-1}(x_0) - \tau(x_0) \circ a.$$

Since $(a, \tau(x_0)) = 0$ and $\tau^{-1} + \tau + \mathrm{Id} = 0$ on $Q^\perp$ we obtain

$$[a, x_0] = a \circ \tau^{-1}(x_0) + a \circ \tau(x_0) = -a \circ x_0.$$

In the same way, $[a, a \circ x_0] = -a \circ (a \circ x_0) = n(a)x_0$; and finally,

$$\begin{aligned}
[x_0, a \circ x_0] &= \tau(x_0) \circ \tau^{-1}(a \circ x_0) - \tau(a \circ x_0) \circ \tau^{-1}(x_0) \\
&= -(x_0 \circ u) \circ \big(x_0 \circ \big(a \circ u^2\big)\big) + \big(x_0 \circ (a \circ u)\big) \circ \big(x_0 \circ u^2\big) \\
&= n(x_0)(a \circ u + u \circ a) = -n(x_0)a
\end{aligned}$$

shows that S is a subalgebra of $(P, [,])$ with $[S, S] = S$; therefore, S is a Lie subalgebra of type A_1.

(ii) First of all we should notice that for any submodule $0 \neq V \subseteq S^\perp$ the orthogonal complement $V' = S^\perp \cap V^\perp$ is a submodule too, and $S^\perp = V \oplus V'$. Thus, $S^\perp$ is completely reducible, and, consequently, it is either irreducible or contains a proper submodule of dimension at most 2. Since S acts as skew-symmetric maps, such a submodule would be trivial. Let us consider $y \in S^\perp$ with $[S, y] = 0$. On one hand $0 = [a, y]$ implies that $\tau(\bar{y}) = a^{-1} \circ \bar{y} \circ a$, and, in particular, $\tau^{-1}(\bar{y}) = \tau^2(\bar{y}) = \bar{y}$, or equivalently $\tau(\bar{y}) = \bar{y} = a^{-1} \circ \bar{y} \circ a$. Since the only elements commuting with a are those in $\mathrm{alg}\{a\}$, this says that $y \in Fe$. But, on the other hand, $[x_0, e] = -\tau(x_0) + \tau^{-1}(x_0) = x_0 \circ (u^2 - u) \neq 0$ forces $y = 0$. □

Given P, S as above, and $\epsilon_i = \pm 1$, $i = 1, 2$, but one of them $\neq 1$, we can define a new product $*$ on P by:

$$x * y = \phi_1(x)\phi_2(y),$$

where ϕ_1, ϕ_2 are the isometries

$$\phi_i|_S = \epsilon_i \,\mathrm{Id}, \qquad \phi_i|_{S^\perp} = \mathrm{Id}, \quad i = 1, 2.$$

The new algebra $(P, *)$ will be denoted by $(P, S, \epsilon_1, \epsilon_2)$, and $\mathcal{S}$ will stand for the family of all these algebras.

Proposition 30. $\mathrm{Der}(P, S, \epsilon_1, \epsilon_2)$ *is a Lie algebra of type* A_1 *and* $(P, S, \epsilon_1, \epsilon_2)$ *is the direct sum of two irreducible modules of dimensions* 3 *and* 5 *for this algebra.*

Proof. Since $\mathrm{ad}_x : y \mapsto [x, y] = xy - yx$ with $x \in S$ commutes with ϕ_1 and ϕ_2, then $\{\mathrm{ad}_x \mid x \in S\} \subseteq L = \mathrm{Der}(P, S, \epsilon_1, \epsilon_2)$. In particular, there are not trivial submodules contained in P. By Proposition 34, this means that either $(P, *)$ is an Okubo algebra or L is of type A_1 and $P = 3 + 5$. However, if $(P, *)$ is an Okubo algebra, it is symmetric, and (4) together with

$$\begin{aligned}(a * a) * a &= \epsilon_1\epsilon_2(aa) * a = -n(a)\epsilon_1\epsilon_2 e * a = \epsilon_1 n(a)a,\\ a * (a * a) &= \epsilon_2 n(a)a\end{aligned}$$

would imply $\epsilon_1 = 1 = \epsilon_2$, which is not possible. Therefore, $(P, *)$ is not an Okubo algebra. □

2. Module structure of A

We include the following result for completeness. We keep the notation $\bar{S}$ for $\bar{F} \otimes_F S$.

Proposition 31. *Let L be a Lie algebra and V an irreducible finite-dimensional L-module. If $\bar{V}$ is a completely reducible $\bar{L}$-module then all the irreducible components of $\bar{V}$ have the same dimension.*

Proof. Since V is irreducible, $D = \mathrm{End}_L(V)$ is a division ring. Take K a maximal subfield of D and $Z(D)$ the center of D. Following [27] we have that $K \otimes_{Z(D)} D \cong \mathrm{Mat}_n(K)$ where $n = [D : K] = [K : Z(D)]$. If k denotes $\dim_F Z(D)$ then $\dim_F D = kn^2$. Now

$$\bar{F} \otimes_{Z(D)} D \cong \bar{F} \otimes_K (K \otimes_{Z(D)} D) \cong \bar{F} \otimes_K \mathrm{Mat}_n(K) \cong \mathrm{Mat}_n(\bar{F}),$$

and

$$\begin{aligned}\bar{F} \otimes_F D &\cong \bar{F} \otimes_F \big(Z(D) \otimes_{Z(D)} D\big) \\ &\cong \big(\bar{F} \otimes_F Z(D)\big) \otimes_{Z(D)} D \cong \bigoplus^k \bar{F} \otimes_{Z(D)} D \cong \bigoplus^k \mathrm{Mat}_n(\bar{F}),\end{aligned} \tag{19}$$

