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Maximal subalgebras of associative superalgebras

Alberto Elduque,[a,*,1] Jesús Laliena,[b,1] and Sara Sacristán [b]

[a] *Department of Mathematics, University of Zaragoza, 50009, Zaragoza, Spain*
[b] *Department of Mathematics and Computation, University of La Rioja, 26004, Logroño, Spain*

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Abstract

The maximal subalgebras of the finite dimensional central simple associative superalgebras, possibly endowed with a superinvolution, are determined. This relies on the corresponding description by M. Racine in the ungraded case, which is completed here too.

1. Introduction

Given an algebraic or geometric structure, the knowledge of its maximal substructures has a great interest. For example, the classical problem of the classification of primitive transformation groups, posed by S. Lie at the end of the last century [6], is equivalent to the determination of certain maximal subgroups in Lie groups. This fact led E. Dynkin in 1952 to describe the maximal subgroups of certain classical groups [3], and also the maximal subalgebras of semisimple Lie algebras [2]. More recently, in 1974, M. Racine determined the maximal subalgebras of finite dimensional central simple algebras for each of the following classes: associative, associative with involution, alternative and special and exceptional Jordan algebras [7,8]. A very subtle case is missing in his determination

* Corresponding author.
E-mail address: elduque@posta.unizar.es (A. Elduque).
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of the maximal $*$-subalgebras of central simple associative algebras with involution. This case will be completed here. The same question for central simple Malcev algebras was solved by the first author in 1986 [4].

This paper is devoted to the determination of the maximal subalgebras of finite dimensional central simple superalgebras which are either associative or associative with superinvolution.

First of all, let us recall some basic features of superalgebras. Let F be a field, a superalgebra A over F is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ over F, endowed with a multiplication $A \times A \to A$ which respects the $\mathbb{Z}_2$-graduation: $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ $(\alpha, \beta \in \mathbb{Z}_2)$. If $a \in A_\alpha$ we say that a is an *homogeneous element* and we use the notation $\bar{a} = \alpha$. A superalgebra A is said to be nontrivial if $A_{\bar{1}} \neq 0$. We remark that the center of A is a superalgebra $Z(A) = Z(A)_{\bar{0}} \oplus Z(A)_{\bar{1}}$. Let $Z = Z(A)_{\bar{0}}$, A is said to be a *central* superalgebra over F if $Z = F$. Given a superalgebra A, it is said to be a *simple superalgebra* if there is no proper nontrivial graded ideal in A and $A^2 \neq 0$. In this case Z is a field.

In [10], Wall described the structure of finite dimensional simple associative superalgebras (see also [1,9]).

Theorem 1.1. *Let A be a finite dimensional nontrivial central simple associative superalgebra over a field F. Then either*:

(i) $Z(A)_{\bar{1}} = 0$, *and this happens if and only if A is central simple as an* (*ungraded*) *algebra over F. Then there exists an element $z \in Z(A_{\bar{0}})$ such that $za_1 = -a_1 z$ for any $a_1 \in A_1$. In this case A is said to be of even type.*

(ii) $Z(A)_{\bar{1}} \neq 0$, *and this happens if and only if $A_{\bar{0}}$ is a central simple algebra over F. Then A is said to be of odd type. In this case $Z(A) = F \oplus Fu$ with $0 \neq u^2 \in F$ and $A = A_{\bar{0}} \oplus A_{\bar{0}} u$.*

Given a superalgebra A over F we say that a graded vector space $M = M_{\bar{0}} + M_{\bar{1}}$ over F is a left *A-supermodule* if it is a left A-module and verifies $A_i M_j \subseteq M_{i+j \,(\mathrm{mod}\ 2)}$ for all $i, j \in \{\bar{0}, \bar{1}\}$. The A-module M is said to be *irreducible* if $AM = M$ and it contains no proper graded submodule.

A unital associative superalgebra A is said to be a *division superalgebra* if all its nonzero homogeneous elements are invertible. If Δ is a division superalgebra with $\Delta_{\bar{1}} \neq 0$ and M is a Δ-supermodule, then M is a free Δ-module: any basis of $M_{\bar{0}}$ as a vector space over $\Delta_{\bar{0}}$ is a basis of M as a module over Δ.

In [9], M. Racine proved the graded version of Schur's Lemma and the Density Theorem for associative superalgebras. Both results are instrumental for the paper:

Theorem 1.2 (Graded Schur's Lemma). *Let A be an associative superalgebra. Let V be an irreducible left A-supermodule. Then* $\mathrm{End}_A(V) = \Delta$ *is a division superalgebra.*

Theorem 1.3 (Graded Density Theorem). *Let M be an irreducible left supermodule for A and let* $\Delta = \mathrm{End}_A(M)$. *Then for every positive integer n, any elements $v_1, \ldots, v_n \in M_\alpha$*

which are linearly independent over $\Delta_{\bar{0}}$, *and any* $w_1, \dots, w_n \in M_\beta$, *there is an element* $a \in A$ *such that* $av_i = w_i$ *for every* $i = 1, 2, \dots, n$.

And, as a consequence:

Theorem 1.4. *Let* V *be an irreducible left* A*-module and* $\Delta = \mathrm{End}_A(V)$. *If* A *is a finite dimensional simple superalgebra, then* $A \cong \mathrm{End}_\Delta(V)$. *Besides, the types of* A *and* Δ *coincide.*

Throughout the paper we will identify, under the conditions of this theorem, A with $\mathrm{End}_\Delta(V)$.

As a general rule, if V is a left module for A, $\Delta = \mathrm{End}_A(V)$ will be assumed to act on the right, so that V becomes a right module for Δ and, therefore, a right module for $A^{\mathrm{op}} \otimes_F \Delta$. Here A^{op} denotes the opposite algebra, while A^{sop} will denote the opposite superalgebra (where $x \cdot y = (-1)^{\bar{x}\bar{y}} yx$, for any homogeneous elements $x, y \in A$).

Finally, let us recall the following version (see [5]) of a basic result in associative algebras, the Double Centralizer Theorem for central simple algebras, that will be used quite often.

Theorem 1.5. *Let* B *be a semisimple subalgebra of a finite dimensional central simple algebra* A. *Then the double centralizer* $C_A(C_A(B))$ *is precisely* B. *If* B *is simple, so is* $C_A(B)$.

Our purpose is to extend to the setting of associative superalgebras the results by Racine on associative algebras [7, Theorems 1–4]. We reproduce below [7, Theorem 1].

Theorem 1.6. *Let* A *be a finite dimensional central simple algebra over the field* F, *let* V *be an irreducible* A*-module and let* $D = \mathrm{End}_A(V)$. *Then a subalgebra* S *of* A *over* F *is maximal if and only if either*:

(i) $S = S(W) = \{a \in A\colon aW \subseteq W\}$, *for* W *a proper* D*-subspace of* V.
(ii) $S = C_A(K) = \{a \in A\colon ak = ka\ \forall k \in A\}$ *where* K/F *is a field extension without intermediate subfields.*

Notice that the subalgebra in item (i) above can be described as $S(W) = eAe + eAf + fAf$, where $0 \neq e \neq 1$ is a projection in $A = \mathrm{End}_D(V)$ onto W, so that e is an idempotent, and $f = 1 - e$. Here $W = eV$.

In Section 2 this will be extended to superalgebras, not "superizing" the proofs in [7], but providing new shorter proofs. Section 3 is devoted to complete [7, Theorem 4], where a very subtle case is missing, providing first a counterexample to the old result. This will turn out to be the most difficult part of the paper. Then, in Section 4, the results for superalgebras which extend [7, Theorems 2–4] will be proved.

2. Maximal subalgebras of associative superalgebras

We begin by studying maximal subalgebras of finite dimensional central simple associative superalgebras. In the following the word subalgebra will be used in the graded sense. First, let us remark the next general result:

Lemma 2.1. *Let F be a finite extension of the field E and S a maximal E-subalgebra of A, a central simple superalgebra over F. Then S contains* 1, *the identity of A.*

Proof. If $1 \notin S$ then the algebra generated by S and 1, that will be denoted by $\mathrm{alg}(S \cup \{1\})$, verifies that $\mathrm{alg}(S \cup \{1\}) = A$, because S is maximal. Then S is a nonzero graded ideal of A. But A is a simple superalgebra, and hence $1 \in S$. □

Now we describe the maximal subalgebras of simple superalgebras of even type.

Theorem 2.2. *Let A be a finite dimensional central simple associative superalgebra over F of even type, let V be an irreducible left A-module and let $\Delta = \mathrm{End}_A(V)$. Let S be a subalgebra of A. Then S is a maximal subalgebra of A if and only if either*:

(i) *There exists a graded proper Δ-submodule W such that $S = \{a \in A \mid aW \subseteq W\}$ (stabilizer of W).*
(ii) *There exists a field K with $F \subsetneqq K \subseteq A_{\bar 0}$, such that there are no proper intermediate subfields between F and K, such that $S = C_A(K)$ (the centralizer of K in A).*
(iii) *There exists $u \in A_{\bar 1}$ with $0 \neq u^2 \in F$ such that $S = C_A(u)$.*

These conditions are mutually exclusive.

