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DIMENSION FILTRATION ON LOOPS

BY

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ABSTRACT

We show that the graded group associated to the dimension filtration on a loop acquires the structure of a Sabinin algebra after being tensored with a field of characteristic zero. The key to the proof is the interpretation of the primitive operations of Shestakov and Umirbaev in terms of the operations on a loop that measure the failure of the associator to be a homomorphism.

1. Introduction

Let G be a group and

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \cdots,$$

its lower central series. Then the graded group $\bigoplus G_i/G_{i+1} \otimes \mathbf{Q}$ is a Lie algebra with the Lie bracket induced by the commutator on G. Its universal enveloping algebra can be identified with the algebra associated to the filtration of the group ring $\mathbf{Q}G$ by the powers of its augmentation ideal [9].

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In this note we generalize these facts to arbitrary loops. It will be convenient to speak of dimension series rather than the lower central series. For groups, both series give rise to the same Lie algebra. However, while there are several inequivalent ways of defining the lower central series for loops, the definition of the dimension series extends to loops without any change.

Now it has become clear that appropriate generalization of Lie algebras to the non-associative context are the Sabinin algebras (called "hyperalgebras" in [4] and [8]). Sabinin algebras were initially introduced in [7] as tangent algebras to local loops. Later, Shestakov and Umirbaev proved that in any (not necessarily associative) bialgebra the set of primitive elements has the structure of a Sabinin algebra [8]. In fact, it was shown in [6] that any Sabinin algebra can be described as the set of primitive elements of some bialgebra.

Our constructions shed some light on the nature of the primitive operations introduced by Shestakov and Umirbaev in [8]. It turns out that the primitive operations in the free non-associative algebra are induced by the associator deviations (in the sense of [5]) in the free loop.

In the appendix we show how the methods of this paper can be applied to the associative case. We obtain a new point of view on the lower central series for groups; this helps to clarify the analogy between the operations of Shestakov–Umirbaev and iterated commutators.

2. Dimension subloops

A set L is called a **loop** if it is equipped with a multiplication $L \times L \to L$ satisfying

(a) for all $a, b \in L$ there exist unique x and y such that $ax = b$ and $ya = b$;

(b) there exists $e \in L$ such that $ea = ae = a$ for all $a \in L$.

The property (a) allows to speak of **left** and **right division**. These are binary operations defined by $b(b\backslash a) = a$ and $(a/b)b = a$ respectively. In general, one cannot speak of inverses in loops; in particular, it may happen that $a\backslash x = b_1 x$ and $a\backslash y = b_2 y$ with $b_1 \neq b_2$. A **homomorphism** of loops is a map that respects the multiplication.

Given a set V, the **free loop** $F(V)$ is defined by its universal property; namely, that any map of the set V into a loop L can be uniquely extended to a loop homomorphism $F(V) \to L$. Elements of $F(V)$ can be represented by non-associative words formed from elements of V and the unit e by applying the multiplication and both divisions. A non-associative word is **reduced** if it does not contain multiplications or divisions by e, or subwords of the types

$(ab)/b$, $b\backslash(ba)$, $(a/b)b$, $b(b\backslash a)$, $b/(b\backslash a)$ and $(a/b)\backslash b$. For each element of $F(V)$ there exists the unique reduced word representing it. We refer to [1] and [2] for more details on loops.

Let L be a loop, $\mathcal{R}$ a commutative unital ring, and $\mathcal{R}L$ the loop ring of L over $\mathcal{R}$. Denote by IL the augmentation ideal, that is, the kernel of the map $\mathcal{R}L \to \mathcal{R}$ that sends $\sum a_i x_i$ with $a_i \in \mathcal{R}$ and $x_i \in L$ to $\sum a_i$. The ideal IL (or simply I) is spanned over $\mathcal{R}$ by elements of the form $x-1$ with $x \in L$. Let $I^m L$ be mth power of I, that is, the submodule of $\mathcal{R}L$ spanned over $\mathcal{R}$ by products of at least m elements of I with any arrangement of the brackets.

Lemma 1: *Let $u \in I^m$ and $a \in L$. Then au, u/a and $a\backslash u$ all lie in I^m.*

Proof: First, $au = (a-1)u + u \in I^m$. Next, $u/a \cdot ((a-1)+1) \in I^m$ implies $u/a \in I$ as $u/a \cdot (a-1)$ is in I. This, in turn, implies $u/a \in I^2$ etc. The same argument works for $a\backslash u$. ∎

Definition: The nth dimension subloop of L over $\mathcal{R}$ is the intersection

$$D_n(L, \mathcal{R}) = L \cap (1 + I^n).$$

We shall sometimes write D_n instead of $D_n(L, \mathcal{R})$.

