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DIMENSION OF BOOLEAN VALUED LATTICES AND RINGS

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A. Joyal initiated the dimension theory of rings in a topos. Joyal's notion of Krull dimension of lattices and rings was considered by the author, who has shown that $\dim K[X]=1$ for any field K in a topos $\mathcal{S}$. The basic aim of this paper is to prove that $\dim R[X]=1$ for any regular ring R in $\mathcal{S}$, that is by working in commutative algebra without choice and excluded middle. Given a regular ring R, let E be the boolean algebra of idempotents of R, and $\mathcal{E}=\mathrm{sh}(E)$ the topos of sheaves over E with the finite cover topology. The Pierce representation $\tilde{R}$ of R is a filed in $\mathcal{E}$, so that $\dim \tilde{R}[X]=1$ and this implies $\dim R[X]=1$ by using preserving properties of the global sections functor $\Gamma:\mathcal{E}\to\mathcal{S}$. Section 1 deals with lattices in the topos $\mathcal{E}=\mathrm{sh}(E)$ of sheaves over a boolean algebra E with the finite cover topology. We characterize lattices in $\mathcal{E}$ as lattice homomorphisms $E\to D$, and we consider the dimension of lattices in this form. In Section 2 we describe rings in $\mathcal{E}$ as boolean homomorphisms $E\to \mathbf{E}(A)$. Here, we discuss the Pierce representation and polynomials. The spectrum of a ring is considered in Section 3, which ends with the aim theorem.

Introduction

In several talks during 1975, A. Joyal initiated the dimension theory of rings in a topos. Later, Joyal's notion of Krull dimension of (distributive with 0 and 1) lattices and (commutative and unitary) rings was considered by the author [1, 2] who has shown that $\dim K[X]=1$ for any field K (geometric field in [3]) in a topos $\mathcal{S}$ (with natural numbers object). The basic aim of this paper is to prove that $\dim R[X]=1$ for any regular ring R in $\mathcal{S}$, that is by working in commutative algebra without choice and excluded middle.

Given a regular ring R, let $E=\mathbf{E}(R)$ be the boolean algebra of idempotents of R, and $\mathcal{E}=\mathrm{sh}(E)$ the topos of sheaves over E with the finite cover topology. The Pierce representation $\tilde{R}$ of R (see [8]) and [7] for a stalk-free approach) is a field in $\mathcal{E}$, so that $\dim \tilde{R}[X]=1$ and this implies $\dim R[X]=1$ by using preserving properties of the global sections functor $\Gamma:\mathcal{E}\to\mathcal{S}$.

The plan of this paper is as follows. Section 1 deals with lattices in the topos $\mathcal{E}=\mathrm{sh}(E)$ of sheaves over a boolean algebra E with the finite cover topology. We characterize lattices in $\mathcal{E}$ as lattice homomorphisms $E\to D$, and we consider the dimension of lattices in this form. In Section 2 we describe rings in $\mathcal{E}$ as boolean homomorphisms $E \to \mathbf{E}(A)$. Here, we discuss the Pierce representation and polynomials. The spectrum of a ring is considered in Section 3, which ends with the aim theorem.

An advance of this paper was communicated in the Seminar on Category Theory and their Applications held in Bogotá, Colombia (August, 1983). Some of the results of Section 2 were our contribution to the Spanish–Portuguese meeting which took place in Salamanca, Spain (April, 1982).

1. Boolean valued lattices

Here we view a boolean algebra E as the category associated to its order, noted $\underline{E}$, which is a site with the *finite cover* topology (first considered by Reyes [9] in a more general context) given by all finite coverings

$$e = e_1 \vee \cdots \vee e_n$$

for each $e \in E$. A functor

$$D : \underline{E}^{\mathrm{op}} \to \mathscr{S}$$

is a *sheaf* if for any covering $e = e_1 \vee \cdots \vee e_n$ and for any family

$$t_i \in D(e_i), \quad 1 \le i \le n$$

which verify

$$D(e_i \wedge e_j \le e_i)(t_i) = D(e_i \wedge e_j \le e_j)(t_j)$$

when $i \ne j$, there is a unique $t \in D(e)$ such that

$$D(e_i \le e)(t) = t_i, \quad 1 \le i \le n.$$

Because the sheaf condition is applied to the empty covering of $0 \in E$, note that $D(0) = 1$ (final object of $\mathscr{S}$).

An important case of covering, called *partition*, is

$$e_i \wedge e_j = 0, \quad i \ne j.$$

Then we write

$$e = e_i + \cdots + e_n.$$

It is not hard to see that a functor $D : \underline{E}^{\mathrm{op}} \to \mathscr{S}$ is a sheaf if and only if it satisfies the sheaf condition for partitions. Using induction, we arrive on the well-known

1.1. Lemma. *A functor $D : \underline{E}^{\mathrm{op}} \to \mathscr{S}$ is a sheaf if and only if*

(i) $D(0) = 1$.

