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Lie structure in semiprime superalgebras with superinvolution ☆

Jesús Laliena *, Sara Sacristán

Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

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Abstract

In this paper we investigate the Lie structure of the Lie superalgebra K of skew elements of a semiprime associative superalgebra A with superinvolution. We show that if U is a Lie ideal of K, then either there exists an ideal J of A such that the Lie ideal $[J \cap K, K]$ is nonzero and contained in U, or A is a subdirect sum of A', A'', where the image of U in A' is central, and A'' is a subdirect product of orders in simple superalgebras, each at most 16-dimensional over its center.

Keywords: Associative superalgebras; Semiprime superalgebras; Superinvolutions; Skewsymmetric elements; Lie structure

1. Introduction

The study of the relationship between the structure of an associative algebra A and that of the Lie algebra A^- was started by I.N. Herstein (see [5,6]) and W.E. Baxter (see [1]). Afterwards, several authors have made different contributions and generalizations to the subject (see for instance [2,9,11]).

Regarding superalgebras, this line of research was motivated by the classification of the finite-dimensional simple Lie superalgebras given by V. Kac [8], particularly the types given from

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* Corresponding author.
E-mail addresses: jesus.laliena@dmc.unirioja.es (J. Laliena), ssacrist@ya.com (S. Sacristán).

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simple associative superalgebras and from simple associative superalgebras with superinvolution. In [3], thinking in simple associative superalgebras with superinvolution, C. Gómez-Ambrosi and I. Shestakov investigated the Lie structure of the set of skew elements, K, of a simple associative superalgebra, A, with superinvolution over a field of characteristic not 2. These results were later extended to prime associative superalgebras with superinvolution [4]. It was specifically proved that the Lie ideals of K and $[K, K]$ which are not contained in the center of A are of the kind $[J \cap K, K]$ for a nonzero ideal J of A, if A is nontrivial, that is with a nonzero odd part, and if A is not a central order in a Clifford superalgebra with at most 4 generators.

This paper is devoted to the description of the Lie ideals of K, the set of skew elements of a semiprime associative superalgebra, A, with superinvolution * over a commutative unital ring ϕ of scalars with $\frac{1}{2} \in \phi$.

We notice that the Lie structure of prime superalgebras and simple superalgebras without superinvolution was studied by F. Montaner (see [12]) and S. Montgomery (see [13]).

For a complete introduction to the basic definitions and examples of superalgebras, superinvolutions and prime and semiprime superalgebras, we refer the reader to [3,12].

Throughout the paper, unless otherwise stated, A will denote a nontrivial semiprime associative superalgebra with superinvolution * over a commutative unital ring ϕ of scalars with $\frac{1}{2} \in \phi$. By a nontrivial superalgebra we understand a superalgebra with nonzero odd part. Z will denote the even part of the center of A, H the Jordan superalgebra of symmetric elements of A, and K the Lie superalgebra of skew elements of A. If P is a subset of A, we will denote by $P_H = P \cap H$ and $P_K = P \cap K$. The following containments are straightforward to check, and they will be used throughout without explicit mention: $[K, K] \subseteq K$, $[K, H] \subseteq H$, $[H, H] \subseteq K$, $H \circ H \subseteq H$, $H \circ K \subseteq K$ and $K \circ K \subseteq H$.

We recall that a superinvolution * is said to be of the first kind if $Z_H = Z$, and of the second kind if $Z_H \neq Z$.

If $Z \neq 0$, one can consider the localization $Z^{-1}A = \{z^{-1}a\colon 0 \neq z \in Z,\ a \in A\}$. If A is prime, then $Z^{-1}A$ is a central prime associative superalgebra over the field $Z^{-1}Z$. We call this superalgebra the central closure of A. We also say that A is a central order in $Z^{-1}A$. While this terminology is not the standard one, for which the definition involves the extended centroid, if $Z \neq 0$ both notions coincide (for more specifications see 1.6 in [12]).

Let A be a prime superalgebra, and let $V = Z_H - \{0\}$ be the subset of regular symmetric elements. Note that if $Z \neq 0$, $Z_H \neq 0$. Also $Z^{-1}A = V^{-1}A$, since for all $0 \neq z \in Z$, $a \in A$ we have $z^{-1}a = (zz^*)^{-1}(z^*a)$. It will be more convenient for us, in order to extend the superinvolution in a natural way, to work with V rather than with Z. We may consider $V^{-1}A$ as a superalgebra over the field $V^{-1}Z_H$. Then the superinvolution on A is extended to a superinvolution of the same kind on $V^{-1}A$ over $V^{-1}Z_H$ via $(v^{-1}a)^* = v^{-1}a^*$. It is then easy to check that $H(V^{-1}A, *) = V^{-1}H$ and $K(V^{-1}A, *) = V^{-1}K$. Moreover, $Z(V^{-1}A)_0 = V^{-1}Z$ and $V^{-1}Z \cap V^{-1}H = V^{-1}Z_H$. We will say that the superalgebra $V^{-1}A$ over the field $V^{-1}Z_H$ is the *-central closure of A.