Proof. Let V be an irreducible left A-module, then V is also an irreducible right $(A^{\mathrm{op}} \otimes_F \Delta)$-module and therefore a right $(S^{\mathrm{op}} \otimes_F \Delta)$-module. Notice that $\mathrm{End}_{A^{\mathrm{op}} \otimes_F \Delta}(V) = \{\varphi \in \mathrm{End}_\Delta(V) = A\colon [\varphi, A] = 0\} = Z(A) = F$, so by density we can identify $A^{\mathrm{op}} \otimes_F \Delta = \mathrm{End}_F(V)$.

If W is a proper graded $(S^{\mathrm{op}} \otimes_F \Delta)$-submodule then $S \subseteq \{a \in A \mid aW \subseteq W\}$. By maximality, $S = \{a \in A \mid aW \subseteq W\}$.

Conversely, with W as before and $S = \{a \in A \mid aW \subseteq W\}$, let us show that S is maximal, even as an ungraded subalgebra of A. Let $e = e^2 \in A_{\bar 0}$ be a projection onto W, then $S = eAe + eAf + fAf$, with $f = 1 - e$. For any homogeneous element $a_\alpha \in fAe$, since A is (graded) simple, $fAfa_\alpha eAe = fAe$ and hence $\mathrm{alg}(S \cup \{a_\alpha\}) = A$ (ungraded).

In this case, notice that $C_A(S) \subseteq C_A(e) = eAe \oplus fAf$ where eAe, fAf are central simple superalgebras (the first one being isomorphic to $\mathrm{End}_\Delta(W)$), so $C_A(S) \subseteq Z(eAe) \oplus Z(fAf) = Fe + Ff$, and since $f, e \notin C_A(S)$ it follows that $C_A(S) = F1$. This shows that such S does not appear in cases (ii) nor (iii).

Now, if V is an irreducible (graded) right module for $S^{\mathrm{op}} \otimes_F \Delta$, let

$$K = \mathrm{End}_{S^{\mathrm{op}} \otimes_F \Delta}(V) \subseteq \mathrm{End}_\Delta(V) = A,$$

which is, by the graded Schur's Lemma (Theorem 1.2), a division superalgebra. Notice that $K = \mathrm{End}_{S^{\mathrm{op}}\otimes_F \Delta}(V) = \{\varphi \in \mathrm{End}_\Delta(V) \mid [\varphi, S] = 0\} = C_A(S)$. Then, by identifying $A^{\mathrm{op}} \otimes_F \Delta = \mathrm{End}_F(V)$, $S^{\mathrm{op}} \otimes_F \Delta$ corresponds to $\mathrm{End}_K(V)$. Since $S \neq A$, also $F \neq K$.

If $K_{\bar 0} \neq F$ and $\widetilde{K}$ is a minimal field with $F \subsetneqq \widetilde{K} \subseteq K_{\bar 0}$, then $S \subseteq C_A(K) \subseteq C_A(K_{\bar 0}) \subseteq C_A(\widetilde{K})$ so, by maximality and since A is central, $S = C_A(\widetilde{K})$. By the Double Centralizer Theorem, $\widetilde{K} = C_A(S) = K$.

On the other hand, if $K_{\bar 0} = F$, then $K = F1 + Fu$ with $u \in A_{\bar 1}$ such that $0 \neq u^2 = \alpha \in F$ and $S \subseteq C_A(K)$ so, by maximality, $S = C_A(K)$.

Conversely, with K either a minimal field extension of F contained in $A_{\bar 0}$ or $K = F1 + Fu$ as above, let $S = C_A(K)$, then $S^{\mathrm{op}} \otimes_F \Delta = C_{A^{\mathrm{op}}\otimes_F \Delta}(K \otimes 1) = \mathrm{End}_K(V)$. Hence V is an irreducible (graded) $S^{\mathrm{op}} \otimes_F \Delta$-module, the graded division algebra K being its centralizer ($K = \mathrm{End}_{S^{\mathrm{op}}\otimes_F \Delta}(V)$). If $S \subseteq T \subseteq A$ for some subalgebra T, then $F \subseteq \mathrm{End}_{T^{\mathrm{op}}\otimes_F \Delta}(V) \subseteq \mathrm{End}_{S^{\mathrm{op}}\otimes_F \Delta}(V) = K$, hence $C_A(T)$ is either F or K and, by density, $T^{\mathrm{op}} \otimes_F \Delta$ is either $\mathrm{End}_K(V) = S^{\mathrm{op}} \otimes_F \Delta$ or $\mathrm{End}_F(V) = A^{\mathrm{op}} \otimes_F \Delta$. Thus either $T = S$ or $T = A$, as required. □

Notice that Theorem 2.2 covers the ungraded case too, thus providing a new proof of [7, Theorem 1]. This also shows that the subalgebras in (i) or (ii) are maximal even as ungraded algebras. We will later use the fact that the subalgebras in (i) above are described as $S = eAe + eAf + fAf$, where e is a nontrivial *even* idempotent and $f = 1 - e$. Therefore there is a basis of V such that, when identifying A with $\mathrm{Mat}_n(\Delta)$, S is formed by the upper block triangular matrices:

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

One can argue that the arguments in the proof above are not "super" arguments. We could have proceeded as follows: V is a module for A^{sop}, where A^{sop} is the opposite superalgebra ($a_\alpha \cdot b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha$), by means of $v_\alpha \cdot a_\beta = (-1)^{\alpha\beta} a_\beta v_\alpha$, and therefore it is a right module for $A^{\mathrm{sop}} \hat{\otimes}_F \Delta$ (where the graded tensor product is used). The centralizer of $S^{\mathrm{sop}} \hat{\otimes}_F \Delta$ centralizes the action of Δ, so it is in $A = \mathrm{End}_\Delta(V)$ and thus, it is the supercentralizer of S. In this way one obtains (i), (ii) or a new (iii)': There exists $u \in A_{\bar 1}$ with $0 \neq u^2 \in F$ such that $S = \widehat{C}_A(u)$, the supercentralizer of u: an homogeneous element a_α is in $\widehat{C}_A(u)$ if $a_\alpha u = (-1)^\alpha u a$ (since u is odd). However, since A is even, there exists $0 \neq z \in Z(A_{\bar 0})$ such that $0 \neq z^2 \in F$ and $za = -az$ for any $a \in A_{\bar 1}$. Then $\widehat{C}_A(u) = C_A(zu)$ and (iii) is recovered.

Corollary 2.3. *Let A be a finite dimensional central simple associative superalgebra over F of even type, and let S be a proper subalgebra of A. Then either S is contained in a maximal subalgebra of type* (i) *in Theorem* 2.2*, or $C_A(S)$ is a division superalgebra strictly containing F.*

Proof. If S is not contained in a subalgebra of type (i), then V is irreducible as a module over $S^{\mathrm{op}} \otimes_F \Delta$, so $C = C_A(S) \cong \mathrm{End}_{S^{\mathrm{op}}\otimes_F \Delta}(V)$ is a division superalgebra by Theorem 1.2 and, by density, $S^{\mathrm{op}} \otimes_F \Delta = \mathrm{End}_C(V)$. Since $S \subsetneqq A$, it follows that $F \subsetneqq C$. □

Later on, the following extension of [7, Proposition 1] will be needed.

Proposition 2.4. *Let A be a finite dimensional central simple superalgebra over F of even type, let V be an irreducible left A-module and let $\Delta = \mathrm{End}_A(V)$. If U and W are different proper Δ-submodules of V, then the only maximal subalgebras of A over F which contain $S(U) \cap S(W)$ are $S(U)$, $S(W)$ and $S(U \cap W)$, $S(U + W)$ if they are maximal, that is, if $W \cap U \neq 0$, $W + U \neq V$.*

Moreover the expression $S(U) \cap S(W)$ is unique, that is, if $S(U) \cap S(W) = B \cap C$ with B, C maximal subalgebras of A over F, then $\{B, C\} = \{S(U), S(W)\}$.