(ii) *For any partition $e = e_1 + e_2 \in E$, the diagram*

$$\begin{array}{ccc} & & D(e_1) \\ & \nearrow & \\ D(e) & & \\ & \searrow & \\ & & D(e_2) \end{array}$$

is a product, with the maps $D(e_i \le e)$, $i = 1, 2$.

Let $\mathscr{E}(E)$ be the full subcategory of $\mathscr{S}^{E^{\mathrm{op}}}$ with objects all the sheaves. If we take $E = \mathbf{2}$, the boolean algebra of two elements, then $\mathscr{E}(\mathbf{2}) \simeq \mathscr{S}$. From now on, we shall consider a fixed boolean algebra E and we shall write $\mathscr{E} = \mathscr{E}(E)$.

It is well known that the global sections functor

$$\Gamma : \mathscr{E} \to \mathscr{S}, \qquad \Gamma(D) = D(1)$$

preserves arbitrary limits (in particular monomorphisms) because there is a geometric morphism $\Delta \dashv \Gamma$.

We are now interested in lattices in $\mathscr{E}$, that is, sheaves D such that $D(e)$ is a lattice for each $e \in E$ and that the restriction maps $D(e_1 \le e_2)$ are lattice homomorphisms. All lattices in this paper are distributive and with 0 and 1 which are preserved by lattice homomorphisms.

The following lemma is a particular case of [5, Lemma 4.1] since $\mathscr{E}$ is equivalent to the topos sh(X) of sheaves over the Stone space $X = \mathrm{spec}(E)$, and lattices are L-structures in the sense of Loullis.

1.2. Lemma. *Let D be a lattice in $\mathscr{E}$ and $e \in E$. For any $t \in D(e)$ there is a global $t' \in \Gamma(D)$ extending t.*

Proof. Take $1 = e + \neg e$ and $(t, 0) \in D(e) \times D(\neg e)$; then use Lemma 1.1. □

Note that the global t' is uniquely extending $t \in D(e)$ and $0 \in D(\neg e)$. In particular, given $e \in E$, there is a unique $\delta(e) = 1' \in \Gamma(D)$ extending $1 \in D(e)$ and $0 \in D(\neg e)$. Hence we have a map

$$\delta : E \to \Gamma(D).$$

Since there is a bijection between the (isomorphic) product decompositions $L = L_1 \times L_2$ of a lattice and the boolean algebra $C(L) \subseteq L$ of the complemented elements of L, we see that $\delta(e)$ is the complemented element of $\Gamma(D)$ corresponding to $\Gamma(1) \simeq D(e) \times D(\neg e)$.

1.3. Lemma. *δ is a lattice homomorphism.*

Proof. For $i=1,2$,

$$D(e_1\wedge e_2\leq 1)(\delta(e_i))=D(e_1\wedge e_2\leq e_i)D(e_i\leq 1)(\delta(e_i))=1,$$

so that $D(e_1\wedge e_2\leq 1)(\delta(e_1)\wedge\delta(e_2))=1$. To prove

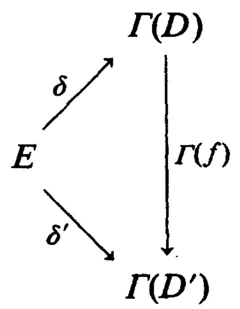
$$\delta(e_1)\wedge\delta(e_2)=\delta(e_1\wedge e_2)$$

we also need $D(\neg(e_1\wedge e_2)\leq 1)(\delta(e_1)\wedge\delta(e_2))=0$. But if we put t for the left-hand side of the last equality and $a=\neg(e_1\wedge e_2)$, then $t\in D(a)$ with $a=\neg e_1\vee\neg e_2$, and for $i=1,2$

$$D(\neg e_i\leq a)(t)=D(\neg e_i\leq 1)(\delta(e_1)\wedge\delta(e_2))=0$$

since $D(\neg e_i\leq 1)(\delta(e_i))=0$. Hence $t=0$ by the sheaf condition. One can prove $\delta(e_1)\vee\delta(e_2)=\delta(e_1\vee e_2)$ similarly. □

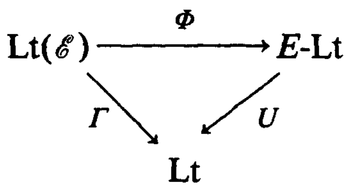
From a lattice homomorphism $f:D\to D'$ in $\mathscr{E}$ we obtain a commutative diagram



and a functor

$$\Phi:\mathrm{Lt}(\mathscr{E})\to E\text{-Lt}$$

where $\mathrm{Lt}(\mathscr{E})$ is the category of lattices in $\mathscr{E}$ and E-Lt is the comma category of lattice homomorphisms (in $\mathscr{S}$) with domain E. If we forget (functor U) the E-structure, Φ is the restriction of Γ to lattices



1.4. Theorem. *Φ is an equivalence.*

Proof. We shall prove (see [6]) that

(i) Φ is full and faithful.