We notice that in every semiprime superalgebra A, the intersection of all the prime ideals P of A is zero. Consequently A is a subdirect product of its prime images. If each prime image of A is a central order in a simple superalgebra at most n^2-dimensional over its center, we say that A verifies $S(n)$.

If M is a subsupermodule of A, we denote by $\bar{M}$ the subalgebra of A generated by M. We will say that M is dense if $\bar{M}$ contains a nonzero ideal of A.

In this paper, we prove that if K is the Lie superalgebra of skew elements of a semiprime associative superalgebra with superinvolution, A, and U is a Lie ideal of K, then one of the following alternatives must hold: either U must contain a nonzero Lie ideal $[J \cap K, K]$, for J

an ideal of A, or A is a subdirect sum of A', A'', where the image of U in A'' is central and A' satisfies $S(4)$.

The following results are instrumental for the paper:

Lemma 1.1. *(See [6, Lemma 1.1.9].) If A is a semiprime algebra and $[a,[a,A]]=0$, then $a \in Z(A)$.*

Lemma 1.2. *(See [12, Lemmata 1.2, 1.3].) If $A = A_0 \oplus A_1$ is a prime superalgebra, then A and A_0 are semiprime and either A is prime or A_0 is prime (as algebras).*

Lemma 1.3. *(See [12, Lemma 1.8].) Let $A = A_0 \oplus A_1$ be a prime superalgebra. Then*

(i) *If $x_1 \in A_1$ centralizes a nonzero ideal I of A_0, then $x_1 \in Z(A)$.*
(ii) *If x_1^2 belongs to the center of a nonzero ideal I of A_0, then $x_1^2 \in Z(A)$.*

Lemma 1.4. *(See [4, Corollary 2].) Let A be a semiprime superalgebra and L a Lie ideal of A. Then either $[L,L]=0$, or L is dense in A.*

Lemma 1.5. *(See [4, Theorem 2.1].) Let A be a prime nontrivial associative superalgebra. If L is a Lie ideal of A, then either $L \subseteq Z$ or L is dense in A, except if A is a central order in a 4-dimensional Clifford superalgebra.*

We remark that the bracket product in Lemma 1.1 is the usual one: $[a,b]=ab-ba$, but the bracket product in Lemmata 1.3, 1.4, 1.5 is the superbracket $[x_i, y_j]_s = x_i y_j - (-1)^{ij} y_j x_i$ for $x_i \in A_i$, $y_j \in A_j$ homogeneous elements. In fact, the superbracket product coincides with the usual bracket if one of the arguments belongs to the even part of A. In the following, to simplify the notation, we will denote both in the usual way [,] but we will understand that it is the superbracket if we are in a superalgebra.

2. Lie structure of K

Let A be an associative superalgebra and M, S be subgroups of A. Define $(M:S)=\{a \in A: aS \subseteq M\}$.

Also we define the following multiplication on A: $u \circ v = uv + (-1)^{\bar{u}\bar{v}} vu$.

Let U be a Lie ideal of K. We recall (see Lemma 4.1 in [3]) that K^2 is a Lie ideal of A.

Lemma 2.1. *If A is semiprime, then either U is dense in A or $[u \circ v, w]=0$ for every $u, v, w \in U$.*

Proof. We have

$$[u \circ v, k] = u \circ [v,k] + (-1)^{\bar{k}\bar{v}} [u,k] \circ v \in \bar{U}$$

for every $u, v \in U$ and $k \in K$. And also for any $u, v \in U$ and $h \in H$ we get

$$[u \circ v, h] = [u, v \circ h] + (-1)^{\bar{u}\bar{v}} [v, u \circ h] \in U,$$

because $K \circ H \subseteq K$. Since $A = H \oplus K$ it follows that $[u \circ v, A] \subseteq \bar{U}$ for any $u, v \in U$. But for any $a \in A$

$$[u \circ v, wa] = [u \circ v, w]a + (-1)^{(\bar{u}+\bar{v})\bar{w}} w[u \circ v, a]$$

and so $[u \circ v, w]A \subseteq \bar{U}$ for every $u, v, w \in U$, that is, $[u \circ v, w] \in (\bar{U} : A)$. We notice that from the above equations we can also deduce that $[u \circ v, w]a \in \bar{K}^2$ and so $[u \circ v, w] \in (\bar{K}^2 : A)$.