The set D_n is indeed a subloop of L. Let $a = 1+u$, $b = 1+v$ with $u, v \in I^n$. Clearly, ab is in D_n. To see that a/b belongs to D_n take $x = a/b - 1$. Then $1+u = (1+x)(1+v)$ and $x = (u-v)/b \in I^n$. In the same manner one shows that D_n is closed with respect to the left division.

It is clear that the dimension subloops are fully invariant subloops of L since the augmentation ideal is mapped into itself by any endomorphism of $\mathcal{R}L$.

3. The main theorem

In this section we shall assume that $\mathcal{R} = \mathbf{k}$ is a field of characteristic zero. We shall denote by $\mathcal{D}$ (or $\mathcal{D}L$) the graded vector space

$$\bigoplus D_n(L, \mathbf{k})/D_{n+1}(L, \mathbf{k}) \otimes \mathbf{k}$$

and by $\mathcal{I}$ (or $\mathcal{I}L$) the graded algebra $\bigoplus I^n L/I^{n+1}L$.

By a **bialgebra** we shall understand a unital, not necessarily associative algebra $\mathcal{A}$ equipped with a non-trivial algebra homomorphism δ: $A \to A \otimes A$. The homomorphism δ is referred to as **comultiplication**. An element $x \in A$ is called **primitive** if

$$\delta(x) = 1 \otimes x + x \otimes 1.$$

For any loop L the graded algebra $\mathcal{I}$ has a non-trivial comultiplication $\mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$. Indeed, the loop algebra $\mathbf{k}L$ has a comultiplication δ that sends g in L to $g \otimes g$. Under δ the element $g-1$ is sent to

$$(g-1) \otimes 1 + 1 \otimes (g-1) + (g-1) \otimes (g-1)$$

and, hence, there is an induced comultiplication on the algebra $\bigoplus I^n/I^{n+1}$.

There is an inclusion map of $\mathcal{D}$ into $\mathcal{I}$ given by sending $x \in D_nL$ to $x-1 \in I^nL$. The image of the class of $g-1$ is primitive for any $g \in L$.

The main result in this article is:

THEOREM 2: *The image of $\mathcal{D}$ in $\mathcal{I}$ coincides with the subspace of primitive elements in $\mathcal{I}$.*

The set of the primitive elements of any bialgebra has the structure of a Sabinin algebra [8]. In fact, if the bialgebra in question is primitively generated, it can be identified with the universal enveloping algebra of the Sabinin algebra of its primitive elements [6]. For any loop the algebra $\mathcal{I}$ is primitively generated since it is generated by the classes of $g-1 \in I$. Therefore, Theorem 2 can be re-stated as follows:

THEOREM 3: *The graded vector space $\mathcal{D}$ is a Sabinin algebra whose universal enveloping algebra is $\mathcal{I}$.*

For groups, Theorem 3 was first proved by Quillen [9]. Quillen's result involves the Lie algebra associated to the lower central series rather than the dimension series; however, for groups these are isomorphic.

4. The dimension subloops of a free loop

Let $x_i \leftrightarrow x_i'$ be a bijection between two sets of variables V and V'. Denote by $\mathcal{R}[[V']]$ the $\mathcal{R}$-algebra of formal power series in m non-associative variables $x_1', \ldots, x_m'$. The power series that start with 1 are readily seen to form a loop $\mathcal{R}_0[[V']]$ under multiplication.

Definition: The **Magnus expansion** is the homomorphism of the free loop $F(V)$ on the generators $x_1, \ldots, x_m$ into $\mathcal{R}_0[[V']]$ that sends the generator x_i to the power series $1 + x_i'$.