(ii) Each lattice homomorphism $\lambda:E\to L$ is isomorphic to $\Phi(D)$ for some lattice D in $\mathscr{E}$.

For (i), given two lattices D, D' in $\mathscr{E}$ with $\Phi(D)=\delta$, $\Phi(D')=\delta'$ and a lattice homomorphism $f_1 : \Gamma(D) \to \Gamma(D')$ such that $f \circ \delta = \delta'$, we must prove that there is a unique lattice homomorphism $f : D \to D'$ with $f(1) = f_1$. For any $e \in E$ we define

$$f(e) : D(e) \to D(e'), \qquad f(e)(t) = D'(e \leq 1)(f_1(t'))$$

where $t' \in \Gamma(D)$ is the unique global extension of $(t, 0) \in D(e) \times D(\neg e)$. It is not difficult to check that $f(e)$ is a lattice homomorphism for any $e \in E$ (the condition $f_1 \circ \delta = \delta'$ is used to see $f(e)(1) = 1$) and thence $f : D \to D'$ is a lattice homomorphism such that $f(1) = f_1$. The uniqueness is trivial.

In order to prove (ii) we take $L_{\leq \lambda(e)} = \{x \in L \mid x \leq \lambda(e)\}$ and we define a functor $D : \underline{E}^{\mathrm{op}} \to \mathscr{S}$ by

$$D(e) = L_{\leq \lambda(e)},$$

$$D(e_1 \leq e_2) : D(e_2) \to D(e_1), \quad t \to t \wedge \lambda(e_1).$$

It is clear that $D(e)$ is a lattice (with unity $\lambda(e)$) and $D(e_1 \leq e_2)$ is a lattice homomorphism. Moreoever $D(0) = 1$ and given $e = e_1 + e_2$ we have $\lambda(e) = \lambda(e_1) + \lambda(e_2)$ so that the map

$$L_{\leq \lambda(e)} \to L_{\leq \lambda(e_1)} \times L_{\leq (e_2)}, \qquad t \to (t \wedge \lambda(e_1), t \wedge \lambda(e_2))$$

is an isomorphism (with inverse $(t_1, t_2) \to t_1 \vee t_2$). Hence D is a sheaf, that is a lattice in $\mathscr{E}$. Finally, $\Gamma(D) = L$ and for any $e \in E$, $\lambda(e)$ is the complemented element of L associated to $L \simeq L_{\leq \lambda(e)} \times L_{\leq \lambda(\neg e)}$ so that $\Phi(D) = \lambda$. $\square$

Recall that the lattice $L_{\leq a}$ is universal among the lattices L' with a lattice homomorphism $f : L \to L'$ such that $f(a) = 1$. For instance, after Theorem 1.4 we have isomorphisms $D(e) \simeq \Gamma(D)_{\leq \delta(e)}$. One can prove this fact directly, since $D(e \leq 1)(\delta(e)) = 1$ and given a lattice homomorphism $l : \Gamma(D) \to L$ such that $l(\delta(e)) = 1$, there is a unique lattice homomorphism $l' : D(e) \to L$ such that $l' \circ D(e \leq 1) = l$. Actually, $l'(t) = l(t')$, so that the definition of $f(e)$ in the proof of Theorem 1.4 is $f(e) = l'$ with $l = D'(e \leq 1) \circ f_1$.

We shall refer to $\lambda : E \to L$ as an *E-lattice*. As usual, we sometimes forget λ and we say that L is an E-lattice. We shall write from now on $\tilde{L}$ for the sheaf $\tilde{L}(e) = L_{\leq \lambda(e)}$ used in the proof of Theorem 1.4. The construction $(\tilde{\cdot})$ is the inverse equivalence of Φ, in particular $\Gamma(\tilde{L}) = L$. Given an E-lattice homomorphism

$$\begin{array}{ccc} & & L \\ & \nearrow^{\lambda} & \\ E & & \Big\downarrow f \\ & \searrow_{\lambda'} & \\ & & L' \end{array}$$

and $e \in E$, we can define a lattice homomorphism

$$f_e : L_{\leq \lambda(e)} \to L'_{\leq \lambda'(e)}, \qquad f_e(x) = f(x)$$

(unique as induced by f) and thence $\tilde{f}(e) = f_e$ is a lattice homomorphism $\tilde{f} : \tilde{L} \to \tilde{L}'$ in $\mathscr{E}$ unique such that $\Gamma(\tilde{f}) = f$.