We claim that $A(\bar{K}^2 : A) \subseteq (\bar{K}^2 : A)$. Indeed, for any $x \in (\bar{K}^2 : A)$, $a, b \in A$

$$axb = (-1)^{(\bar{x}+\bar{b})\bar{a}}(xb)a + [a, xb] \in \bar{K}^2,$$

because $[\bar{K}^2, A] \subseteq \bar{K}^2$ (for any $t, s \in K^2$, $[ts, a] = t[s, a] + (-1)^{\bar{s}\bar{a}}[t, a]s \in \bar{K}^2$). Hence $A(\bar{K}^2 : A)A \subseteq (\bar{K}^2 : A)A \subseteq \bar{K}^2$. But $K(\bar{U} : A) \subseteq (\bar{U} : A)$ because for any $x \in (\bar{U} : A)$, $k \in K$, $a \in A$

$$(kx)a = [k, xa] + (-1)^{(\bar{x}+\bar{a})\bar{k}}(xa)k \in \bar{U},$$

because $[K, \bar{U}] \subseteq \bar{U}$, and so $\bar{K^2}(\bar{U} : A) \subseteq (\bar{U} : A)$. Therefore, we finally get

$$A\big(\bar{K}^2 : A\big)A(\bar{U} : A)A \subseteq \bar{K}^2(\bar{U} : A)A \subseteq \bar{U},$$

and since $[u \circ v, w] \in (\bar{U} : A)$ and also $[u \circ v, w] \in (\bar{K}^2 : A)$ it follows that $A[u \circ v, w]A[u \circ v, w]A \subseteq \bar{U}$. Thus, since A is semiprime, either $[u \circ v, w] = 0$ for any $u, v, w \in U$ or U is dense in A. □

We note that the ideal contained in $\bar{U}$ in the above Lemma, $J = A[u \circ v, w]A[u \circ v, w]A$, is also a $*$-ideal, that is, $J^* \subseteq J$.

Lemma 2.2. *Let A be semiprime, and let U be a Lie ideal of K such that $[U \circ U, U] = 0$. Then*

(i) *$u \circ v \in Z$ for every $u, v \in U_0$.*
(ii) *$u \circ v = 0$ for every $u, v \in U_1$.*

Proof. Assertion (i) is proved as in Theorem 5.3 of [3], and (ii) as in Theorem 3.2 of [4]. □

Next we deal with the second case of Lemma 2.1, that is, when $[u \circ v, w] = 0$ for any $u, v, w \in U$ (and therefore when $u \circ v \in Z$ for every $u, v \in U_0$, and $u \circ v = 0$ for every $u, v \in U_1$), and we will study the prime images of A.

Let P be a prime ideal of A. We will suppose first that $P^* \neq P$. In this case $(P^* + P)/P$ is a nonzero proper ideal of A/P and we claim that $(P^* + P)/P \subseteq (K + P)/P$. Indeed, if $y \in P^*$ then $y + P = (y - y^*) + y^* + P \in (K + P)/P$. Also if U is a Lie ideal of K we have that $(U + P)/P$ is an abelian subgroup of A/P and satisfies

$$\big[(U + P)/P, \big(P^* + P\big)/P\big] \subseteq \big([U, K] + P\big)/P \subseteq (U + P)/P.$$

Therefore $[(U + P)/P, (P^* + P)/P] \subseteq (U + P)/P$, and $(P^* + P)/P$ is an ideal in A/P, a prime superalgebra. Of course if $u \circ v \in Z$ for every $u, v \in U_0$ and $u \circ v = 0$ for any $u, v \in U_1$, then the same property is satisfied in A/P, that is, $(u+P)\circ(v+P)\in Z_0(A/P)$ for every $u+P, v+P\in(U_0+P)/P$, and $(u+P)\circ(v+P)=0$ for any $u+P, v+P\in(U_1+P)/P$. Let us analyze this situation. We notice that the assumption that A/P has a superinvolution is not required. We state first a useful lemma.

Lemma 2.3. *Let A be a prime superalgebra, I a nonzero ideal of A and U a subset of A such that $[U,I]=0$. Then $U\subseteq Z$.*

Proof. For any $u_k\in U_k, a_i\in A_i, y_j\in I_j$, applying $[U,I]=0$ we get

$$u_k(a_iy_j)=(-1)^{(i+j)k}(a_iy_j)u_k=(-1)^{ik}a_i(u_ky_j).$$

Since A is prime it follows that $u_ka_i=(-1)^{ik}a_iu_k$. On the other hand, given $u_1\in U_1$ we have $[u_1,I_0]=0$, and applying Lemma 1.3(i), $u_1\in Z_1(A)$. Hence for every $u_1\in U_1$, $a_1\in A_1$ we have $u_1a_1=a_1u_1=-a_1u_1$, that is, $a_1u_1=0$, and, because $u_1\in Z(A)$ and the primeness of A, $U_1=0$ and $U\subseteq Z$. □