where we have used that $\bar{F} \otimes_F Z(D) \cong \bigoplus^k \bar{F}$. This follows from the fact that, since $\bar{V}$ is completely reducible then $\bar{F} \otimes_F D$ must be semisimple so must be the factor $\bar{F} \otimes_F Z(D)$. Therefore, $\bar{F} \otimes_F D^{\mathrm{opp}} \cong \oplus \mathrm{Mat}_n(\bar{F})$ (where D^{opp} denotes the opposite algebra of D). Now consider R the associative algebra generated by L in $\mathrm{End}(V)$. This is a primitive algebra and by Jacobson's density theorem $R \cong \mathrm{End}_D(V) = \mathrm{Mat}_m(D^{\mathrm{opp}})$ where $m = \dim_D V$. Therefore,

$$\bar{R} = \bar{F} \otimes_F R \cong \bar{F} \otimes \mathrm{Mat}_m\big(D^{\mathrm{opp}}\big) \cong \mathrm{Mat}_m\big(\bar{F} \otimes_F D^{\mathrm{opp}}\big) \cong \bigoplus \mathrm{Mat}_{mn}(\bar{F})$$

and the dimension of the irreducible submodules of $\bar{V}$ are the same. □

Now consider A a division composition algebra of dimension 8 and $L = \mathrm{Der}\, A$,

Corollary 32. *If V is an irreducible L-submodule of A then $\bar{V}$ is completely reducible.*

Proof. Notice that the only place where we have used that $\bar{V}$ is completely reducible is in (19) to get that $\bar{F} \otimes_F Z(D)$ is semisimple. Ignoring this point, the previous proof also proves that $\bar{V}$ is completely reducible. Assume that $\bar{F} \otimes_F Z(D)$ is not semisimple, so $Z(D)$ is not separable and $\mathrm{char}\, F$ divides $[Z(D) : F] = k$. In our case $mn^2k = \dim_F V \leqslant 8$, so $n, m = 1$ and $k = \dim_F V = 5$ or 7. In particular, $D = Z(D)$ is a field, D is not separable over F, and $\dim_D V = 1$. Therefore, $V = Dv$ for any $0 \neq v \in V$. Now $R \cong \mathrm{End}_D(V) = D$ is a field, which means that $L|_V$ must be abelian. In that case, we know from Proposition 12 that $\bar{V}$ must be completely reducible. This gives the contradiction. □

Recall from [4] that over an algebraically closed field of characteristic $p > 2$ any irreducible sl(2)-module is either isomorphic to a module $V(m)$ or its dimension is p. Moreover, $V(m)$ is irreducible if and only if $0 \leqslant m \leqslant p - 1$. These modules have dimension $m + 1$ and the Cartan subalgebra acts with weights $mw_1, (m-2)w_1, \ldots, -(m-2)w_1, -mw_1$ where w_1 denotes the fundamental weight.

The key to find the decomposition of A as a $\mathrm{Der}\, A$ module is that whenever we have a submodule V of A then $V^\perp$ is also a submodule. Moreover, since A does not have isotropic elements, then $V \cap V^\perp = 0$ and $A = V \oplus V^\perp$. In particular A is completely reducible, as $\bar{A}$ is by the corollary.

We will say that the *toral rank* of $\mathrm{Der}\, \bar{A}$ on $\bar{A}$ is r if that is the maximum number of linearly independent weights of any Cartan subalgebra of $\mathrm{Der}\, \bar{A}$ on $\bar{A}$. In [25] it was proved

Theorem 33. *Let $\bar{A}$ be a composition algebra over an algebraically closed field, then the toral rank of* $\mathrm{Der}\, \bar{A}$ *on $\bar{A}$ is $\leqslant 2$. In case it is 2 then either*:

(i) *exists an element $e \in \bar{A}$ with $n(e) = 1$ such that* $(\mathrm{Der}\, \bar{A})e = 0$, *or*
(ii) *$\bar{A}$ is an Okubo algebra and* $\mathrm{Der}\, \bar{A}$ *is a Lie algebra of type A_2.*

This helps us in proving

Proposition 34. *If A does not possess trivial submodules then either*

(i) *A is an irreducible* $\mathrm{Der}\,A$*-module,* $\mathrm{Der}\,A = A_2$ *and A is an Okubo algebra, or*
(ii) $A = 3 + 5$ *and* $\mathrm{Der}\,A = A_1$.

Proof. First of all we should note that, by Corollary 13, $L = \mathrm{Der}\,A$ is not abelian, so we can write it as $[L, L] \oplus Z(L)$, where $Z(L)$ denotes the center of L and $[L, L] \neq 0$ (Theorem 4). Now consider the set $\{x \in A \mid [L, L]x = 0\}$. It is clear that this is a composition subalgebra of dimension 0, 1, 2, or 4. Since $Z(L)$ commutes with $[L, L]$ this subalgebra is invariant under $Z(L)$ and by Proposition 12 either it is 0 or there exists x inside it such that $Z(L)x = 0$. This element x would be killed by L, so A has no trivial $[L, L]$-submodules. In particular it has no submodules of dimension 1 or 2 (notice L acts as skew-symmetric maps).