The restriction of Φ to boolean algebras give us a similar functor $\Psi : \mathrm{Bl}(\mathscr{E}) \to E\text{-}\mathrm{Bl}$ such that

$$\begin{array}{ccc} \mathrm{Bl}(E) & \xrightarrow{\ \Psi\ } & E\text{-}\mathrm{Bl} \\ & {\scriptstyle \Gamma} \searrow \quad \swarrow {\scriptstyle U} & \\ & \mathrm{Bl} & \end{array}$$

1.5. Corollary. *Ψ is an equivalence.*

Proof. It suffices to note that for any boolean algebra L and $a \in L$, the lattice $L_{\leq a}$ is also boolean. □

If D is a lattice in $\mathscr{E}$ and $\mathbf{B}(\cdot)$ means the free boolean algebra generated by $(\cdot)$, then

$$\mathbf{B}(D) = \widetilde{\mathbf{B}(\Gamma(D))}.$$

We can prove this fact in the level of E-lattices. It is an easy routine to check the universal property in the following

1.6. Proposition. *Let $\lambda : E \to L$ be an E-lattice and $j : L \to \mathbf{B}(L)$ the canonical inclusion. Then $\beta = j \circ \lambda : E \to \mathbf{B}(L)$ is the free boolean E-algebra generated by λ and j is the canonical inclusion.*

When we speak about E-lattices $\lambda : E \to L$ we are really concerned with pairs (L, λ) formed by a lattice L and a boolean homomorphism $\lambda : E \to \mathbf{C}(L)$ from E to the center of L, that is the boolean algebra $\mathbf{C}(L)$ of complemented elements of L. We can express E-lattices as $\lambda : E \to L$ since the inclusion $\mathbf{I} : \mathrm{Bl} \to \mathrm{Lt}$ is a left adjoint for $\mathbf{C}$ with unity $\mathbf{CI}(B) = B$ and counity $\mathbf{IC}(L) \hookrightarrow L$. If we consider the functor $\mathbf{C} : \mathrm{Lt}(\mathscr{E}) \to \mathrm{Bl}(\mathscr{E})$ in the level $\mathbf{C} : E\text{-}\mathrm{Lt} \to E\text{-}\mathrm{Bl}$, then $\mathbf{C}(L, \lambda) = \lambda$.

Krull dimension of lattices is based on the following universal construction: given a lattice D and $n > 0$, there is a lattice D_n (the *prime n-chain* lattice of D_n) and lattice homomorphisms

$$p_0, \dots, p_n : D \to D_n$$

which are universal for the property

$$p_0 \leq \cdots \leq p_n$$

i.e., they satisfy this property and for any lattice homomorphisms $l_0 \leq \cdots \leq l_n : D \to L$ there is a unique lattice homomorphism $l : D_n \to L$ such that $lp_i = l_i$, $0 \leq i \leq n$.

Classically, if we take $L=\mathbf{2}$, there is a bijection between prime ideals of D_n and n-chains of prime ideals of D. For details, see [1]. The following lemma says that

$$D_n=\widetilde{\Gamma(D)}_n.$$

1.7. Lemma. *Let $\lambda : E\to L$ be an E-lattice and $l_0\leq\cdots\leq l_n : L\to L_n$ the prime n-chain lattice of L. Then the prime n-chain E-lattice of λ is*

$$\begin{array}{ccc} & & L \\ & \overset{\lambda}{\nearrow} & \\ E & & \Big\downarrow{\scriptstyle l_0\leq\cdots\leq l_n} \\ & \underset{\lambda_n}{\searrow} & \\ & & L_n \end{array}$$

where $\lambda_n=l_0\lambda=\cdots=l_n\lambda$.

Proof. If $l_i\leq l_j$ and $t\in L$ is a complemented element, then $l_i(t)\leq l_j(t)$ and $\neg l_i(t)=l_i(\neg t)\leq l_j(\neg t)=\neg l_j(t)$, so that $l_i(t)=l_j(t)$. Hence $l_0\lambda=\cdots=l_n\lambda$. Now, the universal property is checked easily. $\square$