Proof. Let $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, where all the V_i's are graded and $V_1 = U \cap W$, $U = V_1 \oplus V_2$, $W = V_1 \oplus V_3$ (V_1 and V_4 can be zero). We denote by $e_i \in \mathrm{End}_\Delta V = A$ the projection of V onto V_i associated to this decomposition. Then $e_i \in A_{\bar{0}}$ for any i and $1 = e_1 + e_2 + e_3 + e_4$. Consider the Peirce decomposition of A relative to these idempotents, $A = \bigoplus_{i,j=1}^4 A_{ij}$, where $A_{ij} = e_i A e_j$. One can check, for instance just looking at the expression:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

that $S(U) \cap S(W) = A_{11} \oplus A_{12} \oplus A_{13} \oplus A_{14} \oplus A_{22} \oplus A_{24} \oplus A_{33} \oplus A_{34} \oplus A_{44}$. So $C_A(S(U) \cap S(W)) \subseteq \bigcap_{i=1}^4 C_A(e_i) = A_{11} \oplus A_{22} \oplus A_{33} \oplus A_{44} \subseteq S(U) \cap S(W)$, and hence $C_A(S(U) \cap S(W)) \subseteq \bigoplus_{i=1}^4 Z(A_{ii}) = \bigoplus_{i=1}^4 Fe_i \subseteq A_{\bar{0}}$. For any subalgebra containing $S(U) \cap S(W)$, its centralizer is contained in $C_A(S(U) \cap S(W)) \subseteq \bigoplus_{i=1}^4 Fe_i \subseteq A_{\bar{0}}$, thus avoiding types (ii) and (iii) in Theorem 2.2. Now, if $S(U) \cap S(W) \subseteq S(X)$ for some proper Δ-submodule X, since $e_i A e_i \subseteq S(X)$ for any $i = 1, 2, 3, 4$, X is a sum of some of the V_i's. Since $Ae_4 \subseteq S(X)$, V_4 is not contained in X, and since $e_1 A \subseteq S(X)$, $V_1 \subseteq X$. Therefore X is either $V_1 = U \cap W$, $V_1 \oplus V_2 = U$, $V_1 \oplus V_3 = W$ or $V_1 \oplus V_2 \oplus V_3 = U + W$. The uniqueness of the expression $S(U) \cap S(W)$ now follows easily. □

Next we will describe the maximal subalgebras of the simple superalgebras of odd type.

Theorem 2.5. *Let A be a finite dimensional central simple associative superalgebra over F of odd type. Then $A = A_{\bar{0}} \oplus A_{\bar{0}} u$, with $u \in Z(A)_{\bar{1}}$ such that $0 \neq u^2 \in F$, and $A_{\bar{0}}$ is a central simple algebra. Let S be a subalgebra of A. Then S is a maximal subalgebra of A if and only if either*:

(i) *$S = S_{\bar{0}} \oplus S_{\bar{0}} u$ with $S_{\bar{0}}$ a maximal subalgebra of $A_{\bar{0}}$.*
(ii) *$S = A_{\bar{0}}$.*
(iii) *$A_{\bar{0}}$ is a graded algebra*: *$A_{\bar{0}} = C_{\bar{0}} \oplus C_{\bar{1}}$, and $S = C_{\bar{0}} \oplus C_{\bar{1}} u$.*

This conditions are mutually exclusive.

Proof. Let $\overline{Z} = Z(A) = F1 + Fu$ be the center of A. Since $S \subseteq \overline{Z}S \subseteq A$, it follows by maximality that either $\overline{Z}S = S$ or $\overline{Z}S = A$.

If $S = \overline{Z}S$, $u \in S$ because $1 \in S$. This implies that $S_{\bar{1}} = S_{\bar{0}}u$ and $S = S_{\bar{0}} \oplus S_{\bar{0}}u$. Since S is a maximal subalgebra of A, it follows that $S_{\bar{0}}$ is a maximal subalgebra of $A_{\bar{0}}$. The converse is clear.

If $\overline{Z}S = A$, $A_{\bar{0}} = S_{\bar{0}} + S_{\bar{1}}u$ and $A_{\bar{1}} = S_{\bar{1}} + S_{\bar{0}}u$. Since $S_{\bar{0}} \cap S_{\bar{1}}u$ is an ideal of $A_{\bar{0}}$, because $S_{\bar{0}}(S_{\bar{1}}u) \subseteq S_{\bar{1}}u$ and $(S_{\bar{1}}u)^2 \subseteq S_{\bar{0}}$, and $A_{\bar{0}}$ is simple, it follows that $A_{\bar{0}} = S_{\bar{0}} \oplus S_{\bar{1}}u$ is a graded algebra. If the grading is trivial, that is, $S_{\bar{1}}u = 0 = S_{\bar{1}}$, then $S = A_{\bar{0}}$ and S is a maximal subalgebra of A. Otherwise, $A_{\bar{0}} = C_{\bar{0}} \oplus C_{\bar{1}}$ with $C_{\bar{0}} = S_{\bar{0}}$, $C_{\bar{1}} = S_{\bar{1}}u$ and $S = S_{\bar{0}} \oplus S_{\bar{1}} = C_{\bar{0}} \oplus C_{\bar{1}}u$.

Conversely, if $A_{\bar{0}} = C_{\bar{0}} \oplus C_{\bar{1}}$ and $S = C_{\bar{0}} \oplus C_{\bar{1}}u$, then $A = S \oplus Su \cong S \otimes_F \overline{Z}$ as algebras. We notice that S is a central algebra because $Z(A) = F1 + Fu$ and $Z(S) \subsetneqq Z(A) = F \oplus Fu$. Now we claim that S is a simple algebra. If $\widetilde{F}$ is a splitting field of the polynomial $X^2 - \alpha \in F[X]$, where $\alpha = u^2$, it follows that $\varphi : S \otimes_F \widetilde{F} \to A_{\bar{0}} \otimes_F \widetilde{F}$ given by $\varphi((c_{\bar{0}} + c_{\bar{1}}u) \otimes 1) = c_{\bar{0}} \otimes 1 + c_{\bar{1}} \otimes \alpha^{1/2}$ is an isomorphism. Since $A_{\bar{0}}$ is central simple over F, so is S. Hence S is a maximal ungraded subalgebra of $S \otimes_F \overline{Z} \cong A$ and, therefore, S is a maximal subalgebra of A. $\square$

Later on, also the following extension of [7, Corollary 1] will be needed:

Corollary 2.6. *Let A be a finite dimensional central simple associative superalgebra over a field E and suppose that E/F is a finite field extension. Let S be a subalgebra of A over F. Then S is a maximal subalgebra of A over F if and only if either*:

(i) *$ES \subseteq S$ and S is a maximal subalgebra of A over E.*
(ii) *There exists a field K such that $F \subseteq K \subsetneqq E$ and the extension E/K contains no proper intermediate subfields, such that S is a central simple superalgebra over K and $E \otimes_K S \cong A$ $(\alpha \otimes s \mapsto \alpha s)$ as E-algebras.*

Proof. Let S be a maximal subalgebra of A over F, then $S \subseteq ES \subseteq A$ and, by maximality, either $S = ES$ or $ES = A$. By Lemma 2.1, $1 \in S$.

If $S = ES$ then S is an E-subalgebra and hence S is a maximal subalgebra of A over E.

If $A = ES$ then S is a finite dimensional prime superalgebra over F and so S is a simple superalgebra (one may argue as follows: let I by a minimal left (graded) ideal of S, by primeness I is a faithful and irreducible left module for S, so if $\Delta = \mathrm{End}_S(I)$, by graded density and finite-dimensionality, $S = \mathrm{End}_\Delta(I)$ is simple). Let $K = Z(S)_{\bar{0}}$, then since $ES = A$, $F \subseteq K \subsetneqq E = Z(A)_{\bar{0}}$ and $\varphi : E \otimes_K S \to A : \alpha \otimes s \mapsto \alpha s$, is onto. Therefore φ is an isomorphism because $E \otimes_K S$ is a simple superalgebra. Moreover if $K \subsetneqq K' \subseteq E$, $S = KS \subsetneqq K'S \subseteq A$ and by maximality $K'S = A$ and $K' = E$.

Conversely, if S is a maximal subalgebra of A over E and $S \subseteq T$ with T an F-subalgebra then $E \subseteq ES = S \subseteq T$ and T is E-subalgebra. Therefore $T = S$ or $T = A$. This implies that S is a maximal subalgebra of A over F.

If $F \subseteq K \subsetneqq E$, such that E/K contains no proper intermediate subfields, and S is a central simple superalgebra over K with $E \otimes_K S \cong A$, let T be a maximal subalgebra over F such that $S \subsetneqq T \subsetneqq A$. Then $A = ES \subseteq ET$, so that $E \otimes_{K'} T \cong A$, with $K' = Z(T)_{\bar{0}}$.