As before, A is completely reducible as an $[L, L]$-module. Since all submodules have dimension > 2 the only possible decompositions are: $3 + 5$, $4 + 4$ or 8. Let us rule out the possibility $4 + 4$. By Theorems 33 and 4 we have that $\bar{L} = \mathrm{sl}(2)$. By Proposition 31 we also know that $\bar{A}$ decomposes as either $4 + 4$, $4 + 2 + 2$ or $2 + 2 + 2 + 2$. These modules are either $V(3)$ or $V(1)$. However we can take a Cartan subalgebra $\bar{F}d$ of $\mathrm{Der}\,\bar{A}$ with $d \in \mathrm{Der}\,A$ and, by Proposition 12, 0 should be an eigenvalue of d in $\bar{A}$, which is not true if the only modules in the decomposition of $\bar{A}$ are $V(3)$ or $V(1)$.

If the decomposition is $3 + 5$ then Theorems 33 and 4 say that $L = A_1$. Now assume that $A = 8$. In this case $\bar{A}$ decomposes as either 8, $4 + 4$ or $2 + 2 + 2 + 2$. As before the last two decompositions do not occur, so $\bar{A} = 8$. When the toral rank is 2, the statement follows from Theorem 33. When the toral rank is 1 then $\bar{L} = \mathrm{sl}(2)$ and $\bar{A} = V(7)$. Therefore 0 is not a weight, but this is a contradiction. □

Proposition 35. *If A has trivial submodules then the possible decompositions of A are*:

(i)	$1 + 7$,	(iv)	$1 + 1 + 3 + 3$,	(vii)	$1 + 1 + 1 + 1 + 4$,
(ii)	$1 + 1 + 6$,	(v)	$1 + 1 + 2 + 4$,	(viii)	$1 + 1 + 1 + 1 + 2 + 2$.
(iii)	$1 + 3 + 4$,	(vi)	$1 + 1 + 2 + 2 + 2$,		

In case (iv) $L = A_1$*, and in cases* (vi) *and* (viii) *L is abelian.*

Proof. Notice that $\{x \in A \mid Lx = 0\}$ is a composition subalgebra, so its dimension is 1, 2 or 4.

If A has an irreducible submodule of dimension 7 then we get (i). If it has an irreducible submodule of dimension 6 then the subspace orthogonal to this is a submodule of dimension 2 that contains a trivial submodule. Since A is completely reducible, this submodule is trivial, and we are in (ii).

Now let us assume that there exists an irreducible submodule of dimension 5. This possibility does not appear in the statement, so we should rule it out. The full decomposition of A in this case should be either $1 + 2 + 5$ or $1 + 1 + 1 + 5$. By the remark at the beginning of the proof, $1 + 1 + 1 + 5$ is impossible. Now if $A = 1 + 2 + 5$ then by Corollary 13 L is not abelian, so $L = [L, L] \oplus Z(L)$ with $[L, L] \neq 0$. The three-dimensional module $1 + 2$ is killed by $[L, L]$ and, since the set of elements killed by $[L, L]$ is a composition subalgebra, we should have a trivial $[L, L]$-module of dimension 1 inside of 5. However, since 5 is invariant under $Z(L)$ and $[L, L]$ commutes with $Z(L)$, this one-dimensional $[L, L]$-submodule is in fact an L-module, which is not possible.

If A has an irreducible submodule of dimension 4 then the orthogonal complement of it decomposes as either $1+3$, $1+1+2$ or $1+1+1+1$, which gives cases (iii), (v) and (vii).

Now let us assume that the largest irreducible submodule of A has dimension 3. By the remark at the beginning of the proof the only possible decompositions of the submodule orthogonal to this are $1+1+3$ or $1+2+2$. In the latter case, $\mathrm{alg}\{1+2+2\}$ is a composition subalgebra of dimension $\geqslant 5$, so $\mathrm{alg}\{1+2+2\} = A$. In particular, any derivation that kills $1+2+2$ must be 0. This implies that $\dim_F L \leqslant 2$ and L must be abelian. However, by Corollary 13 this is a contradiction. Therefore, the only case left is $1+1+3+3$, that is, case (iv). Now, any derivation of A killing 3 is trivial, so L embeds as a subalgebra of $A_1 = \mathrm{skew}(3)$. Moreover, if L were abelian then it would have a trivial submodule in 3, which is not possible, so $L = A_1$.

In case that the largest irreducible submodule has dimension 2, it is easy to prove that A decomposes as in cases (vi) or (viii) and that L must be abelian. □

3. Description of division composition algebras

In this section we will prove that any division composition algebra A whose derivation algebra $L = \mathrm{Der}\, A$ is nonabelian belongs to one of the five families studied in Section 1.

Proposition 36. *If $A = 1+7$ then A is standard and $A \in \mathcal{T}$.*

Proof. Take e such that $Le = 0$, $e^2 = e$, $n(e) = 1$ and V the orthogonal complement of Fe. By Proposition 31, V remains irreducible when extending scalars to $\bar{F}$. Thus, by Schur's lemma we have that $\dim \mathrm{End}_L(V) = 1$. Since R_e and L_e are L-homomorphisms we have that $L_e|_V = \lambda\, \mathrm{Id}$ and $R_e|_V = \mu\, \mathrm{Id}$. Moreover, $n(e) = 1$ implies that $\lambda, \mu = \pm 1$. Now it is clear that A is a standard algebra associated to $C = (A, e)$. □