Theorem 2.4. *Let A be a prime superalgebra, and let I be a nonzero proper ideal of A. Suppose that U is a Lie subalgebra of A^- such that $[U,I]\subseteq U$, $u\circ v\in Z$ for every $u,v\in U_0$, and $u\circ v=0$ for every $u,v\in U_1$. Then either A is commutative, or A is a central order in a 4-dimensional simple superalgebra, or $U\subseteq Z$.*

Proof. Let $T=\{x\in A\colon\ [x,A]\subseteq[U,I]\}$. Since

$$\big[\big[[U,I],[U,I]\big],A\big]\subseteq\big[[U,I],\big[[U,I],A\big]\big]\subseteq\big[[U,I],I\big]\subseteq[U,I],$$

we have $[[U,I],[U,I]]\subseteq T$. We notice that T is subring because for any $t,s\in T$,

$$[ts,a]=[t,sa]+(-1)^{\bar{t}\bar{s}+\bar{a}\bar{t}}[s,at]\in[U,I].$$

Let T' be the subring generated by $[[U,I],[U,I]]$. Since

$$\big[\big[[U,I],[U,I]\big],I\big]\subseteq\big[[U,I],\big[[U,I],I\big]\big]\subseteq\big[[U,I],[U,I]\big]$$

it follows that $[T',I]\subseteq T'$. We consider now two cases: (a) $[T',I]=0$, and (b) $[T',I]\neq 0$.

(a) If $[T',I]=0$, then $[[[U,I],[U,I]],I]=0$. By Lemma 2.3 we get $[[U,I],[U,I]]\subseteq Z$, and so $[[U,I],[U,I]]_1=0$.

We claim that in this situation either $U\subseteq Z$, or A is commutative, or A is a central order in a 4-dimensional simple superalgebra. We present the proof of this in 6 steps.

1. $[U,I]_0\subseteq Z$. By hypothesis $u\circ v\in Z$ for any $u,v\in U_0$, so since $[U,I]\subseteq U$ it follows that $uv\in Z$ for any $u,v\in[U,I]_0$. Hence, for any $u,v\in[U,I]_0$, we have

$$[u,v][u,v]=\big[u,v[u,v]\big]-v\big[u,[u,v]\big]=\big[u,[vu,v]\big]-\big[u,[v,v]u\big]=0$$

because $[u,v], vu\in Z$. Therefore, from the primeness of A, $[u,v]=0$ for any $u,v\in[U,I]_0$. So since $[[U,I],[U,I]]_1=0$, $[u,[u,I]]=0$ for any $u\in[U,I]_0$, and therefore, by Lemma 1.2 and Theorem 1 in [7], $u\in Z(I)$, that is $[U,I]_0\subseteq Z$ because A is prime.

2. $[U_0, I_0] = 0$. By step 1 we have $[u_0, [u_0, I_0]] = 0$ for any $u_0 \in U_0$, and again by Theorem 1 in [7] and Lemma 1.2, we obtain that $[U_0, I_0] = 0$.

3. $U_1U_1 \subseteq Z$. Let $u_1 \in U_1$, $y_1 \in I_1$, since $[U_1, I_1] \subseteq [U, I]_0 \subseteq Z$ we get

$$\left[u_1^2, y_1\right] = u_1[u_1, y_1] - [u_1, y_1]u_1 = u_1[u_1, y_1] - u_1[u_1, y_1] = 0.$$

Therefore, since $u_1 \circ v_1 = 0$ for any $u_1, v_1 \in U_1$, $0 = [(u_1 + v_1)^2, y_1] = [u_1v_1 + v_1u_1, y_1] = 2[u_1v_1, y_1]$ for any $y_1 \in I_1$. And, since $[u_1, v_1] = 2u_1v_1 \in U_0$ because $u_1 \circ v_1 = 0$ for any $u_1, v_1 \in U_1$, we have $[u_1v_1, I_0] = 0$ for any $u_1, v_1 \in U_1$ by step 2. So $[u_1v_1, I] = 0$ for any $u_1, v_1 \in U_1$, and then $u_1v_1 \in Z$, because of Lemma 2.3.

4. $I_1(U_1)^3 \subseteq Z$. From the steps 1 and 3 for any $u_1, v_1, w_1 \in U_1$, $y_1 \in I_1$ we get $[u_1, y_1]v_1w_1 \in Z$, but

$$\begin{aligned}[u_1, y_1]v_1w_1 &= [u_1, y_1v_1]w_1 + y_1[u_1, v_1]w_1 \\ &= [u_1w_1, y_1v_1] - u_1[w_1, y_1v_1] + y_1[u_1, v_1w_1] + y_1v_1[u_1, w_1]\end{aligned}$$

and since $[u_1w_1, y_1v_1] = 0$, $y_1[u_1, v_1w_1] = 0$, by step 3 and $u_1[w_1, y_1v_1] \in U_1[U_1, I_0] \subseteq U_1U_1 \subseteq Z$, we obtain that $y_1v_1[u_1, w_1] \in Z$, that is $I_1(U_1)^3 \subseteq Z$ because $u_1 \circ w_1 = 0$.