But K' verifies that $K \subseteq K' \subseteq E$. So either $K' = E$ or $K' = K$ because E/K contains no intermediate subfields. If $K' = E$, T is a subalgebra over E and $T = A$. If $K' = K$, then $E \otimes_K T \cong A \cong E \otimes_K S$ and $S = T$. Therefore S is a maximal subalgebra of A over F. $\square$

3. Maximal subalgebras of associative algebras with involution

Let A be an algebra over a field F, endowed with an involution $*$. A $*$-*ideal* (respectively $*$-*subalgebra*) of A, is an ideal I (respectively subalgebra S) of A which verifies $I^* = I$ ($S^* = S$). For example, the center of A, $Z(A)$, is a $*$-subalgebra of A. Then A is said to be $*$-*simple* if $A^2 \neq 0$ and 0 and A are the only $*$-ideals of A. Suppose now that A is a finite dimensional $*$-simple associative algebra over F. Then either $*|_{Z(A)} = \mathrm{Id}$, and $*$ is said to be an involution of the first kind, or $*|_{Z(A)} \neq \mathrm{Id}$, and then $*$ is said to be of the second kind. A is said to be central, as an algebra with involution, if $\{z \in Z(A)\colon z^* = z\} = F1$. If C is a maximal $*$-subalgebra of A over F, then $C \subseteq B$ with B a maximal subalgebra of A over F. Since C is a $*$-subalgebra, $C \subseteq B \cap B^*$, but $B \cap B^*$ is also a $*$-subalgebra, so the maximality of C as $*$-subalgebra implies $C = B \cap B^*$. Hence, to determine the maximal $*$-subalgebras of A we only need to determine the conditions for $B \cap B^*$ to be a maximal $*$-subalgebra, for B a maximal subalgebra of A over F.

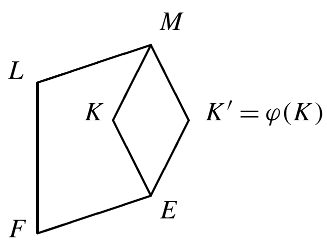
In [7, Theorem 4] the following result is set:

Theorem 3.1. *Let A be a finite dimensional central $*$-simple algebra over E and let B be a maximal subalgebra of A over E. Then $B \cap B^*$ is a maximal $*$-subalgebra of A if and only if either*:

(i) *B is of type* (i) *in Theorem* 1.6, *$B = eAe \oplus eAf \oplus fAf$ with $eV \cap f^*V = 0$, eV or f^*V, where V denotes an irreducible left A-module.*
(ii) *B is of type* (ii) *in Theorem* 1.6, *$B = C_A(K)$ with $K^* = K$.*

Next, a counterexample will be given to show that a case is missing in the above theorem. But first some preliminaries are needed.

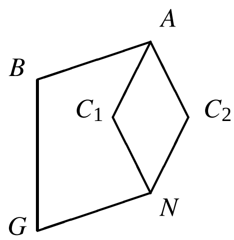
Lemma 3.2. *There exists a finite separable field extension M/F with an order* 2 *automorphism* $\mathrm{Id} \neq \varphi \in \mathrm{Aut}_F M$ ($\varphi^2 = \mathrm{Id}$), *such that the lattice of subfields is*



where $L = \{\alpha \in M\colon \varphi(\alpha) = \alpha\}$.

Proof. Notice that if there exists such extension, then $\varphi(E) = E$ and $\varphi|_E \neq \mathrm{Id}$ must be verified and, therefore, E/F and M/L are Galois field extensions of degree two.

It is well known that there are Galois field extensions with Galois group S_n, the symmetric group of degree n. So it is enough to find a symmetric group S_n with two subgroups A and G, $A \subseteq G$, such that the lattice of subgroups between A and G is as follows:



with N a normal subgroup of G, $[G : N] = 2$ (and therefore $A = N \cap B$ a normal subgroup of B with index 2), and such that B is the semidirect product of A and a cyclic group $\langle x\rangle$ of order two ($x^2 = 1$), and $C_2 = xC_1x^{-1} = xC_1x$. We notice that then $x \notin N$, because $B \nsubseteq N$, therefore G is the semidirect product of N and $\langle x\rangle$.

Actually, if the situation above exists for suitable subgroups, we can take

$$\varphi = x \quad \text{and} \quad F = \mathrm{Fix}(G),$$

the set of elements in a Galois field extension M/F', with Galois group S_n, which are fixed by every automorphism of G. Also we take $E = \mathrm{Fix}(N)$ (and then E/F is a Galois extension of degree two),

$$K = \mathrm{Fix}(C_1), \qquad K' = \mathrm{Fix}(C_2)$$

(notice then that for any α in the extension field, $\varphi(\alpha) \in \mathrm{Fix}(xC_1x^{-1})$ if and only if $\alpha \in \mathrm{Fix}(C_1)$, therefore $\varphi(K) = K'$),

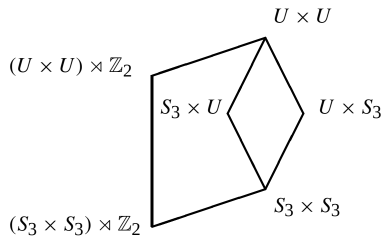
$$L = \mathrm{Fix}(B) \quad \text{and} \quad M = \mathrm{Fix}(A)$$

(and therefore $L = \{\alpha \in M\colon \varphi(\alpha) = \alpha\}$). Then these fields satisfy the requirements of the lemma.

Let G be the semidirect product of $S_3 \times S_3$ and the cyclic group $\langle x\rangle$ of order two where $(\sigma,\tau)x = x(\tau,\sigma)$ for every $\sigma,\tau \in S_3$. G is imbedded in S_6 identifying $S_3 \times 1$ with the subgroup of S_6 formed by the permutations of the set $\{1,2,3\}$, $1 \times S_3$ with the subgroup of S_6 of the permutations of the set $\{4,5,6\}$, and x with the permutation $(14)(25)(36)$ in S_6. Let $U = \langle(12)\rangle$, which is a maximal subgroup of S_3, and consider $A = U \times U$. We claim that if H is a group such that $U \times U \subsetneqq H \subsetneqq S_3 \times S_3$, then either $H = S_3 \times U$ or $H = U \times S_3$. Suppose there exists an element $(\sigma,\tau) \in H$ with $\sigma \notin U$. If $\tau \in U$, then $(\sigma,1) \in H$ and, since U is maximal subgroup of S_3, then $S_3 \times 1 \subseteq H$ and so $H = S_3 \times U$. If $\tau \notin U$, multiplying if necessary by $(1,(12))$, we can suppose that there exists $(\sigma,\tau) \in H$

such that $\sigma, \tau \notin U$ with σ and τ having different signature. So the order of σ is either 2 or 3 and the order of τ is either 3 or 2, respectively. Then $(\sigma, \tau)^2$ is either $(1, \mu)$ or $(\mu, 1)$ with $\mu \notin U$ and so either $1 \times S_3 \subseteq H$ or $S_3 \times 1 \subseteq H$, that is, either $H = S_3 \times U$ or $H = U \times S_3$.

Similar arguments show that the lattice of subgroups between $A = U \times U$ and $G = (S_3 \times S_3) \rtimes \mathbb{Z}_2$ is



as required. □

Lemma 3.3. *Under the conditions of the lemma above, $A = \mathrm{End}_E(M)$ has an involution of the second kind.*

Proof. We consider the nondegenerate E-bilinear form $\langle\, ,\rangle : M \times M \to E$ such that $\langle x, y\rangle = T_{M/E}(xy)$, where $T_{M/E}$ denotes the trace in the extension M/E, and we define $h : M \times M \to E$ by $h(x, y) = \langle \varphi(x), y\rangle$ for any $x, y \in M$. Then h is an F-bilinear map and an E-linear map in the second component. Moreover $h(y, x) = \langle \varphi(y), x\rangle = T_{M/E}(\varphi(y)x) = T_{M/E}(\varphi(\varphi(x)y)) = \varphi(h(x, y))$, therefore h is hermitian and h determines the involution of the second kind given by $h(ax, y) = h(x, a^*y)$ for every $a \in A$ and $x, y \in M$. □

The M above is realized as a subalgebra of A by means of $L : M \to A$ such that $L_x(y) = xy$ for any $x, y \in M$. For any $\tau \in M$, $h(\tau x, y) = T_{M/E}(\varphi(\tau x)y) = T_{M/E}(\varphi(x)\varphi(\tau)y) = h(x, \varphi(\tau)y)$, that is, $\tau^* = \varphi(\tau)$. In particular, $K' = \varphi(K) = K^*$.

Theorem 3.4. *Let M/F be a field extension satisfying the conditions in Lemma* 3.2*, and let $A = \mathrm{End}_E(M)$. Then $M = C_A(M)$ is a maximal $*$-subalgebra of A over E. Moreover if $B = C_A(K)$, then B is a maximal subalgebra of A over E, $B \cap B^* = C_A(M)$ and B is neither of type* (i) *nor type* (ii) *in Theorem* 3.1.