For any $n\geq 0$ $(D_0=D)$ the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{p_0\leq\cdots\leq p_{n+1}} & D_{n+1} \\ {\scriptstyle p_0\leq\cdots\leq p_i\leq p_i\leq\cdots\leq p_n}\Big\downarrow & \overset{s_i}{\nearrow} & \\ D_n & & \end{array}$$

defines a unique s_i, $0\leq i\leq n$, and we have a lattice homomorphism

$$(s_0,\ldots,s_n): D_{n+1}\to\prod_{n+1} D_n$$

from D_{n+1} to the product of $n+1$ copies of D_n. The Krull dimension of D is defined as follows:

$$\dim D\leq n \quad\text{iff}\quad (s_0,\ldots,s_n)\text{ is monic.}$$

Classically a prime ideal of $\prod_{n+1} D_n$ is a pair (P,i) formed by a prime ideal P of D_n and $i\in\{0,1,\ldots,n\}$. Moreover $(s_0,\ldots,s_n)$ monic is equivalent to $(s_0,\ldots,s_n)^{-1}$ surjective over prime ideals, with $(s_0,\ldots,s_n)^{-1}(P,i)=s_i^{-1}(P)$. Hence $\dim D\leq n$ if and only if each $(n+1)$-chain of prime ideals of D is degenerated.

1.8. Theorem. *For any lattice D in $\mathscr{E}$,* $\dim D = \dim \Gamma(D)$.

Proof. Since Γ is a right adjoint, it preserves the construction $(s_0, \ldots, s_n)$ by Lemma 1.7. Moreover, products and monics in E-Lt are as in Lt, so that it suffices to use Theorem 1.4. □

2. Boolean valued rings

All rings we consider are commutative and unitary, and ring homomorphisms preserve the unity. We start with a lemma like 1.2.

2.1. Lemma. *Let A be a ring in $\mathscr{E}$ and $e \in E$. For any $a \in A(e)$ there is a global $a' \in \Gamma(A)$ extending a.*

This global a' is unique extending $a \in A(e)$ and $0 \in A(\neg e)$. Taking $a = 1 \in A(e)$ we obtain a global

$$\alpha(e) \in \Gamma(A)$$

uniquely extending $1 \in A(e)$ and $0 \in A(\neg e)$.

2.2. Lemma. *$\alpha(e)$ is idempotent.*

Proof. $A(e \leq 1)(\alpha(e)^2) = A(e \leq 1)(\alpha(e))^2 = 1^2 = 1$ and also $A(\neg e \leq 1)(\alpha(e)^2) = 0$. □

In each category with finite limits we have a functor $\mathbf{E}$ which associates to any ring A the boolean algebra $\mathbf{E}(A)$ of idempotents of A (with $x \wedge y = xy$, $x \vee y = x + y - xy$, $\neg x = 1 - x$). For any $e \in \mathbf{E}(A)$ we have

$$A \simeq Ae \times A(1-e)$$

and conversely, given $A \simeq A_1 \times A_2$ the element $e \in A$ corresponding to $(1,0)$ is idempotent.

It is clear that if A is a ring in $\mathscr{E}$, then

$$\Gamma(\mathbf{E}(A)) = \mathbf{E}(\Gamma(A)).$$

2.3. Lemma. *$\alpha : E \to \mathbf{E}(\Gamma(A))$ is a boolean homomorphism.*

Proof. It is similar to the proof of Lemma 1.3 and we omit the details. □

For any ring homomorphism $f : A \to A'$ in $\mathscr{E}$, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathbf{E}(\Gamma(A)) & & \Gamma(A) \\ & {}^{\alpha}\nearrow & & & \\ E & & \Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma(f)} \\ & {}_{\alpha'}\searrow & & & \\ & & \mathbf{E}(\Gamma(A')) & & \Gamma(A') \end{array}$$

so that we have a functor

$$\Lambda : \mathrm{An}(\mathscr{E}) \to E\text{-An}$$

from rings in $\mathscr{E}$ to the obvious comma category. The composition of Λ with the forgetful functor $U : E\text{-An} \to \mathrm{An}$ is the restriction of Γ to rings

$$\begin{array}{ccc} \mathrm{An}(\mathscr{E}) & \xrightarrow{\ \Lambda\ } & E\text{-An} \\ & {}_{\Gamma}\searrow \quad \swarrow_{U} & \\ & \mathscr{E} & \end{array}$$

An object of E-An, called *E-ring*, is a pair (R, ϱ) where R is a ring in $\mathscr{S}$ and $\varrho : E \to \mathbf{E}(R)$ a boolean homomorphism. We cannot forget the ring R because sometimes $\mathbf{E}(R) = \mathbf{E}(R')$ but $R \neq R'$. But usually we shall say that R is an E-ring, omitting ϱ.

2.4. Theorem. *Λ is an equivalence.*

Proof. We only sketch the proof, after Theorem 1.4. Given a ring homomorphism $f_1 : \mathbf{E}(\Gamma(A) \to \mathbf{E}(\Gamma(A'))$ such that $f_1\alpha = \alpha'$, an element $e \in E$ and $a \in A(e)$, we define a unique ring homomorphism $f : A \to A'$ such that $f(1) = f_1$ by

$$f(e)(a) = A'(e \leq 1)(f_1(a'))$$

and hence Λ is full and faithful.