Proposition 37. *If $A = 1+1+6$ then $A \in \mathcal{Q}$.*

Proof. Let us denote $1+1$ by K and 6 by V. Take $e_1, e_2 \in \bar{K}$ with $n(e_1) = 0 = n(e_2)$ and $n(e_1+e_2) = 1$. We have $\ker \bar{L}_{e_1} \cap \ker \bar{L}_{e_2} \subseteq \ker \bar{L}_{e_1+e_2} = 0$. On one hand, $\ker \bar{L}_{e_i}|_{\bar{V}}$ and $\mathrm{Im}\, \bar{L}_{e_i}|_{\bar{V}}$ (the subspace image of $\bar{L}_{e_i}|_{\bar{V}}$) are totally isotropic and so, $\dim \ker \bar{L}_{e_i}|_{\bar{V}}, \dim \mathrm{Im}\, \bar{L}_{e_i}|_{\bar{V}} \leqslant 3$. On the other hand, $\dim \ker \bar{L}_{e_i}|_{\bar{V}} + \dim \mathrm{Im}\, \bar{L}_{e_i}|_{\bar{V}} = 6$, hence $\dim \ker \bar{L}_{e_i}|_{\bar{V}} = 3$ and $\bar{V} = \ker \bar{L}_{e_1}|_{\bar{V}} \oplus \ker \bar{L}_{e_2}|_{\bar{V}}$. Since $\bar{L}$ kills e_1 and e_2 these subspaces are also L-modules. It follows by Proposition 31 that $\bar{V}$ splits as $3+3$ and, in particular, $\dim \mathrm{End}_L(V) = 2$ (with the notation in that proposition, $kmn^2 = 6$ and $k = 2$ implies $n = 1$ and $m = 3$). Therefore $\mathrm{End}_L(V) = \mathrm{span}\{L_b|_V \mid b \in K\} = \mathrm{span}\{R_a|_V \mid a \in K\}$. Now take $a, b \in K$ such that $R_a|_V = \mathrm{Id} = L_b|_V$ and consider a Cayley–Dickson algebra C with product $x \circ y = (R_a^{-1}x)(L_b^{-1}y)$. It is clear that $(K, \circ)$ is a subalgebra of C. We can complete K to Q, a quaternion subalgebra of C. We just need to define $\phi_{Q,1} = R_a$ and $\phi_{Q,2} = L_b$ to get the statement. □

Proposition 38. *If $A = 1+3+4$ then $A \in \mathcal{T}$.*

Proof. Let e be such that $Le = 0$, $e^2 = e$ and $n(e) = 1$, and consider $C = (A, e)$. Note that if 3 is not an irreducible $[L, L]$-module then it is trivial for $[L, L]$. Since $L = [L, L] \oplus Z(L)$ with $\dim Z(L) \leqslant 1$ and, by skew-symmetry, $Z(L)$ has trivial submodules on 3, this implies that 3 is not longer irreducible. Therefore, 3 is an irreducible $[L, L]$-module.

Take $d \in L$ such that $d|_3 \neq 0$. Since d is a skew-symmetric map $\dim \ker d|_3 = 1$ and $\ker d|_3$ is invariant under L_e and R_e. Therefore we have a nonzero element $x \in 3$ such that $ex, xe = \pm x$. Since the eigenspaces of L_e and R_e are L-submodules we have that $L_e|_3, R_e|_3 = \pm \mathrm{Id}$.

Take the ideal $I = \{h \in L \mid h(1+3) = 0\}$. If $I \neq 0$ then $1+3 = \ker h$ for some $h \in L$, and $1+3$ is a subalgebra of A. If $I = 0$ then the map $h \mapsto h|_3$ is injective, which says that $\dim L \leqslant 3$. Therefore, in this case $L = A_1$. If we extend scalars, by Proposition 31 $\bar{A}$ decomposes either as $1+3+4$ or $1+3+2+2$, and $\bar{L} = \mathrm{sl}(2)$. It follows that $\bar{A} = V(0) \oplus V(2) \oplus V(3)$ or $V(0) \oplus V(2) \oplus V(1) \oplus V(1)$. Since $V(2) \otimes V(2) = V(4) \oplus V(2) \oplus V(0)$ we also get in this case that $1+3$ is a subalgebra. We will denote it by Q.

Consider now $\lambda, \mu \in \bar{F}$ eigenvalues of $\bar{L}_e$ and $\bar{R}_e$ respectively over $\bar{V}$, where $V = 4$. If both of them are equal to ± 1 then they are also eigenvalues of L_e and R_e; so, $L_e|_V, R_e|_V = \pm \mathrm{Id}$. It is easy to see that $R_e = J^{\delta_1} \tau_{\pm e}$ and $L_e = J^{\delta_2} \tau_{\pm e}$, where J denotes the standard involution of C and $\tau_{\pm e}$ are automorphism of C fixing the quaternion subalgebra $(Q, \circ)$. This proves the proposition in this case. So in the remainder we will assume that, for instance, $\lambda \neq \pm 1$.