Proof. The field extension K/E has no proper intermediate subfields, so $B = C_A(K)$ is a maximal subalgebra of A over E and $B \cap B^* = C_A(K) \cap C_A(K^*) = C_A(M) = M$ because $\mathrm{alg}(K, K^*) = M$ and $\dim_E A = (\dim_E M)^2$. If S is a maximal subalgebra of A over E such that $M \subseteq S$, then since M (which is imbedded in A) stabilizes no subspace of M (see Theorem 1.6), it follows that S is of type (ii) in Theorem 1.6, that is, $S = C_A(D)$ with D/E a field extension without intermediate subfields. Hence $E \subsetneqq D \subseteq C_A(M) = M$ and D is either K or K^* because of Lemma 3.2. Therefore $C_A(K)$ and $C_A(K^*)$ are the unique maximal subalgebras of A over E containing M, and since $C_A(K)$ and $C_A(K^*)$ are not $*$-subalgebras, M is a maximal $*$-subalgebra of A. □

To give a correct version of Theorem 3.1, still some extra preliminary results are needed.

Lemma 3.5. *Let A be a finite dimensional central simple algebra over F, and let K be a field such that $F \subsetneqq K \subseteq A$. Then $C_A(K)$ is not contained in any maximal subalgebra of A of type* (i) *in Theorem* 1.6.

Proof. Without loss of generality, suppose that $A = \mathrm{End}_D(V)$, with V a right vector space over the division algebra D. Since $C_A(K)^{\mathrm{op}} \otimes_F D$ is a simple algebra, V is completely reducible as a module over it. If W is a nonzero $(C_A(K)^{\mathrm{op}} \otimes_F D)$-submodule of V, take a $(C_A(K)^{\mathrm{op}} \otimes_F D)$-submodule $\widetilde{W}$ of V such that $V = W \oplus \widetilde{W}$. Let $e \in \mathrm{End}_F(V)$ the projection of V onto W. Then e is a nonzero idempotent and $e \in \mathrm{End}_{C_A(K)^{\mathrm{op}} \otimes_F D}(V) \subseteq \mathrm{End}_D(V) = A$. Therefore $e \in C_A(C_A(K)) = K$ and so $e = 1$, that is, $W = V$ and there are no D-vector subspaces of V stabilized by $C_A(K)$. □

Corollary 3.6. *Let A be a finite dimensional central simple algebra over a field E, endowed with an involution $*$, and let M/E be a field extension such that $E \subseteq M \subseteq A$, $M^* = M$ and such that there is no $*$-stable intermediate subfields between E and M. Then $C_A(M)$ is a maximal $*$-subalgebra of A.*

Proof. By Lemma 3.5, if S is a maximal subalgebra of A containing $C_A(M)$, then $S = C_A(K)$, with K/E a minimal field extension such that $K \subseteq M$. It has to be proved that $C_A(M) = S \cap S^*$.

If $K = M$ then $S = C_A(M)$ is maximal in A and $S^* = S$, therefore $C_A(M)$ is a maximal $*$-subalgebra of A. If $K \neq M$, then K and K^* are subfields of M, with $K = E(c)$ and $K^* = E(c^*) \neq K$. Therefore $c + c^* \notin E$ and $M = E(c, c^*) = E(c + c^*)$, because M has no subfields which are stable under $*$. In particular, $S \cap S^* = C_A(K) \cap C_A(K^*) = C_A(c, c^*) = C_A(M)$. □

Proposition 3.7. *Let M/E be a finite field extension, and let $\varphi \in \mathrm{Aut}(M)$ such that $\varphi \neq \mathrm{Id}$, $\varphi^2 = \mathrm{Id}$, $\varphi(E) \subseteq E$, and M/E contains no proper φ-invariant intermediate subfields. Then either*:

(i) *M/E has no proper intermediate subfields, or*
(ii) *$\varphi|_E \neq \mathrm{Id}$ and, in this case, if $L = \{x \in M\colon \varphi(x) = x\}$ and $F = E \cap L$, it follows that M/F is a separable field extension, L/F has no intermediate subfields and if K is a minimal subfield such that $E \subsetneqq K \subsetneqq M$ then $\varphi(K) \neq K$ and $M = \mathrm{alg}(K, \varphi(K))$.*

Proof. Suppose that M/E has intermediate subfields and let K be a minimal one, that is, K is a field such that $E \subsetneqq K \subsetneqq M$. The hypotheses imply then that $\varphi(K) \neq K$, $K = E(c)$, $\varphi(K) = E(\varphi(c))$, with $c + \varphi(c) \notin E$ and $M = E(c, \varphi(c)) = E(c + \varphi(c))$.

If $\varphi|_E = \mathrm{Id}$, since $\varphi(c + \varphi(c)) = c + \varphi(c)$ it follows that $\varphi = \mathrm{Id}$, a contradiction. Therefore $\varphi|_E \neq \mathrm{Id}$. Let $L = \{x \in M \mid \varphi(x) = x\}$, $F = E \cap L$. If M' is a field such that $F \subsetneqq M' \subsetneqq L$, then $E \subsetneqq E(M') \subsetneqq M$ and $E(M')$ is φ-invariant, a contradiction. Therefore L/F has no intermediate subfields and, therefore, either L/F is a purely inseparable field extension of degree p with $p = \mathrm{char}\, F$, but then $[M : E] = p$, and $p = [M : E] = [M : K][K : E]$, a contradiction, or L/F is separable, that is, M/F is separable because M/L is a Galois extension. □

Lemma 3.2 shows that the situation in case (ii) of Proposition 3.7 actually occurs.
Finally, [7, Theorem 4] is completed to:

Theorem 3.8. *Let A be a finite dimensional central $*$-simple algebra over E and let B be a maximal subalgebra of A over E. Then $B \cap B^*$ is a maximal $*$-subalgebra of A if and only if either*:

(i) *B is of type* (i) *in Theorem* 1.6, $B = eAe \oplus eAf \oplus fAf$ *with* $eV \cap f^*V = 0$, eV *or* f^*V, *where V denotes an irreducible left A-module.*
(ii) *B is of type* (ii) *in Theorem* 1.6: $B = C(K)$, *with* $K^* = K$.
(iii) *$*$ is of the second kind, $B = C_A(K)$ with K/E a separable field extension without intermediate subfields, $K \neq K^*$ and* $\mathrm{alg}(K, K^*) = M$ *is a field such that M/E has no $*$-stable intermediate subfields. In this case $B \cap B^* = C_A(K) \cap C_A(K^*) = C_A(M)$.*

The three possibilities above are mutually exclusive.

Proof. Assume first that $B \cap B^*$ is a maximal $*$-subalgebra of A. If B is of type (i) in Theorem 1.6, the argument in [7, Theorem 4] gives (i). Otherwise $B = C_A(K)$, with K/E a field extension without intermediate subfields. If $K = K^*$ we are in case (ii) (so $B \cap B^* = B$ is maximal). If $K \neq K^*$ and $G = \mathrm{alg}(K, K^*)$, $B \cap B^* = C_A(G)$. Now, if $B \cap B^*$ were contained in $S(W)$ for some proper subspace W, then $B \cap B^* \subseteq S(W) \cap S(W)^*$ and, by maximality, $B \cap B^* = S(W) \cap S(W)^*$, a contradiction with Proposition 2.4. Thus $C_A(B \cap B^*)$ ($\supseteq G$) is a division algebra by Corollary 2.3, and so is G. As in the proof of [7, Theorem 4] we pick a minimal field L such that $F \subsetneqq L \subseteq F(c + c^*)$ where $F = \{x \in E\colon x^* = x\}$ and $K = E(c)$. Now $L = F(d)$ with $d^* = d$. Consider the field extension $E(d)$ of E, which is $*$-stable. If $K \nsubseteq E(d)$, by the Double Centralizer Theorem it follows that $C_A(E(d)) \nsubseteq C_A(K) = B$, and $B \cap B^* = C_A(K) \cap C_A(K^*) \subsetneqq C_A(E(d))$, because $E(d) \subseteq E(c+c^*) \subseteq \mathrm{alg}(K, K^*)$. But this is a contradiction with the maximality of $B \cap B^*$, because $C_A(E(d))$ is a $*$-subalgebra. Therefore $K \subseteq E(d)$ and so $K^* \subseteq E(d)^* = E(d)$. Hence $c + c^* \in E(d)$, that is, $c + c^* \in L = F(d) = \{x \in E(d)\colon x^* = x\}$ and $L = F(c + c^*)$, $E(d) = E(c + c^*) = E(c, c^*) = \mathrm{alg}(K, K^*)$ and $F \neq E$ (otherwise $L = E(d)$ would contain proper subfields: K and K^*). Finally the result follows by applying Corollary 3.6 and Proposition 3.7 with $M = E(d) = \mathrm{alg}(K, K^*)$. The uniqueness follows easily from Lemma 3.5 and the Double Centralizer Theorem. □

Theorem 3.4 gives examples of the situation in case (iii) above. In both types (ii) and (iii), $B = C_A(M)$ where $M = M^*$ is a field such that $E \subsetneqq M \subset A$ and M/E has no proper intermediate $*$-subfields.