On the other hand, given a boolean homomorphism $\varrho : E \to \mathbf{E}(R)$ we can take for any $e \in E$

$$A(e) = R\varrho(e)$$

where $R\varrho(e) = \{x\varrho(e) \mid x \in R\}$ is a ring with unity $\varrho(e)$. If $e_1 \leq e_2$, we define

$$A(e_1 \leq e_2) : A(e_2) \to A(e_1), \qquad a \to a\varrho(e_1)$$

and so we obtain a ring in $\mathscr{E}$ such that $\Lambda(A) = \varrho$. □

Given an E-ring (R, ϱ), we shall write $\tilde{R}$ for the sheaf

$$\tilde{R}(e) = R\varrho(e).$$

So we have a functor $(\tilde{\cdot})$ which is the inverse equivalence of Λ. Moreover, $\Gamma(\tilde{R})=R$. Like in lattices, given an E-ring homomorphism $f:(R,\varrho)\to(R,\varrho')$, there is a unique ring homomorphism $\tilde{f}:\tilde{R}\to\tilde{R}'$ such that $\Gamma(\tilde{f})=f$: take $\tilde{f}(e)(x\varrho(e))=f(x)\varrho'(e)$.

2.5. Remark. One can prove Corollary 1.5 directly and then derive jointly Theorems 1.4 and 2.4 by using some common properties of the functors $\mathbf{C}:\mathrm{Lt}\to\mathrm{Bl}$ and $\mathbf{E}:\mathrm{An}\to\mathrm{Bl}$. The functor $\mathbf{E}$ has also a left adjoint $\mathbf{Z}_{(\cdot)}$ with $\mathbf{E}\mathbf{Z}_B\simeq B$ and $\mathbf{Z}_{\mathbf{E}(R)}\to R$ monic. From a well-known fact about adjunctions, $\mathbf{Z}_E$ is the initial E-ring, that is the ring of integers in $\mathscr{E}$. In particular, the ring of integers in $\mathscr{S}$ is $\mathbf{Z}\simeq\mathbf{Z}_2$. As we can see in [7] the ring of integers in the topos sh(X) over a space X is the sheaf of locally constant functions from X to $\mathbf{Z}$. Hence we can describe $\mathbf{Z}_E$ as follows: elements are blocks

$$\begin{pmatrix} e_1\cdots e_n \\ r_1\cdots r_n\end{pmatrix} \quad \text{with} \quad \begin{cases} e_1+\cdots+e_n=1, \\ r_i\in\mathbf{Z},\ 1\le i\le n\end{cases}$$

and with an evident relation of equality for blocks. The addition is

$$\begin{pmatrix} e_1\cdots e_n \\ r_1\cdots r_n\end{pmatrix}+\begin{pmatrix} e_1'\cdots e_m' \\ r_1'\cdots r_m'\end{pmatrix}=\begin{pmatrix} e_1\wedge e_1'\cdots e_n'\wedge e_m' \\ r_1+r_1'\cdots r_n+r_m'\end{pmatrix}$$

and similarly for the product. The adjunction

$$\mathbf{Z}_{(\cdot)}\dashv\mathbf{E}, \qquad \frac{E\xrightarrow{\varrho}\mathbf{E}(R)}{\mathbf{Z}_E\xrightarrow{\xi}R}$$

is given by

$$\xi\begin{pmatrix} e_1\cdots e_n \\ r_1\cdots r_n\end{pmatrix}=\sum_{i=1}^{n} r_i\varrho(e_1), \qquad \varrho(e)=\xi\begin{pmatrix} e & \neg e \\ 1 & 0\end{pmatrix}$$

and it gives the equivalence between boolean algebras and boolean rings.

We can extend this adjunction to $\mathscr{E}$ and so we obtain

$$\mathbf{Z}_{(\cdot)}\dashv\mathbf{E}, \qquad E\text{-An}\rightleftarrows E\text{-Bl}$$

where $\mathbf{E}(R,\varrho)=\varrho$ and $\mathbf{Z}_{(E\to B)}=(\mathbf{Z}_B, E\to B)$ because $B\simeq\mathbf{E}\mathbf{Z}_B$.