Denote by $S(\lambda)$ the subspace $\{x \in \bar{V} \mid ex = \lambda x\}$. Since $n(x) = n(ex) = \lambda^2 n(x)$ $\forall x \in S(\lambda)$ we have that $S(\lambda)$ is totally isotropic and $\dim S(\lambda) \leqslant 2$. If the dimension is 1 then, by Proposition 31, $\bar{V}$ decomposes as $1+1+1+1$ and $[L, L]$ kills V. In such a case, $[L, L]$ kills a subalgebra of dimension $\geqslant 5$ and so $[L, L] = 0$, which is not possible. Therefore, $\dim S(\lambda) = 2$ Recall that L_e is an isometry of $n()$; with this in mind it is not difficult to check that $(S^*(\lambda), S^*(\lambda')) = 0$ unless $\lambda' = \lambda^{-1}$, where $S^*(\lambda) = \{x \in \bar{V} \mid \exists n \text{ s.t. } (L_e - \lambda \mathrm{Id})^n x = 0\}$. Since $\lambda \neq \lambda^{-1}$ this forces the existence of another eigenvalue λ^{-1} of L_e on $\bar{V}$. As before, $\dim S(\lambda^{-1}) = 2$ and $\bar{V} = S(\lambda) \oplus S(\lambda^{-1})$ is the decomposition into irreducible $\bar{L}$-modules. Any $0 \neq d \in [L, L]$ acts semisimply on $S(\lambda)$ and its trace is 0, so its eigenvalues are $\pm\alpha$. By skew-symmetry this is also true on $S(\lambda^{-1})$. If in addition $d|_3 \neq 0$ then, by Proposition 12, the eigenvalues of d on $\bar{Q}$ are 0 and $\pm 2\alpha$. So, by Lemma 14, we have a $\mathbb{Z}_3$-gradation of $\bar{A}$ given by $\bar{A}_0 \oplus (\bar{A}_\alpha \oplus \bar{A}_{-2\alpha}) \oplus (\bar{A}_{-\alpha} \oplus \bar{A}_{2\alpha})$. This gradation carries over to $\bar{C}$, and by Proposition 16, we have a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of $\bar{C}$ with $\bar{A}_\alpha \oplus \bar{A}_{-2\alpha} = \mathrm{span}\{u_1, u_2, u_3\}$. In $S(\lambda)$ and $S(\lambda^{-1})$ we have eigenvectors of eigenvalue α. Up to a change in the canonical basis we can assume the $u_2 \in S(\lambda)$ and $u_3 \in S(\lambda^{-1})$. We also have eigenvectors of eigenvalue $-\alpha$ in $S(\lambda)$ and $S(\lambda^{-1})$. Moreover, since these subspaces are totally isotropic it follows that $v_3 \in S(\lambda)$ and $v_2 \in S(\lambda^{-1})$. In particular we can pick δ_2 such that $J^{\delta_2} L_e$ acts as the identity on $\mathrm{span}\{e_1, e_2\}$ (on Q to be precise), it fixes $\mathrm{span}\{u_1, u_2, u_3\}$ and $\mathrm{span}\{v_1, v_2, v_3\}$, and it acts with determinant 1 over these spaces. By (6) this says this map is an automorphism of C which restricted to Q is the identity, so $L_e = J^{\delta_2} \tau_v$ for some $v \in B$. Analogously we obtain that $R_e = J^{\delta_1} \tau_u$ for some $u \in B$. □

Proposition 39. *If $A = 1+1+3+3$ then $A \in \mathcal{W}$.*

Proof. Denote the copies of 3 by W and W' and take $K = 1+1$. We can assume W and W' to be orthogonal. Notice that any derivation which kills a copy of 3 kills a subalgebra of dimension $\geqslant 5$ and consequently it must be 0. Now take $0 \neq d \in L$. Since d is skew-symmetric $\dim \ker d|_W = 1 = \dim \ker d|_{W'}$. So given $0 \neq x \in \ker d|_W + \ker d|_{W'}$ we have $\ker d|_W + \ker d|_{W'} = Kx = xK$. In particular, with $x \in \ker d|_W$ we can find $a, b \in K$ such that $xa = x = bx$. Since the eigenspaces of L_b and R_a are L-submodules we get $R_a|_W = \mathrm{Id} = L_b|_W$. Take $x' \in \ker d|_{W'}$. Since $0 = (x, x') = (xa, x'a) = (x, x'a)$ it follows that $x'a = \lambda x'$. Similarly $bx' = \mu x'$. Moreover, from the fact that $n(a) = 1 = n(b)$ we get $\lambda, \mu = \pm 1$. If both of them are equal to 1 then A is also an algebra of type $(C, \phi_{Q,1}, \phi_{Q,2})$, which is a contradiction by Proposition 28. So at least one of them is -1. Now it is enough to change to the Cayley–Dickson algebra given by $x \circ y = (R_a^{-1}x)(L_b^{-1}y)$. $\square$

Proposition 40. *If $A = 1+1+2+4$ or $A = 1+1+1+1+4$ then $A \in \mathcal{Q}$.*

Proof. Since we assume L is not abelian, it follows that $0 \neq [L, L]$ kills any submodule of dimension 2. Therefore as an $[L, L]$-module, in both cases A decomposes as $1+1+1+1+4$. Let us denote $1+1+1+1$ by B and 4 by $B^{\perp}$. By Proposition 31 we have that the $[\bar{L}, \bar{L}]$-module $\bar{B}^{\perp}$ splits as either $2+2$ or 4. Therefore, the dimension of $\mathrm{End}_{[L,L]}(B^{\perp})$ is at most 4. Now, notice that $\{R_a|_{B^{\perp}} \mid a \in B\}$ and $\{L_b|_{B^{\perp}} \mid b \in B\}$ are four-dimensional subspaces contained in $\mathrm{End}_{[L,L]}(B^{\perp})$ we conclude that $\dim \mathrm{End}_{[L,L]}(B^{\perp}) = 4$, and that we can pick $a, b \in B$ such that $R_a|_{B^{\perp}} = \mathrm{Id} = L_b|_{B^{\perp}}$. Take C to be the Cayley–Dickson algebra defined by $x \circ y = (R_a^{-1}x)(L_b^{-1}y)$, which contains B as a quaternion subalgebra, and define $\phi = R_a$ and $\psi = L_b$. $\square$