4. Associative superalgebras with superinvolution

In this section, the maximal superalgebras in associative superalgebras with superinvolution will be studied. If A is an associative superalgebra over a field F, let us remind that a superinvolution $*$ over A is an even linear map (that is, a map that applies $A_{\bar 0}$ in $A_{\bar 0}$ and $A_{\bar 1}$

in $A_{\bar{1}}$), $*: A \to A$, such that for every $a, b \in A$, $a^{**} = a$ and $(ab)^* = (-1)^{\bar{a}\bar{b}} b^* a^*$. A superinvolution $*$ is said to be of the first kind if $*|_Z = \mathrm{Id}$ (recall that $Z = Z(A)_{\bar{0}}$), otherwise it is said to be of the second kind. If A is an associative superalgebra with superinvolution $*$, $(A, *)$ is said to be simple if A has no proper (graded) $*$-stable ideals and central if it is unital and $Z(A, *) (= \{z \in Z\colon z^* = z\}) = F1$.

From [9, Lemma 11] it is known that if A is an associative superalgebra with superinvolution $*$, $A_{\bar{0}}$ is an artinian algebra and $(A, *)$ is simple, then either:

(1) there exists an ideal B of A such that B is simple and $A = B \oplus B^*$. In this case $A_{\bar{0}} = B_{\bar{0}} \oplus B^*_{\bar{0}}$ is artinian and so is $B_{\bar{0}}$. Therefore B is artinian and simple and $(A, *) \cong (B \oplus B^{\mathrm{sop}}, \mathrm{exch})$ (where exch denotes the exchange superinvolution), or
(2) A is an artinian simple superalgebra and then, either $A \cong M_n(\Delta)$ with Δ a division superalgebra, or $A \cong M_{p,q}(D)$ with D a division algebra.

But something can be added. If $A \cong M_{p,q}(D)$, with D a division algebra, since $A_{\bar{0}} \cong M_p(D) \oplus M_q(D)$ and $1^* = 1$, then either $e_i^* = e_i$ for $i = 1, 2$ or $e_1^* = e_2$ and $e_2^* = e_1$, where e_1 and e_2 are the unital elements of the simple ideals of $A_{\bar{0}}$: $M_p(D)$ and $M_q(D)$.

So in case (2) above the following possibilities appear:

(i) $A \cong M_n(\Delta)$ with Δ a division superalgebra.
(ii) $A \cong M_{p,q}(D)$, $A_{\bar{0}} \cong M_p(D) \times M_q(D)$ and $M_p(D)$, $M_q(D)$ are $*$-stable simple ideals of $A_{\bar{0}}$. In this case $A_{\bar{1}} = e_1 A_{\bar{1}} e_2 \oplus e_2 A_{\bar{1}} e_1$, where e_1 and e_2 are the unital elements of $M_p(D)$ and $M_q(D)$ respectively, and since $e_i^* = e_i$, it follows that $*$ exchanges the two irreducible $A_{\bar{0}}$-subbimodules of $A_{\bar{1}}$.
(iii) $A \cong M_{p,q}(D)$, $A_{\bar{0}} \cong M_p(D) \times M_q(D)$, but $*$ exchanges the simple ideals of $A_{\bar{0}}$: $M_p(D)$ and $M_q(D)$. Then $M_p(D) \cong M_q(D)^{\mathrm{op}}$ and hence $p = q$ and $A \cong M_{p,p}(D)$. Here $*$ fixes the two irreducible $A_{\bar{0}}$-subbimodules of $A_{\bar{1}}$.

Lemma 4.1. *Let $(A, *)$ be a superalgebra with superinvolution over a field F. If C is a maximal $*$-subalgebra of A, then $C = B \cap B^*$ with B a maximal subalgebra of A.*

Proof. If C is a maximal $*$-subalgebra of A over F then $C \subseteq B$, with B a maximal F-subalgebra of A. But $C = C^* \subseteq B^*$, therefore $C \subseteq B \cap B^*$. Since $B \cap B^*$ is a $*$-subalgebra of A, the maximality of C implies $C = B \cap B^*$. □

Thus to determine the maximal $*$-subalgebras of a finite dimensional central superalgebra with superinvolution $(A, *)$ over F it suffices to determine the conditions for $B \cap B^*$ to be a maximal $*$-subalgebra of A, for a maximal F-subalgebra B of A.

The following theorem can be proved following verbatim the non graded case (see [7]). Thus the proof is omitted.

Theorem 4.2. *Let $(A, *)$ be a superalgebra with superinvolution over a field F. If $(A, *) \cong (B \oplus B^{\mathrm{sop}}, \mathrm{exch})$ with B a finite dimensional central simple superalgebra, then a subalgebra S of A is a maximal $*$-subalgebra of A if and only if either*:

(i) $S = C \oplus C^{\text{sop}}$, *for a maximal subalgebra* C *of* B.
(ii) $(B, -)$ *is a central simple superalgebra over* F *with superinvolution* $-$ *of the first kind and* $S = \{(b, \bar{b})_\alpha \in B \oplus B^{\text{sop}}\colon\ b_\alpha \in B_\alpha\}$.

Proposition 4.3. *Let* $(A, *)$ *be a finite dimensional central superalgebra with superinvolution of the second kind over a field* F. *Assume that* A *is simple, and let* $E = Z$ (*a quadratic Galois field extension of* F). *Let* S *be a* $*$*-subalgebra of* A *over* F. *Then* S *is a maximal* $*$*-subalgebra of* A *over* F *if and only if either*:

(i) S *is a maximal* $*$*-subalgebra of* A *over* E.
(ii) S *is a central simple superalgebra over* F, *and* $(E \otimes_F S, \sigma \otimes *|_S) \cong (A, *)$ $(\alpha \otimes s \mapsto \alpha s)$, *where* σ *is the Galois automorphism of the extension* E/F.

Proof. Suppose that S is a maximal $*$-subalgebra of A over F and consider ES. Since $S \subseteq ES \subseteq A$ and ES is $*$-stable, by maximality either $S = ES$ or $ES = A$. If $S = ES$ then S is an E-subalgebra and therefore S is a maximal subalgebra of A over E such that $S^* = S$. If $A = ES$, then as in the proof of Corollary 2.6, S is a prime finite dimensional superalgebra, and hence a simple superalgebra. Let $K = Z(S)_{\bar{0}}$. Since $A = ES$, $F \subseteq K \subsetneqq E$. Hence $K = F$ and $E \otimes_F S \cong A$.

The converse follows as in Corollary 2.6. □

Therefore, it is enough to deal with E-subalgebras and check under what conditions, given a maximal E-subalgebra B, $B \cap B^*$ is a maximal $*$-subalgebra. The last results deal with this problem for the different possibilities for B.

Theorem 4.4. *Let* A *be a finite dimensional central simple superalgebra of even type over the field* E *and let* V *be an irreducible* A*-module. Let* $*$ *be a superinvolution on* A *and let* B *be a maximal subalgebra of* A *over* E *of type* (i) *in Theorem* 2.2. *Then* $B \cap B^*$ *is a maximal* $*$*-subalgebra of* A *if and only if* $B = eAe \oplus eAf \oplus fAf = S(eV)$ *with* $eV \cap f^*V = 0$, eV *or* f^*V. *These conditions are equivalent to* $V = f^*V \oplus eV$, $e^*e = 0$, *or* $ff^* = 0$, *respectively.*

Proof. First notice that if $B = eAe \oplus eAf \oplus fAf = S(eV)$ (with e a nontrivial even idempotent, $f = 1 - e$ and V an irreducible module), then $B^* = e^*Ae^* \oplus f^*Ae^* \oplus f^*Af^* = S(f^*V)$. Suppose that $B \cap B^*$ is a maximal $*$-subalgebra of A. Let $\Delta = \operatorname{End}_A(V)$. If $eV \cap f^*V \neq 0, eV, f^*V$, then $eV \cap f^*V$ is a $((B \cap B^*)^{\text{op}} \otimes_E \Delta)$-submodule of V. Hence $B \cap B^* \subseteq S(eV \cap f^*V)$, with $S(eV \cap f^*V)$ a maximal subalgebra of A of type (i) in Theorem 2.2. Since $B \cap B^*$ is maximal, $B \cap B^* = S(eV \cap f^*V) \cap S(eV \cap f^*V)^*$. But this is a contradiction with the last statement of Proposition 2.4.

Conversely, if $eV \cap f^*V = 0$, eV or f^*V, then from Proposition 2.4 the only maximal subalgebras of A containing $B \cap B^* = S(eV) \cap S(f^*V)$ are $S(eV)$ and $S(f^*V)$, because the condition $eV \cap f^*V = 0$ implies, by dimension count, that $eV \oplus f^*V = V$, since f^*V and fV have the same dimension (for instance, as modules for the division algebra $\Delta_{\bar{0}}$). Therefore $B \cap B^*$ is a maximal $*$-subalgebra of A.