5. Either $U_1U_1 = 0$ or A is commutative. By step 4 we have an ideal of A_0, $I_1u_1^3$, contained in Z, and so $[A_1, I_1u_1^3] = 0$, and by Lemma 1.3 either $A_1 \subseteq Z(A)_1$ or $I_1u_1^3 = 0$ for any $u_1 \in U_1$.

If $A_1 \subseteq Z_1(A)$ then $A_1^2 \subseteq Z$, and since A is prime and $A_1 + A_1^2$ is a nonzero ideal contained in $Z(A)$, because A is nontrivial, we deduce that A is commutative.

If $I_1u_1^3 = 0$, since $0 = I_1u_1^3 = (I_1u_1)(u_1^2A)$ and u_1^2A is an ideal of A because $u_1^2 \in Z$ by step 3, then from the primeness of A either $I_1u_1 = 0$ or $u_1^2 = 0$ for any $u_1 \in U_1$. But if $u_1^2 = 0$ for every $u_1 \in U_1$ we get $U_1U_1 = 0$ because $u_1 \circ v_1 = 0$ for any $u_1, v_1 \in U_1$ and if $I_1u_1 = 0$ then $0 = I_1(u_1v_1)$ for every $u_1, v_1 \in U_1$. From step 3 and because A is prime we obtain that either $U_1U_1 = 0$ or $I_1 = 0$. But $I_1 = 0$ contradicts that A is prime because then $IA_1 = 0$ and so $I(A_1 + A_1^2) = 0$ with $A_1 + A_1^2$ a nonzero ideal of A. Therefore $U_1U_1 = 0$ in any case, when $I_1u_1^3 = 0$.

6. Either $U \subseteq Z$, or A is commutative, or A is a central order in a 4-dimensional simple superalgebra. We consider $[v_1, z_1]I$ with $v_1 \in U_1$, $z_1 \in I_1$. It is an ideal of A by step 1. For any $u_0 \in U_0$, $v_1 \in U_1$ and $y_1, z_1 \in I_1$ we have

$$[u_0, y_1][v_1, z_1]I = [u_0, y_1]v_1z_1I + [u_0, y_1]z_1v_1I$$

with $[u_0, y_1]v_1z_1I = 0$ by step 5 and

$$[u_0, y_1]z_1v_1I = -y_1[u_0, z_1]v_1I + [u_0, y_1z_1]v_1I = 0$$

by steps 2 and 5. Since A is prime we obtain that either (i) $[U_1, I_1] = 0$ or (ii) $[U_1, I_1] \neq 0$, and then $[U_0, I_1] = 0$.

(i) If $[U_1, I_1] = 0$ then for any $u_1 \in U_1$, u_1I_1 is a nilpotent ideal of A_0 because by step 5 $(u_1I_1)(u_1I_1) = u_1^2I_1 = 0$, and since A_0 is semiprime by Lemma 1.2, we deduce that $u_1I_1 = 0$. But then $u_1I_0A_1 \subseteq u_1I_1 = 0$ and also $u_1I_0A_1^2 = 0$, that is, $u_1I_0(A_1 + A_1^2) = 0$ with $A_1 + A_1^2$ a nonzero ideal of A. By the primeness of A, $u_1I_0 = 0$, and so $u_1I = 0$ and $U_1 = 0$. Therefore

$[U, I] = [U_0, I] = [U_0, I_0]$ because $[U, I] \subseteq U$, and $[U_0, I_0] = 0$ by step 2, so by Lemma 2.3, $U_0 \subseteq Z$.

(ii) If $[U_1, I_1] \neq 0$, then $[U_0, I_1] = 0$ and so by step 2, $[U_0, I_0] = 0$, so from Lemma 2.3, $U_0 \subseteq Z$. Also $Z \neq 0$ and we may localize A by Z and consider in $Z^{-1}A$, the Lie subalgebra $Z^{-1}(ZU)$ and the ideal $Z^{-1}I$, which satisfy the hypothesis of the theorem. Now we have also that $0 \neq Z^{-1}Z$ is a field. By step 1, $[U_1, I_1] \subseteq Z$, and hence