2.6. Example. Let $E=\mathbf{E}(R)$ be the boolean algebra of idempotents of a ring R in $\mathscr{S}$. The Pierce representation of R is the ring R in $\mathscr{E}$ given by

$$\tilde{R}(e)=Re,$$

so that the associated E-ring is (R,ϱ) with $\varrho:E\to E$ the identity. After [8] we know that

$$R \text{ regular} \quad\Leftrightarrow\quad \tilde{R} \text{ field}$$

(see also [3]). In fact, R is regular if $\neg(1=0)$ and

$$\forall x,\ \exists x'\ (x^2x'=x\wedge xx'^2=x')$$

and it is a field if $\neg(1=0)$ and

$$\forall x\,(x=0 \vee (\exists y\; xy=1)).$$

The field condition for $\tilde{R}$ means that we have in R, $\neg(1=0)$ and for each $x \in R$ there is a partition $e_1 + \cdots + e_n = 1$ in E such that for any index $i=1,\ldots,n$ there is $x_i \in R$ with $xe_i = 0$ or $xx_ie_i = e_i$. If R is regular, then $e = xx' \in E$ and we have the partition $1 = e + (1-e)$ with $x(1-e) = 0$ and $xx'e = e$, so that $\tilde{R}$ is a field.

Finally we consider polynomials.

2.7. Proposition. *Let $\varrho: E \to \mathbf{E}(R)$ be an E-ring and $\bar{\varrho} = i\varrho: E \to \mathbf{E}(R[X])$ where $i: R \to R[X]$ is the inclusion. Then $(R[X], \bar{\varrho})$ is the E-ring of polynomials over (R, ϱ) with inclusion i.*

Proof. We must verify the following universal property: For any E-ring homomorphism $f: (R, \varrho) \to (R', \varrho')$ and for any $x' \in R'$ there is a unique E-ring homomorphism $\bar{f}: (R[X], \bar{\varrho}) \to (R', \varrho')$ such that $\bar{f} \circ i = f$ and $\bar{f}(X) = x'$. But given $f: R \to R'$ and $x' \in R$ there is a unique $\bar{f}: R[X] \to R'$ such that $\bar{f} \circ i = f$ and $\bar{f}(X) = x'$. Moreover $\bar{f}$ is an E-ring homomorphism because f is so. □

Hence, if A is a ring in $\mathscr{E}$, the polynomial ring $A[X]$ in $\mathscr{E}$ is given by

$$A[X] = \widetilde{\Gamma(A)[X]}$$

where we consider $\Gamma(A)[X]$ as an E-ring in the form

$$E \xrightarrow{\alpha} \mathbf{E}(\Gamma(A)) \xrightarrow{i} \mathbf{E}(\Gamma(A)[X])$$

with α corresponding to A.

If R is a ring such that $E = \mathbf{E}(R) = \mathbf{E}(R[X])$ (for instance if R is regular), then the Pierce representation of R and $R[X]$ are related by

$$\widetilde{R[X]} = \tilde{R}[X]$$

and they correspond to the E-rings (R, id), $(R[X], \mathrm{id})$ respectively.

3. Spectrum and dimension of rings

In a topos $\mathscr{E}$ the *spectrum* of a ring A is (see [4] and [2]) a lattice $D(A)$ with a map

$$d_A: A \to D(A)$$

which is universal among the maps $d: A \to D$ from A to a lattice D such that

(i) $d(0) = 0$, $\quad d(1) = 1$,
(ii) $d(ab) = d(a) \wedge d(b)$,
(iii) $d(a+b) \le d(a) \vee d(b)$.

We shall say that a map $d:A\to D$ with these properties is a *support* of A, so that d_A is a universal support of A.

Given a ring homomorphism $f:A\to A'$, the composition $d_A f:A\to D(A')$ is a support of A and thence there is a unique $D(f):D(A)\to D(A')$ such that $D(f)d_A = d_{A'}f$. So we have a functor

$$\mathbf{D}:\mathrm{An}(\mathscr{E})\to \mathrm{Lt}(\mathscr{E}).$$

3.1. Lemma. *For any support $d:R\to L$ the restriction $E(R)\to L$ of d to idempotents is a lattice homomorphism.*

Proof. If $e_1, e_2 \in E(R)$, then $d(e_1\wedge e_2)=d(e_1e_2)=d(e_1)\wedge d(e_2)$ so that for any $e\in E(R)$, $d(\neg e)=d(1-e)=\neg d(e)$ because

$$d(e)\wedge d(1-e)=d(e(1-e))=d(0)=0,$$

$$d(e)\vee d(1-e)\geq d(e+(1-e))=d(1)=1.$$

Now we conclude $d(e_1\vee e_2)=d(e_1)\vee d(e_2)$ by De Morgan's laws. □

3.2. Lemma. *For any support $d:R\to L$ and $a\in R$ there is a unique support d_a; $R[a^{-1}]\to L_{\leq d(a)}$ such that the following diagram is commutative*

$$\begin{array}{ccc} R & \xrightarrow{\ d\ } & L \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle l} \\ R[a^{-1}] & \xrightarrow[d_a]{} & L_{\leq d(a)} \end{array}$$

where r and l are the canonical homomorphisms. Furthermore, given a support $d':R[a^{-1}]\to L'$ and a lattice homomorphism $f:L\to L'$ such that $d'r=fd$ (i.e., f such that $fd(a)=1$) there is a unique lattice homomorphism $f_a:L_{\leq d(a)}\to L'$ such that $f_a l=f$ and $f_a d_a=d'$.