The last assertion follows as in [7, Theorem 4]. □

Therefore, with B as is the previous theorem, there is an idempotent $0 \neq e \in A_{\bar{0}}$ such that $B \cap B^* = S(eV) \cap S((1-e^*)V)$ with either $eV \cap (1-e^*)V = 0$ or (changing e by f^* if necessary) $e^*e = 0$. As in [7], $B \cap B^*$ can be better described by an adequate election of this idempotent e involved. In [9, Theorem 7] it is proved that if $A = \mathrm{End}_\Delta(V)$ for a division superalgebra Δ and a right graded Δ-module V, any superinvolution in A is induced by a nondegenerate hermitian or skew-hermitian superform $h_\nu : V \times V \to \Delta$; that is, h_ν is a biadditive map satisfying:

$$h_\nu(v_\alpha, w_\beta) \in \Delta_{\alpha+\beta+\nu},$$
$$h_\nu(v_\alpha d_\delta, w_\beta) = (-1)^{(\alpha+\nu)\delta}\bar{d}_\delta h_\nu(v_\alpha, w_\beta),$$
$$h_\nu(v_\alpha, w_\beta d_\delta) = h_\nu(v_\alpha, w_\beta)d_\delta,$$
$$h_\nu(v_\alpha, w_\beta) = (-1)^{\alpha\beta}\varepsilon\,\overline{h_\nu(w_\beta, v_\alpha)},$$

for any $v_\alpha \in V_\alpha$, $w_b \in V_\beta$, $d_\delta \in \Delta_\delta$, where $\varepsilon = 1$ if h_ν is hermitian and $\varepsilon = -1$ if h_ν is skew-hermitian, and where $\bar{}$ is a superinvolution of Δ. This superform h_ν is said to be *tracic* if for any $\alpha = 0, 1$ and any $v_\alpha \in V_\alpha$, $h_\nu(v_\alpha, v_\alpha) = c + (-1)^\alpha \varepsilon \bar{c}$ with $c \in \Delta_\nu$. Notice that if the characteristic is $\neq 2$ and $h_\nu(v_\alpha, v_\alpha) = d$, then $d = (-1)^\alpha \varepsilon \bar{d}$, so $h_\nu(v_\alpha, v_\alpha) = d/2 + (-1)^\alpha \varepsilon \bar{d}/2$, thus any superform is then tracic.

Lemma 4.5. *Let V be a finite dimensional right module over a division superring Δ and let $h_\nu : V \times V \to \Delta$ be a nondegenerate hermitian or skew-hermitian tracic form such that $V = U \oplus W$ with U and W subspaces such that $h_\nu(U, U) = 0$ and $\dim U = \dim W$. Then there is a subspace $\widetilde{W}$ of V such that $V = U \oplus \widetilde{W}$ and $h_\nu(\widetilde{W}, \widetilde{W}) = 0$.*

Proof. It is enough to give an homogeneous basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V (as a Δ-module) such that $\{x_1, \ldots, x_n\}$ is a basis of U, $h_\nu(x_i, y_j) = \delta_{ij}$ and $h_\nu(x_i, x_j) = h_\nu(y_i, y_j) = 0$ for any i, j. If $\Delta_{\bar{1}} \neq 0$ and $\nu = 1$, take $0 \neq \mu \in \Delta_{\bar{1}}$ with $\bar{\mu} = \pm\mu$ (this is always possible) and define $\tilde{h}$ by $\tilde{h}(x, y) = \mu h_\nu(x, y)$. Then, if $d_\delta^\sigma = (-1)^\delta \mu \bar{d}_\delta \mu^{-1}$ for any $d_\delta \in \Delta$, $\delta = \bar{0}, \bar{1}$, σ is a new superinvolution of Δ and $\tilde{h}$ satisfies $\tilde{h}(v_\alpha, w_\beta d_\delta) = \tilde{h}(v_\alpha, w_\beta)d_\delta$ and, if $\bar{\mu} = \varepsilon'\mu$ with $\varepsilon' = \pm 1$:

$$\begin{aligned}
\tilde{h}(v_\alpha, w_\beta) = \mu h_\nu(v_\alpha, w_\beta) &= (-1)^{\alpha\beta}\mu\varepsilon\overline{h_\nu(w_\beta, v_\alpha)} \\
&= (-1)^{\alpha\beta}(-1)^{\nu+\beta+\alpha}\varepsilon h_\nu(w_\beta, v_\alpha)^\sigma \mu \\
&= (-1)^{\alpha\beta}(-1)^{1+\beta+\alpha}\big(-\varepsilon\varepsilon'\big)h_\nu(w_\beta, v_\alpha)^\sigma \mu^\sigma \\
&= (-1)^{\alpha\beta}\big(-\varepsilon\varepsilon'\big)\big(\mu h_\nu(w_\beta, v_\alpha)\big)^\sigma
\end{aligned}$$

so $\tilde{h}$ is $-\varepsilon\varepsilon'$-hermitian. Besides, if $h_\nu(v_\alpha, v_\alpha) = c + (-1)^\alpha \varepsilon \bar{c}$, for $c \in \Delta_{\bar{1}}$, then $\tilde{h}(v_\alpha, v_\alpha) = \mu(c + (-1)^\alpha \varepsilon \bar{c}) = \mu c + (-1)^\alpha \varepsilon \mu \bar{c} \mu^{-1} \mu = \mu c - (-1)^\alpha \varepsilon c^\sigma \mu = \mu c + (-1)^\alpha \varepsilon\varepsilon' c^\sigma \mu^\sigma = \mu c + (-1)^\alpha(-\varepsilon\varepsilon')(\mu c)^\sigma$, so $\tilde{h}$ is tracic too.

Hence, if $\Delta_{\bar 1} \neq 0$, we may assume $\nu = 0$; but then $h = h_\nu|_{V_{\bar 0}\times V_{\bar 0}}$ is ε-hermitian and by the proof of [7, Lemma 3], there is a $\Delta_{\bar 0}$-basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ of $V_{\bar 0}$ (and hence a Δ-basis of V) with the required properties.

Now assume $\Delta_{\bar 1} = 0$. If $\nu = 0$, $h_\nu|V_{\bar 0} \times V_{\bar 0}$ is ε-hermitian and $h_\nu|V_{\bar 1} \times V_{\bar 1}$ is $(-\varepsilon)$-hermitian, so the arguments in [7, Lemma 3] apply to both situations and one obtains the required basis by joining the obtained bases in $V_{\bar 0}$ and $V_{\bar 1}$.

Finally, assume $\Delta_{\bar 1} = 0$ and $\nu = 1$. Then $h_{\bar 1}(V_{\bar 0}, V_{\bar 0}) = 0 = h_{\bar 1}(V_{\bar 1}, V_{\bar 1})$ (so any such $h_{\bar 1}$ is trivially tracic). Take $\{x_1, \dots, x_r\}$ to be any Δ-basis of $U_{\bar 0}$ and, since $h_{\bar 1} : U_{\bar 0} \times W_{\bar 1} \to \Delta$ is a nondegenerate sesquilinear form, a basis $\{y_1, \dots, y_r\}$ can be chosen in $W_{\bar 1}$ such that $h_{\bar 1}(x_i, y_j) = \delta_{ij}$ for any i, j (notice that $h_{\bar 1}(x_i, x_j) = 0 = h_{\bar 1}(y_i, y_j)$ for any i, j since $h_{\bar 1}(V_{\bar 0}, V_{\bar 0}) = 0 = h_{\bar 1}(V_{\bar 1}, V_{\bar 1})$). In the same way, take bases $\{x_{r+1}, \dots, x_n\}$ of $U_{\bar 1}$ and $\{z_{r+1}, \dots, z_n\}$ of $W_{\bar 0}$ with $h_{\bar 1}(x_i, z_j) = \delta_{ij}$ for any $i, j = r+1, \dots, n$. Finally, for any $i = 1, \dots, n-r$ consider the element $y_{r+i} = z_{r+i} - \varepsilon \sum_{l=1}^{r} x_l h_{\bar 1}(y_l, z_{r+i})$, which satisfies also that $h_{\bar 1}(x_{r+j}, y_{r+i}) = \delta_{ij}$ for any $j = 1, \dots, r$. But now, for any $j = 1, \dots, r$ and $i = 1, \dots, n-r$,

$$\begin{aligned} h_{\bar 1}(y_j, y_{r+i}) &= h_{\bar 1}(y_j, z_{r+i}) - \varepsilon \sum_{l=1}^{r} h_{\bar 1}(y_j, x_l) h_{\bar 1}(y_l, z_{r+i}) \\ &= h_{\bar 1}(y_j, z_{r+i}) - \varepsilon h_{\bar 1}(y_j, x_j) h_{\bar 1}(y_j, z_{r+i}) \\ &= h_{\bar 1}(y_j, z_{r+i}) - \varepsilon^2 h_{\bar 1}(y_j, z_{r+i}) = 0. \qquad \square \end{aligned}$$