We denote the Magnus expansion of $x \in F(V)$ by $\mathcal{M}(x)$. It follows from the definition that

$$\mathcal{M}(x_i \backslash 1) = 1 - x_i' + {x_i'}^2 - x_i'{x_i'}^2 + x_i'(x_i'{x_i'}^2) - \cdots$$

and

$$\mathcal{M}(1/x_i) = 1 - x_i' + {x_i'}^2 - {x_i'}^2 x_i' + ({x_i'}^2 x_i')x_i' - \cdots .$$

The elements of $F(V)$ whose Magnus expansion has no terms of non-zero degree less than n form a normal subloop of $F(V)$. These subloops, in fact, are precisely the dimension subloops:

Lemma 4: *The Magnus expansion of $x \in F(V)$ begins with terms of degree n if and only if $x \in D_n(F(V), \mathcal{R})$ and $x \notin D_{n+1}(F(V), \mathcal{R})$.*

The Magnus expansion can be extended linearly to the loop ring $\mathcal{R}F(V)$. It is clear that the Magnus expansion of any element of $IF(V)$ has no term of degree zero, and, therefore, the expansion of any element of $I^kF(V)$ starts with terms of degree at least k. The converse is established with the help of the Taylor formula for free loops.

Define $u_i \in I$ by $u_i = x_i - 1$. We shall call the u_i **monomials of degree** 1 in $\mathcal{R}F(V)$. A **monomial of degree** k in $\mathcal{R}F(V)$ is a product (with any arrangement of the brackets) of k monomials of degree 1.

Lemma 5 (The Taylor formula): *Given a positive integer n, any $x \in F(V)$ can be uniquely written as*

$$x = 1 + \sum_{j \le n, \alpha} a_{j,\alpha}\mu_{j,\alpha} + r,$$

where $\mu_{j,\alpha}$ are monomials of degree j, $a_{j,\alpha}$ are elements of $\mathcal{R}$ and $r \in I^{n+1}F(V)$.

Let us first establish the existence of such formula for $n = 1$. For all the generators x_i it is obvious. Assume now that the set of such $x \in F(V)$ that $x-1$ cannot be written as a linear combination of the u_i modulo I^2, is non-empty. Let w be a reduced word of minimal possible length (number of operations used to form the word) representing such x. If $w = ab$ with a, b reduced words then

$$w - 1 = ab - 1 = ((a-1)+1)((b-1)+1) - 1 \equiv (a-1) + (b-1) \bmod I^2$$

and we come to a contradiction since a and b are of smaller length then w. Assume that $w = a/b$ with a, b reduced words. Clearly, $(a(1-b))(1-b) \in I^2$. By Lemma 1

$$((a(1-b))(1-b))/b = a/b - a(1-(b-1))$$

is also in I^2, hence

$$w - 1 = a/b - 1 \equiv a(1-(b-1)) - 1 \equiv (a-1) - (b-1) \bmod I^2$$

and we have a contradiction again. Similarly, w cannot be of the form $b\backslash a$; hence $w-1$ is a linear combination of the u_i modulo I^2 for all w.

Now, since elements of the form $g-1$ span I, it follows that any $r \in I^k$ is of the form

$$r = \sum_\alpha a_{k,\alpha}\mu_{k,\alpha} + \tilde{r}$$

where $\tilde{r} \in I^{k+1}F(V)$; this establishes the existence of the Taylor expansion with the remainder in $I^{n+1}F(V)$. The coefficients $a_{i,\alpha}$ are uniquely defined as the Magnus expansion of a monomial μ in $\mathcal{R}F(V)$ starts with the monomial μ', obtained from μ by replacing each u_i with x'_i. This proves Lemma 5. Lemma 4 follows immediately from Lemma 5.

5. Primitive operations and associator deviations

Let $\mathcal{A}$ be a bialgebra over a field of characteristic zero. Assume that A is generated by a set S of its primitive elements. Then the space of all primitive elements of $\mathcal{A}$ is the minimal vector subspace of $\mathcal{A}$ that contains S and is closed with respect to the commutators and the primitive operations $p_{r,s}$ defined by Shestakov and Umirbaev in [8].

Let x'_i, y'_j and z' be variables from the set V'. Denote by $\mathcal{R}(V')$ the free non-associative algebra generated by V'.