The case $A = 3+5$ and L of type A_1 requires more elaborate arguments. First of all, notice that passing to the algebraic closure $\bar{F}$ of F we have that, by Proposition 31, $\bar{A} = \bar{F} \otimes_F A = 3+5$ and $\bar{L} = \mathrm{sl}(2)$. From the restriction $\mathrm{char}\, F \neq 2, 3$ it follows that $3 = V(2)$. Moreover, 3 is not a subalgebra of $\bar{A}$, so we have a nonzero A_1-projection $V(4) \oplus V(2) \oplus V(0) = V(2) \otimes V(2) \to 5$, that forces $5 = V(4)$ (notice that over fields of characteristic 5 there are more five-dimensional irreducible sl(2)-modules than $V(4)$ [4]). We can look at our product on A as a linear combination of sl(2)-projections, which are completely determined:

Lemma 41. *Let $\bar{F}$ be an algebraically closed field of characteristic $\neq 2, 3$ then*

(i) $\dim \mathrm{Hom}_{\mathrm{sl}(2)}(V(i) \otimes V(j), V(k)) = 1$, $i, j, k = 2, 4$,
(ii) $\dim \mathrm{Hom}_{\mathrm{sl}(2)}(V(i) \otimes V(i), V(0)) = 1$, $i = 2, 4$,
(iii) $\dim \mathrm{Hom}_{\mathrm{sl}(2)}(V(2) \otimes V(4), V(0)) = 1$.

Proof. This is an immediate consequence of Theorem 1.11 in [4] and the fact that the modules $Q(l)$ appearing there have only one irreducible quotient, which is isomorphic to $V(l)$. $\square$

As candidates for these projections we can use the product $*$ on $\mathrm{P}_8(\bar{F})$ defined in (1). Consider $\bar{S} = \{a \in \mathrm{P}_8(\bar{F}) \mid a^t = -a\}$ and $\bar{S}^{\perp} = \{b \in \mathrm{P}_8(\bar{F}) \mid b^t = b\}$ the skew-symmetric and symmetric traceless matrices, respectively (the use of bars here will be natural in a moment). Clearly, $\bar{S}$ is a Lie subalgebra of $(\mathrm{P}_8(\bar{F}), [,])$ isomorphic to sl(2), and $\bar{S}^{\perp}$ is an $\bar{S}$-module of type $V(4)$. Therefore, $\mathrm{P}_8(\bar{F}) = \bar{S} \oplus \bar{S}^{\perp} = V(2) \oplus V(4)$, and we can identify $\bar{A}$ with $\mathrm{P}_8(\bar{F})$ as sl(2)-modules. This is, we can think of $\bar{A}$ as $\mathrm{P}_8(\bar{F})$ with other product, another bilinear form and A as a form of this algebra. If we keep $(,)$ for the bilinear form of $\bar{A}$ and denote the bilinear form of $\mathrm{P}_8(\bar{F})$ by $((,))$, by Lemma 41 we have that

$$(,)|_{\bar{S}\otimes\bar{S}} = \alpha((,))|_{\bar{S}\otimes\bar{S}} \quad \text{and} \quad (,)|_{\bar{S}^{\perp}\otimes\bar{S}^{\perp}} = \beta((,))|_{\bar{S}^{\perp}\otimes\bar{S}^{\perp}}.$$

Hence, if $\varphi : \mathrm{P}_8(\bar{F}) \to \bar{A}$ is an isomorphism as modules, then φ' defined by $\varphi'(a) = \sqrt{\alpha}\varphi(a)$ and $\varphi'(b) = \sqrt{\beta}\varphi(b)$ for any $a \in S$ and $b \in S^{\perp}$ is not only a module isomorphism but an isometry too. Therefore, we can assume that the bilinear forms are the same, and we will do so. Now, consider the projection π_2 and π_4 of $\mathrm{P}_8(\bar{F})$ on $\bar{S}$ and $\bar{S}^{\perp}$, respectively. By Lemma 41 the product xy on $\bar{A}$ can be written as:

$$
\begin{aligned}
aa' &= C_1\pi_2(a*a') + C_2\pi_4(a*a'),\\
bb' &= C_3\pi_2(b*b') + C_4\pi_4(b*b'),\\
ab &= C_5\pi_2(a*b) + C_6\pi_4(a*b),\\
ba &= C_7\pi_2(b*a) + C_8\pi_4(b*a),
\end{aligned}
\tag{20}
$$

for any $a, a' \in \bar{S}$ and $b, b' \in \bar{S}^\perp$ and C_i some constants to be determined.

Fix a basis $\{a_{-2}, a_0, a_2\}$ of $\bar{S}$ and $\{b_{-4}, b_{-2}, b_0, b_2, b_4\}$ of $\bar{S}^\perp$ of eigenvectors (with obvious eigenvalues) for $ad_{a_0} : x \mapsto a_0 * x - x * a_0$. Observe that since ad_{a_0} is skew-symmetric then $(a_i, a_j) = 0 = (b_i, b_j)$ if $i + j \neq 0$. A straightforward argument using weights and the bilinear form (or just a direct computation) proves the following properties:

Lemma 42.