$$0 \neq \big[Z^{-1}(ZU)_1, Z^{-1}I_1\big] \subseteq Z^{-1}I_0 \cap Z^{-1}Z.$$

Therefore $Z^{-1}I$ has invertible elements and so $Z^{-1}I = Z^{-1}A$. But then $Z^{-1}(ZU)$ is a Lie ideal of $Z^{-1}A$. Since $[Z^{-1}(ZU),\ Z^{-1}(ZU)] = 0$ because $U_0 \subseteq Z$ and because of step 5, it follows from Theorem 3.2 and its proof in [12] that either $Z^{-1}(ZU) \subseteq Z^{-1}Z$ or A is a central order in the matrix algebra $M_{1,1}(Z^{-1}Z)$. In the last case A is a central order in a 4-dimensional simple superalgebra, and in the first case $Z^{-1}(ZU) \subseteq Z^{-1}Z$ and we can deduce from the primeness of A that $U \subseteq Z$.

Therefore in case (a) we have obtained that either $U \subseteq Z$, or A is commutative, or A is a central order in a 4-dimensional simple superalgebra

(b) We recall that $[T', I] \subseteq T'$. Consider $[T', T']$. We claim that $I[T', T']I \subseteq T'$. Indeed, let $x \in T'$, $y \in T'$ and $a \in I$. Since $[T', I] \subseteq T'$ and T' is a subring,

$$[x, y]a = [x, ya] - (-1)^{\bar{x}\bar{y}} y[x, a] \in T'.$$

Now, let $b \in I$; we get

$$\begin{aligned} b[x, y]a &= \big[b, [x, y]\big]a + (-1)^{(\bar{x}+\bar{y})\bar{b}}[x, y]ba \\ &= -(-1)^{\bar{y}(\bar{b}+\bar{x})}\big[y, [b, x]\big]a - (-1)^{\bar{b}\bar{x}+\bar{b}\bar{y}}\big[x, [y, b]\big]a \\ &\quad + (-1)^{(\bar{x}+\bar{y})\bar{b}}[x, y]ba \in T'. \end{aligned}$$

Therefore, by the primeness of A, T' is dense if $[T', T'] \neq 0$.

If $[T', T'] = 0$, then

$$\big[\big[[U, I], [U, I]\big], \big[[U, I], [U, I]\big]\big] = 0.$$

We denote $V = [[U, I], [U, I]]$ and we have that $[V, V] = 0$, and so for V, instead of U, we would immediately have case (a). So we obtain that either A is commutative, or A is a central order in a 4-dimensional simple superalgebra, or $V \subseteq Z$. But if $V \subseteq Z$ we can apply case (a) and we obtain that either $U \subseteq Z$, or A is commutative, or A is a central order in a 4-dimensional simple superalgebra.

It remains to consider the case when T' is dense in A. We denote by $J = I[[T', I], T']I$ and so $J \subseteq T'$. From the definition of T and because $T' \subseteq T$ we know that $[T', A] \subseteq [U, I]$, and therefore $[J, A] \subseteq [U, I] \subseteq U$. By hypothesis $u \circ v \in Z$ for any $u, v \in U_0$, so $u \circ v \in Z$ for any $u, v \in [J, A]_0$.

We assume first that $u \circ v = 0$ for any $u, v \in [J, A]_0$. Then $1/2(u \circ u) = u^2 = 0$ for any $u \in [J, A]_0$ and since A_0 is semiprime by Lemma 1.2, we can apply Lemma 1 in [10] and we have $[J, A]_0 = 0$. Therefore $[J, A] = [J, A]_1$ and then $[J, A]$ is a Lie ideal of A such that $[[J, A], [J, A]] = 0$. From Theorem 3.2 and its proof in [12] it follows that either $[J, A] \subseteq Z$ or

A is a central order in a 4-dimensional matrix superalgebra. If $[J, A] \subseteq Z$, since $[J, A] = [J, A]_1$, we get $[J, A] = 0$ and now by Lemma 2.3, $J \subseteq Z$, and so A is commutative.

Suppose now that there exist $u, v \in [J, A]_0$ such that $u \circ v \neq 0$. Then $Z \neq 0$, and we may form the localization $Z^{-1}A$. Since $[J, A] \subseteq [U, I] \subseteq U$ we have $[Z^{-1}J, Z^{-1}A] \subseteq [Z^{-1}(ZU), Z^{-1}I] \subseteq Z^{-1}(ZU)$, and so from the hypothesis of the theorem for any $u, v \in [Z^{-1}J, Z^{-1}A]_0$ we get $u \circ v \in Z^{-1}Z \cap Z^{-1}J$. But $Z^{-1}Z$ is a field and so $Z^{-1}J$ has some invertible element forcing $Z^{-1}J = Z^{-1}A$. Therefore $[Z^{-1}J, Z^{-1}A] = [Z^{-1}A, Z^{-1}A] \subseteq Z^{-1}(ZU)$ and again by the hypothesis of the theorem it follows that $[Z^{-1}A, Z^{-1}A]_1 \circ [Z^{-1}A, Z^{-1}A]_1 = 0$. We apply now Lemma 2.6 in [12] and we obtain that $Z^{-1}A$ is commutative (superalgebras of the type (b) and (c) in the lemma do not satisfy the condition $[Z^{-1}A, Z^{-1}A]_1 \circ [Z^{-1}A, Z^{-1}A]_1 = 0$), and so A is commutative. This finishes the proof. □

Next we consider the cases when $P^* = P$ and the involution on A/P is of the second kind or of the first kind.