Let $x' = (\cdots(x'_1x'_2)\cdots)x'_r$ and $y' = (\ldots(y'_1y'_2)\ldots)y'_s$ where $r, s \geq 1$. Specifying the products x' and y' together with the numbers r and s is equivalent to giving the sequences x'_i and y'_i. The operations

$$p_{r,s}(x'_1,\ldots,x'_r;y'_1,\ldots,y'_s;z') = p_{r,s}(x';y';z')$$

are defined by the formula

$$(x'y')z' - x'(y'z') = \sum x'_{(1)}y'_{(1)} \cdot p_{\alpha,\beta}(x'_{(2)};y'_{(2)};z')$$

where the sum is taken over all decompositions of the sequences $x'_1,\ldots,x'_r$ and $y'_1,\ldots,y'_s$ into complementary subsequences

$$x'_{(1)} = (\ldots(x'_{i_1}x'_{i_2})\ldots)x'_{i_p}, \quad x'_{(2)} = (\ldots(x'_{i_{p+1}}x'_{i_{p+2}})\ldots)x'_{i_r}$$

and

$$y'_{(1)} = (\ldots(y'_{j_1}y'_{j_2})\ldots)y'_{j_q}, \quad y'_{(2)} = (\ldots(y'_{j_{q+1}}y'_{j_{q+2}})\ldots)y'_{j_s}.$$

For example, if $r = s = 1$ the operation $p_{1,1}(x'_1;y'_1;z')$ is just the associator

$$(x'_1,y'_1,z') = (x'_1y'_1)z' - x'_1(y'_1z').$$

Also,

$$p_{2,1}(x_1', x_2'; y_1'; z') = (x_1'x_2', y_1', z') - x_1'(x_2', y_1', z) - x_2'(x_1', y_1', z),$$
$$p_{1,2}(x_1'; y_1', y_2'; z') = (x_1', y_1'y_2', z') - y_1'(x_1', y_2', z') - y_2'(x_1', y_1', z').$$

Consider the map

$$\widetilde{\mathcal{M}}\colon \mathcal{R}F(V) \to \mathcal{R}(V')$$

defined by taking the lowest-degree term of the Magnus expansion. Our method of proving Theorem 2 consists in finding operations in the free loop $F(V)$ which correspond to the operations $p_{r,s}$ in the free algebra under the above map.

Such operations were introduced in [5] under the name of **associator deviations**. Associator deviations (or simply **deviations**) are functions from L^{n+3} to L where L is an arbitrary loop and n is a non-negative integer called the **level** of the deviation. There exists one deviation of level zero, namely the loop associator

$$(a, b, c) = (a(bc))\backslash((ab)c).$$

In general, there are $(n+2)!/2$ associator deviations of level n. Given $n > 0$ and an ordered set $\alpha_1, \ldots, \alpha_n$ of not necessarily distinct integers satisfying $1 \leq \alpha_k \leq k+2$, the deviation $(a_1, \ldots, a_{n+3})_{\alpha_1,\ldots,\alpha_n}$ is defined inductively by

$$(a_1, \ldots, a_{n+3})_{\alpha_1,\ldots,\alpha_n} := (A(a_{\alpha_n})A(a_{\alpha_n+1}))\backslash A(a_{\alpha_n}a_{\alpha_n+1}),$$

where $A(x)$ stands for $(a_1, \ldots, a_{\alpha_n-1}, x, a_{\alpha_n+2}, \ldots, a_{n+3})_{\alpha_1,\ldots,\alpha_{n-1}}$. In particular, there are three deviations of level one:

$$(a, b, c, d)_1 = ((a, c, d)(b, c, d))\backslash(ab, c, d),$$
$$(a, b, c, d)_2 = ((a, b, d)(a, c, d))\backslash(a, bc, d),$$
$$(a, b, c, d)_3 = ((a, b, c)(a, b, d))\backslash(a, b, cd).$$

Let us write $P_{m,n}(a_1, \ldots, a_m, b_1, \ldots, b_n, c)$ for the deviation

$$(a_1, \ldots, a_m, b_1, \ldots, b_n, c)_{\underbrace{1, \ldots, 1}_{m-1}, \underbrace{m+1, \ldots, m+1}_{n-1}}.$$

PROPOSITION 6: *In a free loop generated by $x_1, \ldots, x_m, y_1, \ldots, y_n$ and z*

$$\mathcal{M}(P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = 1 + p_{m,n}(x_1', \ldots, x_m'; y_1', \ldots, y_n'; z') + O(n+m+2)$$

where $O(n+m+2)$ *consists of terms of degree at least* $n+m+2$*, each term containing all the variables* x_i', y_i' *and* z' *at least once.*

A similar statement holds for the commutators:

Proposition 7: *In a free loop generated by* x_1 *and* x_2*, the Magnus expansion of the loop commutator* $(x_2x_1)\backslash(x_1x_2)$ *is of the form*

$$1 + x_1'x_2' - x_2'x_1' + O(3)$$

where $O(3)$ *consists of terms of degree* 3 *and more, each term containing* x_1' *and* x_2' *at least once.*