(P_1) $a_0 * a_0, b_0 * b_0 \in \bar{F} b_0$ *and* $a_0 * b_0, b_0 * a_0 \in \bar{F} a_0$.
(P_2) $a_0 * b_{\pm4}, b_{\pm4} * a_0, b_0 * b_{\pm4}, b_{\pm4} * b_0 \in \bar{F} b_{\pm4}$.
(P_3) $\pi_4(a_{-2} * a_0) \neq 0 \neq \pi_4(b_0 * a_2)$.
(P_4) $\pi_2(b_{-2} * a_2) \neq 0$.

With these properties at hand, we can derive some relations between the C_i's.

Lemma 43. *The constants C_i in* (20) *satisfy*:

(i) $C_i^2 = 1$, $i = 1, \ldots, 8$.
(ii) $C_6C_8 = C_2C_4 = C_5C_7$, $C_3C_7 = C_4C_8$ *and* $C_1C_7 = C_2C_8$.

Proof. By (P_1) we have $n(a_0)^2 = n(a_0a_0) = n(C_2a_0 * a_0) = C_2^2 n(a_0)^2$ and $C_2^2 = 1$. A similar argument with b_0 instead of a_0 gives $C_4^2 = 1$. Now with (P_2) we obtain that $n(a_0)(b_{-4}, b_4) = (a_0b_{-4}, a_0b_4) = C_6^2(a_0 * b_{-4}, a_0 * b_4) = C_6^2 n(a_0)(b_{-4}, b_4)$, from which $C_6^2 = 1$. Working with $(b_{-4}a_0, b_4a_0)$ we get $C_8^2 = 1$. Comparing

$$
\begin{aligned}
n(a_{-2}a_2) &= C_1^2 n\big(\pi_2(a_{-2} * a_2)\big) + C_2^2 n\big(\pi_4(a_{-2} * a_2)\big),\\
n(a_{-2}a_2) &= n(a_{-2})n(a_2) = n(a_{-2} * a_2)\\
&= n\big(\pi_2(a_{-2} * a_2)\big) + n\big(\pi_4(a_{-2} * a_2)\big)
\end{aligned}
$$

by means of (P_4), we conclude that $C_1^2 = 1$.

The full linearization $(xy, wz) + (xz, wy) = 2(x, w)(y, z)$ of $n(xy) = n(x)n(y)$ and (P_1), (P_2) provide

$$
\begin{aligned}
(a_0b_{-4}, b_4a_0) &= -\big(a_0^2, b_4b_{-4}\big) = -C_2C_4(a_0 * a_0, b_4 * b_{-4}),\\
(a_0b_{-4}, b_4a_0) &= C_6C_8(a_0 * b_{-4}, b_4 * a_0) = -C_6C_8(a_0 * a_0, b_4 * b_{-4}),
\end{aligned}
$$

what says that $C_6C_8 = C_2C_4$. The equalities $C_3C_7 = C_4C_8$ and $C_2C_4 = C_5C_7$ arise from similar computations with $(b_{-4}a_0, b_0b_4)$ and (a_0b_0, b_0a_0). Finally,

$$\begin{aligned}(a_{-2}a_0, b_0a_2) &= -(a_{-2}a_2, b_0a_0) = -C_1C_7(a_{-2} * a_2, b_0 * a_0)\\ &= C_1C_7(a_{-2} * a_0, b_0 * a_2)\\ &= C_1C_7 \sum_{i=2,4} \bigl(\pi_i(a_{-2} * a_0), \pi_i(b_0 * a_2)\bigr),\\ (a_{-2}a_0, b_0a_2) &= C_1C_7\bigl(\pi_2(a_{-2} * a_0), \pi_2(b_0 * a_2)\bigr)\\ &\quad + C_2C_8\bigl(\pi_4(a_{-2} * a_0), \pi_4(b_0 * a_2)\bigr),\end{aligned}$$

where we have used (P_1) and (P_3), implies $C_1C_7 = C_2C_8$. From all the relations we have, it is immediate to conclude that $C_i^2 = 1$ for all $i = 1, \ldots, 8$. □

Proposition 44. *There exist* $\epsilon_1, \epsilon_2 = \pm 1$*, but one of them* $\neq 1$*, such that* $\bar{A}$ *with the product* $x * y = \phi_1(x)\phi_2(y)$ *is isomorphic to* $\mathrm{P}_8(\bar{F})$*, where*

$$\phi_i|_{\bar{S}} = \epsilon_i \,\mathrm{Id}, \qquad \phi_i|_{\bar{S}^\perp} = \mathrm{Id}.$$

Moreover, $\operatorname{Der} \bar{A} = \{d \in \operatorname{Der}(\bar{A}, *) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$.

Proof. First of all, we should notice that the matrix transpose gives an isomorphism between $\mathrm{P}_8(\bar{F})$ and $\mathrm{P}_8(\bar{F})^{\mathrm{opp}}$ (the opposite algebra) which keeps invariant the subspaces $\bar{S}$ and $\bar{S}^\perp$. So, if we use $\mathrm{P}_8(\bar{F})^{\mathrm{opp}}$ to derive new constants C_i' in (20) instead of $\mathrm{P}_8(\bar{F})$, then we obtain $C_3' = -C_3$ and $C_4' = C_4$. Therefore, changing $\mathrm{P}_8(\bar{F})$ by $\mathrm{P}_8(\bar{F})^{\mathrm{opp}}$ if necessary, we can assume that $C_3 = C_4$. In fact, we can also assume that $C_3 = C_4 = 1$ because otherwise, we can consider $\mathrm{P}_8(\bar{F})$ with the product $x \bullet y = -x * y$, which is isomorphic to $\mathrm{P}_8(\bar{F})$ and leads to new constants $-C_i$. Now, the relations in the previous lemma become

$$C_1 = C_2, \qquad C_3 = C_4 = 1, \qquad C_5 = C_6, \qquad C_7 = C_8 \quad \text{and} \quad C_2C_6 = C_8.$$