Lemma 2.5. *Let A be a prime superalgebra with a superinvolution $*$ of the second kind. Let U be a Lie ideal of K such that $u \circ v \in Z$ for every $u, v \in U_0$, and $u \circ v = 0$ for every $u, v \in U_1$. Then either $U \subseteq Z$ or A satisfies $S(2)$.*

Proof. If $*$ is of the second kind we know that $Z_H = \{x \in Z\colon\ x^* = x\} \neq Z$. We may localize A by V and replace U by $V^{-1}(Z_H U)$ and A by $V^{-1}A$. The hypothesis remains unchanged, so we keep for this superalgebra the same notation A, and now Z is a field. Let $0 \neq t \in Z_K$. Then $H = tK$ and $A = tK + K$. It follows that $[ZU, A] \subseteq ZU$, $u \circ v \in Z$ for every $u, v \in ZU_0$, and $u \circ v = 0$ for every $u, v \in ZU_1$. By Theorem 2.4, either $ZU \subseteq Z$, which implies that $U \subseteq Z$, or A satisfies $S(2)$. □

Lemma 2.6. *Let A be a prime superalgebra with a superinvolution $*$ of the first kind. Let U be a Lie ideal of K such that $u \circ v \in Z$ for every $u, v \in U_0$, and $u \circ v = 0$ for every $u, v \in U_1$. Then either $U \subseteq Z$ or A satisfies $S(4)$.*

Proof. If $u^2 = 0$ for every $u \in U_0$, applying Theorem 3.3 in [4] we obtain that $U = 0$. Suppose then that $u^2 \neq 0$ for some $u \in U_0$. By Theorem 3.4 in [4] we get that either $U \subseteq Z$ or A is a central order in a Clifford algebra with either 2 or 4 generators. □

Combining the above results we obtain

Theorem 2.7. *Let A be a semiprime superalgebra and U a Lie ideal of K with $u \circ v \in Z$ for every $u, v \in U_0$, and $u \circ v = 0$ for every $u, v \in U_1$. Then A is the subdirect sum of two semiprime homomorphic images A', A'', such that A' satisfies $S(4)$ and the image of U in A'' is central.*

Proof. Let $T' = \{P\colon\ P$ is a prime ideal of A such that A/P satisfies $S(4)\}$ and let $T'' = \{P\colon\ P$ is a prime ideal of A such that the image of U in A/P is central$\}$.

If we consider P a prime ideal of A such that $P^* \neq P$ we know from Theorem 2.8 that either A/P is a central order in a simple superalgebra at most 4-dimensional over its center, or $(U + P)/P$ is central. If we consider P a prime ideal of A such that $P^* = P$, it follows from Lemmata 2.5, 2.6 that either A/P is a central order in a simple superalgebra at most 16-dimensional over its center, or the image of U in A/P is central.

So every prime ideal of A belongs either T' or T''. Then A' is obtained by taking the quotient of A by the intersection of all the prime ideals in T', and A'' is obtained by taking the quotient of A by the intersection of all the prime ideals in T''. This proves the theorem. □

We finally arrive at the main theorem on the Lie structure of K.

Theorem 2.8. *Let A be a semiprime superalgebra with superinvolution $*$, and let U be a Lie ideal of K. Then either A is a subdirect sum of two semiprime homomorphic images A', A'', with A' satisfying $S(4)$ and the image of U in A'' being central, or $U \supseteq [J \cap K, K] \neq 0$ for some ideal J of A.*

Proof. From Lemmata 2.1 and 2.2 we know that either U is dense in A, and so there exist a nonzero ideal J such that $J \subseteq \bar{U}$, or $u \circ v \in Z$ for every $u, v \in U_0$, and $u \circ v = 0$ for every $u, v \in U_1$. In the second case we obtain by Theorem 2.7 the first part of the theorem. So suppose that $J \subseteq \bar{U}$.

The identity

$$[xy, z] = [x, yz] + (-1)^{\bar{x}\bar{y}+\bar{x}\bar{z}}[y, zx]$$

can be used to show that $[\bar{U}, A] = [U, A]$. Hence $[J \cap K, K] \subseteq [\bar{U}, A] = [U, A] = [U, H] + [U, K]$. But $[U, H] \subseteq H$, and $[U, K] \subseteq K$, so $[J \cap K, K] \subseteq [U, K] \subseteq U$.