To prove Proposition 7 note that

$$(1 + x_1' + x_2' + x_2'x_1')\mathcal{M}((x_2x_1)\backslash(x_1x_2)) = 1 + x_1' + x_2' + x_1'x_2'.$$

Denote by m_k the terms of degree k in the Magnus expansion of $(x_2x_1)\backslash(x_1x_2)$. Then, if $k > 2$ we have

$$m_k + (x_1' + x_2')m_{k-1} + (x_2'x_1')m_{k-2} = 0.$$

It follows that if m_{k-1} contains both variables, m_k also does. Calculation shows that $m_2 = x_1'x_2' - x_2'x_1'$ and Proposition 7 follows.

The proof of Proposition 6 is more complicated; it is given in the next section. Here we show how Proposition 6 implies Theorem 2.

First, let us establish Theorem 2 for finitely generated free loops. Consider the filtration on $\mathbf{k}(V')$ by be subspaces of elements of degree at least i. The graded algebra associated to this filtration is clearly $\mathbf{k}(V')$ itself. The map $\widetilde{\mathcal{M}}$, when restricted to $\mathcal{I}F(V)$, becomes a homomorphism of algebras. It maps the generators $x_i - 1$ of $\mathcal{I}F(V)$ to the generators x_i' of the algebra $\mathbf{k}(V')$. Since $\mathbf{k}(V')$ is free, $\widetilde{\mathcal{M}}$ gives an isomorphism between $\mathcal{I}F(V)$ and $\mathbf{k}(V')$.

Now, assume that all primitive elements of $\mathcal{I}F(V)$ of degree less than k are contained in $\mathcal{D}$; this is certainly true for $k = 2$. Any primitive element of degree k is a linear combination of terms of the form $p_{\alpha,\beta}(u_1, \ldots, u_\alpha; v_1, \ldots, v_\beta; w)$ and $[u, v]$ where u, v, u_i, v_i, w are primitive elements of $\mathcal{I}F(V)$ of degree smaller than k. Without loss of generality we can assume that these elements belong to

$$\bigoplus D_i/D_{i+1} \subset \mathcal{D}F(V) \subset \mathcal{I}F(V);$$

this implies that there exist $\hat{u}, \hat{v}, \hat{u}_i, \hat{v}_i$ and $\hat{w}$ in the loop $F(V)$ such that $\widetilde{\mathcal{M}}(\hat{u}) = u$ and similarly for v, u_i, v_i and w. Now, by Proposition 6

$$\widetilde{\mathcal{M}}(P_{\alpha,\beta}(\hat{u}_1, \ldots, \hat{u}_\alpha; \hat{v}_1, \ldots, \hat{v}_\beta; \hat{w}) = p_{\alpha,\beta}(u_1, \ldots, u_\alpha; v_1, \ldots, v_\beta; w).$$

Also, by Proposition 7

$$\widetilde{\mathcal{M}}((\hat{v}\hat{u})\backslash(\hat{u}\hat{v})) = [u, v],$$

and, hence, any primitive element of degree k also belongs to $\mathcal{D}$.

Now, let L be an arbitrary loop. Any primitive element u of $\mathcal{I}L$ can be obtained from a finite number of elements (say, m) of the form $g_i - 1 \in IL/I^2L$ with $g_i \in L$, by applying commutators, the $p_{\alpha,\beta}$'s and taking linear combinations. Consider the homomorphism of the free loop $F(V)$ on m generators to L that sends the generators of $F(V)$ to the g_i. It is clear that u is the image of a primitive element $w \in \mathcal{I}F(V)$ under the induced map $\mathcal{I}F(V) \to \mathcal{I}L$. However, since w is in the image of $\mathcal{D}F(V)$ inside $\mathcal{I}F(V)$ we see that u is in the image of $\mathcal{D}L$ inside $\mathcal{I}L$.

6. Proof of Proposition 6

Given a subset $S' \subseteq V'$, we shall say that a monomial in $\mathbf{k}[[V']]$ is **balanced** (with respect to S') if it contains each element of S' at least once. For $S \subseteq V$, we shall say that an element $\phi \in F(S)$ is balanced (with respect to S) if every non-zero term in $\mathcal{M}(\phi) - 1$ is balanced with respect to S'.