From this, it is immediate to conclude that $x * y = \phi_1(x)\phi_2(y)$ where

$$\phi_1|_{\bar{S}} = C_6 \,\mathrm{Id}, \qquad \phi_1|_{\bar{S}^\perp} = \mathrm{Id}, \qquad \phi_2|_{\bar{S}} = C_8 \,\mathrm{Id}, \qquad \phi_2| - \bar{S}^\perp = \mathrm{Id}.$$

So, in the statement $\epsilon_1 = C_6$ and $\epsilon_2 = C_8$. Notice if $\epsilon_1 = \epsilon_2 = 1$, then $\bar{A}$ is an Okubo algebra. □

Now, let us descend to A. Since ϕ_1 and ϕ_2 are defined over F, $(A, *)$ with $x * y = \phi_1(x)\phi_2(y)$ is an Okubo algebra and $\operatorname{Der} A = \{d \in \operatorname{Der}(A, *) \mid [d, \phi_1] = 0 = [d, \phi_2]\}$. Moreover, since $\operatorname{Der}(A, *) = \{\mathrm{ad}_x : y \mapsto x * y - y * x \mid x \in (A, *)\}$ we have that S is a Lie subalgebra of $\operatorname{Der}(A, *)$ isomorphic to $\operatorname{Der} A$ which decomposes A as $3 + 5$.

Proposition 45. *If* $A = 3 + 5$ *then* $A \in \mathcal{S}$.

Proof. Let us check that $(A, *)$ has a nonzero idempotent. Consider $a \in S$. $\bar{F}a$ is a Cartan subalgebra of $\bar{S}$. Using this Cartan subalgebra to obtain the bases $\{a_{-2}, a_0, a_2\}$ and $\{b_{-4}, b_{-2}, b_0, b_2, b_4\}$, we have that $b_0 \in \bar{F}b$ with $b \in S^\perp$. By (P_1), $b * b = \lambda b$, with $\lambda \in F$;

hence, $e = b/\lambda$ is a nonzero idempotent of $(A, *)$. So far, by (3) we can define a Hurwitz product $x \circ y$ on A with unit e and such that

$$x * y = \tau(\bar{x}) \circ \tau^{-1}(\bar{y})$$

for an automorphism τ of $(A, \circ)$ of order 3 which fixed elements form a quaternion subalgebra Q. Therefore, the result will follow once we had proved that S is as constructed in Section 1.3.

We claim that $S \cap Q \neq 0$. To see this, observe that for any $a \in S$, $[a, e] = a * e - e * a = -\tau(a) + \tau^{-1}(a) \in S^\perp \cap Q^\perp$. If $[a, e] = 0$ then $\tau(a) = \tau^{-2}(a) = a$ and $a \in Q$. Otherwise, $\{[a, e] \mid a \in S\} \subseteq S^\perp \cap Q^\perp$ is 3-dimensional and $S, Q \subseteq (S^\perp \cap Q^\perp)^\perp$. By dimensions, $S \cap Q \neq 0$.

Take $0 \neq a \in S \cap Q$, $x \in S$, $x \perp a$ and $x = x' + x''$ with $x' \in Q$ (in fact $x' \perp e$) and $x'' \in Q^\perp$. We have

$$\begin{aligned}
[a, x] &= \tau(a) \circ \tau^{-1}(x) - \tau(x) \circ \tau^{-1}(a) = a \circ \tau^{-1}(x) - \tau(x) \circ a \\
&= -\big(\tau^{-1}(x) + \tau(x)\big) \circ a = -(2x' - x'') \circ a \in S.
\end{aligned}$$

In a similar way, $[a, [a, x]] = -n(a)(4x' + x'')$. Therefore, a, x' and $x' \circ a \in S$. However, if $x' \neq 0$ then these elements are linearly independent and hence, $S = \operatorname{span}\{a, x', x' \circ a\} \subseteq Q$; consequently, $[S, e] = 0$ and Fe is a trivial submodule, which is not possible. Therefore, $S = \operatorname{span}\{a, x_0, x_0 \circ a\}$ with $x_0 \in Q^\perp$. Now, take $u \in Q$ such that $\tau(x_0) = x_0 \circ u$ with $u \circ u + u + e = 0$. On one hand, $[x_0, x_0 \circ a] \in Fa$ but, on the other hand,

$$\begin{aligned}
[x_0, x_0 \circ a] &= \tau(x_0) \circ \tau^{-1}(x_0 \circ a) - \tau(x_0 \circ a) \circ \tau^{-1}(x_0) \\
&= (x_0 \circ u) \circ \big(x_0\big(a \circ u^2\big)\big) - \big(x_0 \circ (a \circ u)\big) \circ \big(x_0 \circ u^2\big) \\
&= -n(x)\big[\big(a \circ u^2\big) \circ u^2 + u^2 \circ \big(u^2 \circ a\big)\big] = -n(x)(a \circ u + u \circ a) \\
&= -n(x)\big(2(e, u)a - 2(u, a)e\big) = n(x)a + 2(u, a)n(x)e
\end{aligned}$$

implies that $(u, a) = 0$. This concludes the proof. $\square$

The only case left is solved by Proposition 34:

Proposition 46. *If* $A = 8$ *then* $A \in \mathcal{O}$.

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