Finally, suppose that $[J \cap K, K] = 0$, then $[u \circ v, w] = 0$ for any $u, v, w \in J \cap K$ because $[uv, w] = u[v, w] + (-1)^{\bar{v}\bar{w}}[u, w]v = 0$. So by Lemmata 2.1, 2.2 and Theorem 2.7 it follows that for each prime image, A/P, of A either its center contains $((J \cap K) + P)/P$, or A/P is a central order in a simple superalgebra at most 16-dimensional over its center.

We claim that if the image of $J \cap K$ in A/P for some prime ideal P of A is central, then A is as described in the first part of the conclusion of the theorem.

Let P be a prime ideal such that $P^* \neq P$. If $(J + P)/P \neq 0$, then since A/P is a prime superalgebra we get $((J \cap P^*) + P)/P \neq 0$, and so we have $((J \cap P^*) + P)/P \subseteq ((J \cap K) + P)/P \subseteq Z_0(A/P)$, that is, A/P is commutative. So A/P is commutative unless $J \subseteq P$. And if $J \subseteq P$, then by the proof of Lemma 2.1 we know that $A[u \circ v, w]A[u \circ v, w]A \subseteq P$ for any $u, v, w \in U$, and because P is a prime ideal we deduce that $[u \circ v, w] \in P$ for any $u, v, w \in U$. But now by Lemma 2.2 and since $[u \circ v, w] + P = 0$ for any $u, v, w \in U$, it follows that A/P satisfies the conditions $u \circ v \in Z$ for any $u, v \in ((U + P)/P)_0$ and $u \circ v = 0$ for any $u, v \in ((U + P)/P)_1$. By Theorem 2.4 we obtain that either $(U + P)/P \subseteq Z_0(A/P)$, or A/P satisfies $S(4)$.

And if P is a prime ideal such that $P^* = P$ then A/P has a superinvolution induced by $*$ and $K(A/P) = (K + P)/P$. In this case if $((J \cap K) + P)/P = 0$ we get $(J + P)/P \subseteq (H + P)/P = H(A/P)$, and therefore $(J + P)/P$ is supercommutative. But then for any $a, b \in A/P$ and $y, z \in (J + P)/P$ it follows that

$$\begin{aligned} yabz &= (-1)^{(\bar{b}+\bar{z})(\bar{y}+\bar{a})}(bz)(ya) = (-1)^{\bar{b}(\bar{y}+\bar{a})}b(ya)z \\ &= (-1)^{\bar{b}\bar{y}+\bar{b}\bar{a}+(\bar{a}+\bar{z})\bar{y}}b(az)y = (-1)^{\bar{b}\bar{a}}ybaz, \end{aligned}$$

and since A/P is prime $ab = (-1)^{\bar{a}\bar{b}}ba$, that is, A/P is supercommutative. Now from Lemma 1.9 in [12], A/P is a central order in a simple superalgebra at most 4-dimensional over its center. And if $((J \cap K) + P)/P \neq 0$ then $Z_0(A/P) \neq 0$, so by localizing at $V = (Z_0(A/P) \cap H(A/P)) - \{0\}$ we can suppose that $Z_0(A/P)$ is a field, which we denote by Z. We will replace $V^{-1}(A/P)$ by A/P and $V^{-1}((J+P)/P)$ by $(J+P)/P$. Then if $0 \neq t \in ((J \cap K) + P)/P$ we have $tH = K$ with $H = H(A/P)$, $K = K(A/P)$, so $K = tH \subseteq K \cap J \subseteq Z$, and also $tH = K \subseteq Z$ and $H \subseteq t^{-1}Z \subseteq Z$. Therefore A/P is a field. □

Finally we have

Corollary 2.9. *Let A be a semiprime superalgebra with superinvolution $*$, and let U be a Lie ideal of K. Then either $[J \cap K, K] \subseteq U$ where J is a nonzero ideal of A or there exists a semiprime ideal T of A such that $A/\operatorname{Ann} T$ satisfies $S(4)$ and $(U+T)/T \subseteq Z_0(A/T)$.*

Proof. By Theorem 2.8 we have that either the first conclusion holds, or, for each prime ideal P of A, either A/P satisfies $S(4)$ or $(U+P)/P \subseteq Z_0(A/P)$. Let T be the intersection of the prime ideals P of A such that $(U+P)/P \subseteq Z_0(A/P)$. Then $\operatorname{Ann} T$ contains the intersection of those prime ideals P such that A/P satisfies $S(4)$. So we get that $A/\operatorname{Ann} T$ satisfies $S(4)$, and this proves the result. □

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