Lemma 8: *Given a balanced $\phi \in F(S)$, $x \in S$ (so $\phi = \phi(x)$) and $y \in V \setminus S$ then the expression*

$$\phi(x, y) = (\phi(x)\phi(y))\backslash\phi(xy) \in F(S \cup \{y\})$$

is balanced with respect to $S \cup \{y\}$.

Proof:

$$\mathcal{M}(\phi(xy)) - 1 = (\mathcal{M}(\phi(x)) - 1) + (\mathcal{M}(\phi(y)) - 1) + M'$$

where M' is an (infinite) sum of balanced monomials. Hence,

$$\mathcal{M}(\phi(x, y)) - 1 = (\mathcal{M}(\phi(x))\mathcal{M}(\phi(y)))\backslash(M' - (\mathcal{M}(\phi(x)) - 1)(\mathcal{M}(\phi(y)) - 1)).$$

It follows that all the monomials contained in $\mathcal{M}(\phi(x, y)) - 1$ with non-zero coefficients are balanced. Indeed, all of the monomials in the lowest-degree term of $\mathcal{M}(\phi(x, y)) - 1$ are balanced since every monomial in

$$M' - (\mathcal{M}(\phi(x)) - 1)(\mathcal{M}(\phi(y)) - 1)$$

is balanced. It follows that all the other terms of $\mathcal{M}(\phi(x, y)) - 1$ are balanced since they are expressed via the lowest-degree term with the help of a recurrent relation. ∎

Given $f \in \mathbf{k}[[V']]$, let $L_{S'}(f)$ be the part of f which contains all the variables in S' with multiplicity one, but no variables in $V' \backslash S'$. Similarly, given $\phi \in F(V)$, $L_S(\phi)$ will stand for $L_{S'}(\mathcal{M}(\phi))$.

Lemma 9: *Let $S \subseteq \hat{S} \subseteq V$ with $|S| \geq 2$, $x \in S$ and $y \in V \setminus \hat{S}$. Given $\phi = \phi(x) \in F(S)$ balanced, $\phi(x, y)$ as above and $w \in F(\hat{S})$ then*

$$L_{\hat{S}\cup\{y\}}(\phi(w, y)) = L_{\hat{S}\cup\{y\}}(\phi(wy)).$$

Proof: First we decompose $S = \{x\} \sqcup S_0$ ($\sqcup$ denotes disjoint union), $\mathcal{M}(\phi(x)) = \sum_I A_I(x)$ with I the multidegree of $A_I(x)$ on S_0' and $\mathcal{M}(\phi(x, y)) = \sum_K A_K(x, y)$. With this notation we have that

$$A_M(xy) = \sum_{I+J+K=M} (A_I(x)A_J(y))A_K(x, y).$$

Hence,

$$A_M(x, y) = A_M(xy) - A_M(x) - A_M(y) - \sum_{\substack{I+J+K=M;\\ I,J,K\neq M}} (A_I(x)A_J(y))A_K(x, y)$$

and, since $\phi(x)$ and $\phi(x, y)$ are balanced,

$$A_{(1,\ldots,1)}(x, y) = A_{(1,\ldots,1)}(xy) - A_{(1,\ldots,1)}(x) - A_{(1,\ldots,1)}(y).$$

Therefore, using that $L_{\hat{S}'\cup\{y'\}}(A_{(1,\ldots,1)}(w)) = 0$ and $L_{\hat{S}'\cup\{y'\}}(A_{(1,\ldots,1)}(y)) = 0$, we get

$$\begin{aligned} L_{\hat{S}\cup\{y\}}(\phi(w, y)) &= L_{\hat{S}'\cup\{y'\}}(A_{(1,\ldots,1)}(w, y)) \\ &= L_{\hat{S}'\cup\{y'\}}(A_{(1,\ldots,1)}(wy)) = L_{\hat{S}\cup\{y\}}(\phi(wy)) \end{aligned}$$

as desired. ∎

Now we are in the position to prove Proposition 6. It follows from Lemma 8 that $P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)$ is balanced with respect to the set

$$S = \{x_1, \ldots, x_m, y_1, \ldots, y_n, z\}.$$

Therefore it suffices to show that

$$L_S(P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = p_{m,n}(x_1', \ldots, x_m'; y_1', \ldots, y_n'; z').$$