Proof. We define

$$d_a(x/a^n)=d(x)\wedge d(a).$$

This definition is correct because if $x/a^n=y/a^m$, then $a^h(a^m x + a^n y)=0$ for some h and thence

$$d(x)\wedge d(a)=d(x)\wedge d(a^{h+m})=d(xa^{h+m})=d(ya^{h+n})$$
$$=d(y)\wedge d(a^{h+n})=d(y)\wedge d(a).$$

It is easy to prove that d_a is a support. For instance

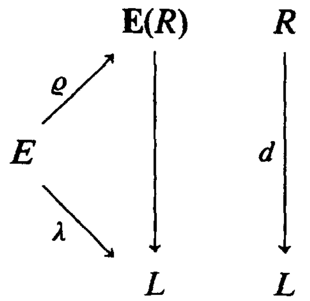
$$d_a\left(\frac{x}{a^n}+\frac{y}{a^m}\right)\leq d_a\left(\frac{x}{a^n}\right)\vee d_a\left(\frac{y}{a^m}\right)$$

since

$$d(a^m x+a^n y)\leq d(a)\wedge[d(x)\vee d(y)].$$

It is clear that $d_a r=ld$ and also the rest of the proof. □

Given an E-ring (R,ϱ) and a support $d:R\to L$ by Lemma 3.1 we can obtain an E-lattice (L,λ) with $\lambda=d\varrho$, and we say that $d:(R,\varrho)\to(L,\lambda)$ is an *E-support*



If $d:A\to D$ is a support in $\mathscr{E}$, $A(A)=(\Gamma(A),\alpha)$ and $\Phi(D)=(\Gamma(D),\delta)$, then it is clear that $\Gamma(d):\Gamma(A)\to\Gamma(D)$ is an E-support. Conversely we have by Lemma 3.2 the following

3.3. Proposition. *For any E-support $d:(R,\varrho)\to(L,\lambda)$ there is a unique support $\tilde{d}:\tilde{R}\to\tilde{L}$ in $\mathscr{E}$ such that $\Gamma(\tilde{d})=d$.*

Proof. Let us recall that $\tilde{R}(e)=R\varrho(e)$ and $\tilde{L}(e)=L_{\leq\lambda(e)}$. But since $\varrho(e)$ is idempotent, $R\varrho(e)=R[\varrho(e)^{-1}]$ and $\lambda(e)=d(\varrho(e))$, we can define a support

$$\tilde{d}(e)=d_e$$

following Lemma 3.2, that is

$$\tilde{d}(e)(x\varrho(e))=d(x)\wedge\lambda(e).$$

It is easy to verify the naturality, so that we have a support $\tilde{d}:\tilde{R}\to\tilde{L}$ with $\Gamma(\tilde{d})=\tilde{d}(1)=d_1=d$. The uniqueness follows also from Lemma 3.2. □

3.4. Proposition. *The universal E-support of an E-ring (R,ϱ) is $d_R:(R,\varrho)\to(D(R),\delta)$ where d_R is the universal support of R and $\delta=d_R\varrho$.*

Proof. It is a simple exercise to check the universal property required. □

Thus the spectrum of a ring A in $\mathscr{E}$ is

$$D(A)=\widetilde{\mathbf{D}(\Gamma(A))}.$$

Finally we are ready to prove our aim

3.5. Theorem. *If R is a regular ring in a topos $\mathscr{S}$, then* $\dim R[X]=1$.

Proof. We suppose the theorem is true for a field, as was proved in [2].

The Pierce representation $\tilde{R}$ of R is a field (Example 2.6) in the topos $\mathscr{E}=\mathrm{sh}(E)$ of sheaves over the boolean algebra $E=\mathbf{E}(R)$, so that $\dim \tilde{R}[X]=1$. Now $\dim \widetilde{R[X]}=1$ by Proposition 2.7, that is $\dim D(\widetilde{R[X]})=1$ since the dimension of a ring is by definition the dimension of its spectrum. Furthermore, $D(\widetilde{R[X]})=\widetilde{D(R[X])}$, hence by Theorem 1.8, $\dim D(R[X])=1$, that is $\dim R[X]=1$. □

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