Set $x' = ((x_1'x_2')\cdots)x_m'$, $y' = ((y_1'y_2')\cdots)y_n'$, and, similarly, $x = ((x_1x_2)\cdots)x_m$ and $y = ((y_1y_2)\cdots)y_n$. As a corollary of Lemma 9 we have

$$L_S(P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = L_S((x, y, z)).$$

Since $(xy)z = x(yz) \cdot (x, y, z)$, we have

$$L_S((xy)z)) = \sum_{S_1 \sqcup S_2 = S} L_{S_1}(x(yz)) L_{S_2}((x, y, z))$$

with the convention $L_\emptyset(\cdot) = 1$. The right–hand side of this equality is

$$x'(y'z') + \sum_{S_1 \sqcup S_2 = S \setminus \{z\}} L_{S_1}(xy) L_{S_2 \sqcup \{z\}}((x, y, z))$$

since z' appears in any term of positive degree of $\mathcal{M}((x, y, z))$. Moreover, setting

$$\hat{p}(x'_{i_1}, \ldots, x'_{i_k}, y'_{j_1}, \ldots, y'_{j_l}, z') = L_{\{x'_{i_1}, \ldots, x'_{i_k}, y'_{j_1}, \ldots, y'_{j_l}\}}(\mathcal{M}((x, y, z)))$$

and $\hat{p}(1, \cdot, z') = \hat{p}(\cdot, 1, z') = \hat{p}(1, 1, z') = 0$, we obtain

$$\begin{aligned}(x'y')z' - x'(y'z') &= \sum_{S_1 \sqcup S_2 = S \setminus \{z\}} L_{S_1}(xy) L_{S_2 \sqcup \{z\}}((x, y, z)) \\ &= \sum (x'_{(1)} y'_{(1)}) \hat{p}(x'_{(2)}, y'_{(2)}, z'),\end{aligned}$$

which shows that the operators $\hat{p}$ agree with the primitive operations of Shestakov and Umirbaev since they satisfy the same recurrence and initial conditions. Therefore,

$$L_S((x, y, z)) = \hat{p}(x'_1, \ldots, x'_m, y'_1, \ldots, y'_n, z') = p_{m,n}(x'_1, \ldots, x'_m; y'_1, \ldots, y'_n; z').$$

7. Miscellaneous remarks

1. All the results of this paper can be stated without change for left loops (binary systems with left division and a two-sided unit) since the definition of the associator deviations does not involve right division.

2. If all elements of a loop L satisfy some identity, then the bialgebra $\mathcal{I}$ satisfies a "linearized" version of the same identity. In particular, if L is Moufang loop, $\mathcal{D}L$ is readily seen to be a Malcev algebra. Similarly, if L is a Bol loop, $\mathcal{D}L$ is Bol algebra.

3. Our results are stated only over fields of characteristic zero since the necessary general results from the theory of non-associative algebras are only available for such fields. In particular, the notions corresponding to Lie rings and restricted Lie algebras are not yet axiomatized. It is clear, however, that the direct sum $\bigoplus D_i(L, \mathcal{R})/D_{i+1}(L, \mathcal{R})$ always has a rich algebraic structure. Using the same argument as in the proof of Lemma 8 one sees that

- $[D_p, D_q] \subset D_{p+q}$;
- $(D_p, D_q, D_r) \subset D_{p+q+r}$;
- $(D_{p_1}, \ldots, D_{p_n})_{\alpha_1,\ldots,\alpha_{n-3}} \subset D_{p_1+\cdots+p_n}$ for all combinations $\alpha_1, \ldots, \alpha_{n-3}$.

Therefore, $\bigoplus D_i(L, \mathcal{R})/D_{i+1}(L, \mathcal{R})$ carries multilinear operations induced by the commutator and the associator deviations, and all these operations respect the grading.

4. For groups, the dimension filtration is closely related to the lower central series. In particular, the Lie algebras over the field of rational numbers, associated to both filtrations, are isomorphic. This is no longer true for general loops, at least if one uses Bruck's definition of lower central series [1]. Instead, the dimension filtration is related to the commutator-associator filtration [5]; this connection will be discussed elsewhere.

5. A series closely related to the dimension series for loops (with $\mathcal{R}$ a finite field) has appeared in [3] under the name of **subtree series**. In particular, the subtree series for the loop $F/D_n(F, \mathcal{R})$, where F is a free loop, is exactly the dimension series. The connection between the subtree-counting process of [3] and the dimension series becomes clear if one notices that the Magnus expansion of a word in a free loop that is formed using only multiplications, is just the sum of all its subwords.

8. Appendix. Deviations in groups

The appearance of deviations in nilpotency theory may seem to be a specific feature of the non-associative case. In fact, the usual lower central series for groups can be constructed using deviation-like operations. These deviations are defined using the function $f(x) = x^2$ instead of the associator.

Namely, let the **x^2-deviation** be the function

$$\{a, b\}_1 = b^{-2}a^{-2}(ab)^2.$$

The two x^2-deviations of the second level are defined as

$$\{a, b, c\}_{11} = \{b, c\}_1^{-1}\{a, c\}_1^{-1}\{ab, c\}_1$$

and

$$\{a, b, c\}_{12} = \{a, c\}_1^{-1}\{a, b\}_1^{-1}\{a, bc\}_1.$$

In general, given $n > 0$, called **level** and an ordered set $\alpha_1, \ldots, \alpha_n$ of not necessarily distinct integers satisfying $1 \leq \alpha_k \leq k$, the deviation $\{a_1, \ldots, a_{n+1}\}_{\alpha_1,\ldots,\alpha_n}$

is defined inductively by

$$\{a_1, \dots, a_{n+1}\}_{\alpha_1,\dots,\alpha_n} := A(a_{\alpha_n+1})^{-1}A(a_{\alpha_n})^{-1}A(a_{\alpha_n}a_{\alpha_n+1}),$$

where $A(x)$ stands for $\{a_1, \dots, a_{\alpha_n-1}, x, a_{\alpha_n+2}, \dots, a_n\}_{\alpha_1,\dots,\alpha_n}$.

Let G be an arbitrary group and n — a positive integer.

PROPOSITION 10: *The subgroup of G generated by all x^2-deviations of level n coincides with $\gamma_{n+1}G$ — the $n+1$st term of the lower central series of G. The x^2-deviations of the form $\{a_1, a_2 \dots, a_{n+1}\}_{11\cdots 1}$ are sufficient to generate $\gamma_{n+1}G$.*

Notice that for $n = 2$ this is rather obvious since the group commutator of a and b is equal to $\{b^{-1}ab, b\}_1$.

It is sufficient to prove Proposition 10 for the case when G is finitely generated and free. For free groups the dimension series coincides with the lower central series and one can use the Magnus expansion. An analogue of Lemma 8 is valid in the associative setup, so every x^2-deviation of level n belongs to $\gamma_{n+1}G$. Proposition 6 is replaced by the following statement:

$$\mathcal{M}(\{x_1, x_2, \dots, x_n, z\}_{11\cdots 1}) = 1 + [\dots[[z', x_1'], x_2'], \dots, x_n'] + O(n+2). \tag{1}$$

This formula implies that the groups $\gamma_iG/\gamma_{i+1}G$ are generated by the classes of the deviations of the form $\{a_1, \dots, a_i\}_{1\cdots 1}$. Moreover, the subgroups generated by deviations of such form are normal, being verbal subgroups; hence, Proposition 10 follows.

The formula (1) can be proved in the same fashion as Proposition 6 and we do not repeat the argument. The key point is the identity

$$z'x' - x'z' = \sum x'_{(1)}q_\alpha(x'_{(2)}; z') \tag{2}$$

where $x' = x_1'x_2'\cdots x_n'$, the sum is taken over all decompositions of the sequence $x_1'x_2'\cdots x_n'$ into two complementary subsequences whose products are $x'_{(1)}$ and $x'_{(2)}$ respectively and $q_k(x_1'x_2'\cdots x_k'; z')$ is the iterated commutator

$$[\dots[[z', x_1'], x_2'], \dots, x_k'].$$

This identity shows that the iterated commutators in associative algebras can be defined in the same way as the Shestakov–Umirbaev operations, starting with the commutator instead of the associator.

To establish (2), use induction on n. Let $x'_* = x'_1 x'_2 \cdots x'_{n-1}$, and assume that for x'_* the identity (2) holds. Then the right-hand side of (2) can be written as

$$\sum x'_{*(1)}[q(x'_{*(2)};z'),x'_n] + \sum x'_{*(1)}x'_n q(x'_{*(2)};z') + x'_*[z',x'_n]$$
$$= [z',x'_*]x'_n + x'_*[z',x'_n] = [z',x'].$$

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