Proposition 4.6. *Let A be a finite dimensional central simple superalgebra of even type over the field E with a superinvolution $*$, and let B be a maximal subalgebra of A over E of type* (i) *in Theorem* 2.2 *such that $B \cap B^*$ is a maximal $*$-subalgebra of A. Then*:

(i) *If there is an idempotent $0 \neq e \in A_{\bar 0}$ such that $B \cap B^* = S(eV) \cap S((1-e^*)V)$ and $eV \cap (1-e^*)V = 0$, then e can be chosen such that $e = e^*$. In this case $B \cap B^* = eAe \oplus (1-e)A(1-e)$.*

(ii) *If there is an idempotent $0 \neq e \in A_{\bar 0}$ such that $B \cap B^* = S(eV) \cap S((1-e^*)V)$ and $e^*e = 0$ and if the involution $*$ on A is induced by a nondegenerate tracic hermitian or skew-hermitian superform on V, then e can be chosen such that e, e^* and $(1-e)(1-e^*)$ are orthogonal idempotents with $1 = e + e^* + (1-e)(1-e^*)$ ($(1-e)(1-e^*)$ may be* 0). *In this case*

$$\begin{aligned} B \cap B^* &= \left(eA + Ae^*\right) \oplus (1-e)\left(1-e^*\right)A(1-e)\left(1-e^*\right) \\ &= eAe \oplus e^*Ae^* \oplus (1-e)\left(1-e^*\right)A(1-e)\left(1-e^*\right) \oplus eAe^* \\ &\quad \oplus eA(1-e)\left(1-e^*\right) \oplus (1-e)\left(1-e^*\right)Ae^*. \end{aligned}$$

Proof. In case (i), as in [7, Lemma 2], $V = eV \oplus (1-e^*)V$ and if h_ν is a nondegenerate hermitian or skew-hermitian superform on V inducing $*$, $h_\nu(eV, (1-e^*)V) = h_\nu(V, e^*(1-e^*)V) = 0$, so eV and $(1-e^*)V$ are orthogonal subspaces. Take the projection $g \in A_{\bar 0}$ of V onto eV parallel to $(1-e^*)V$. By orthogonality, $g = g^2 = g^*$, $gV = eV$ and g is the idempotent looked for.

In case (ii), let $f = 1 - e$, then $e^* f = e^*(e + f) = e^* 1 = e^*$, so $f^* e = e$. Hence

$$ff^* f = ff^*(1-e) = ff^* - ff^* e = ff^* - fe = ff^*,$$
$$\big(ff^*\big)^2 = ff^* ff^* = ff^* f^* = ff^*.$$

Also $e(ff^*) = 0$, $(ff^*)e = fe = 0$, and $e^*(ff^*) = e^* f^* = 0 = (ff^*)e^*$, so ff^* is an idempotent orthogonal to both e and e^*. Besides, since $eV \subseteq f^* V$, one has $ef^* V \subseteq eV = eeV \subseteq ef^* V$, so $ef^* V = eV$. Moreover, $1 - e^* - ff^* = f^* - ff^* = ef^*$, so $V = e^* V \oplus (ff^*)V \oplus (1 - e^* - ff^*)V = (eV \oplus e^* V) \oplus ff^* V$. Let h_ν be an hermitian or skew-hermitian nondegenerate superform on V inducing $*$. Since $e^* ff^* = 0 = eff^*$, $eV \oplus e^* V$ is orthogonal to $ff^* V$, and since $e^* e = 0$, $h_\nu(eV, eV) = 0$. By the previous lemma, there is another isotropic subspace U of $eV \oplus e^* V$ such that $eV \oplus e^* V = eV \oplus U$, so $V = eV \oplus U \oplus ff^* V$. Let g be the projection onto eV parallel to $U \oplus ff^* V$. Then $g = g^2 \in A_{\bar 0}$ and $gV = eV$. Since $h_\nu(V, g^* gV) = h_\nu(gV, gV) = h_\nu(eV, eV) = 0$, $g^* g = 0$. Let a be the orthogonal projection onto $eV \oplus e^* V = gV \oplus U$ parallel to $ff^* V$. By orthogonality $a = a^2 = a^*$. Also, $h_\nu((a-g)V, (a-g)V) = h_\nu(U, U) = 0$, so $0 = (a - g^*)(a - g) = a - g - g^*$ and $g^* = a - g$ is the projection onto U parallel to $eV \oplus ff^* V$. Finally, $gg^* = g(a - g) = 0$ and $(1-g)(1-g^*) = 1 - g - g^* = 1 - a$ is the projection onto $ff^* V$ parallel to $eV \oplus e^* V$. Thus, g, g^* and $(1-g)(1-g^*)$ are mutually orthogonal idempotents whose sum is 1 and g is the idempotent looked for. □

Now, let us consider the maximal subalgebras of types (ii) and (iii) in Theorem 2.2.

Theorem 4.7. *Let A be a finite dimensional central simple even superalgebra over E with a superinvolution $*$. Let $B = C_A(K)$ be a maximal superalgebra of A over E of type* (ii) *in Theorem* 2.2*. Then $B \cap B^*$ is a maximal $*$-subalgebra of A if and only if $M^* = \mathrm{alg}(K, K^*)$ is a field such that M/E has no $*$-stable intermediate subfields.* (*Notice that $B \cap B^* = C_A(K) \cap C_A(K^*) = C_A(M)$.*)

Proof. If B is of type (ii) in Theorem 2.2 and $B = C_A(K)$ then $B^* = C_A(K^*)$.

If $K^* = K$ then $B \cap B^* = B$ is a maximal subalgebra of A and hence a maximal $*$-subalgebra of A.

If $K^* \neq K$, we follow the steps in the proof of Theorem 3.8: by Corollary 2.3, $B \cap B^*$ is contained in a maximal subalgebra of type (i) in Theorem 2.2 unless $C_A(B \cap B^*)$ is a division superalgebra. But if $B \cap B^* \subseteq S(W)$ for some proper subspace W, then $B \cap B^* \subseteq S(W) \cap S(W)^*$. Since $B \cap B^*$ is a maximal $*$-subalgebra, it follows that $B \cap B^* = S(W) \cap S(W)^*$. But $S(W)^*$ is again a maximal subalgebra of A of type (i) in Theorem 2.2 (see the proof of Theorem 4.4), and this gives a contradiction with Proposition 2.4. Hence $C_A(B \cap B^*) = G$, a division superalgebra. Now write $K = E(c)$, then $K^* = E(c^*)$. Since $K \neq K^*$, $c + c^* \notin E$. Therefore $*$ acts as the identity on the field $F(c + c^*) \subseteq G$ with $F = \{x \in E \mid x^* = x\}$. As in the proof of Theorem 3.8, pick a minimal field L such that $F \subsetneq L \subseteq F(c + c^*)$ and deduce that $M = \mathrm{alg}(K, K^*) = E(c + c^*)$ satisfies that M/E has no proper intermediate $*$-stable subfields (thus being in the situation of Lemma 3.2).

The converse is a consequence of the "super" versions of Lemma 3.5 and Corollary 3.6 (with A an even finite dimensional central simple superalgebra, $*$ a superinvolution and K or M contained in $A_{\bar 0}$). $\square$

Theorem 4.8. *Let A be a finite dimensional central simple even superalgebra over E. Let B be a maximal subalgebra of type* (iii) *in Theorem* 2.2*, so $B = C_A(u)$ with $u \in A_{\bar 1}$, $0 \neq u^2 \in E$. Then $B \cap B^*$ is a maximal $*$-subalgebra of A if and only if $u^* \in Eu$.*

Proof. Suppose $B \cap B^*$ is a maximal $*$-subalgebra and $u^* \notin Eu$. Therefore $uu^* \notin E$ and $B \cap B^* = C_A(u, u^*) \subseteq C_A(uu^*)$.

As in the previous proof, $B \cap B^*$ is not contained in a maximal subalgebra of type (i) in Theorem 2.2 and, therefore, $K = C_A(B \cap B^*)$ is a division superalgebra.

Hence $E(uu^*) \subseteq C_A(B \cap B^*) = K$ and $E(uu^*)$ is a $*$-stable field extension of E. Let L be a minimal $*$-stable subfield such that $E \subsetneqq L \subseteq E(uu^*)$. Then $B \cap B^* = C_A(u, u^*) \subseteq C_A(uu^*) \subseteq C_A(L)$. By maximality, $B \cap B^* = C_A(L) \subseteq C_A(u)$. Then $E1 + Eu = C_A(C_A(u)) \subseteq C_A(C_A(L)) = L$ by the Double Centralizer Theorem, and thus $u \in L \subseteq A_{\bar 0}$, a contradiction. Hence $u^* \in Eu$. Besides, if $u^* = \lambda u,\ \lambda \in E$ then $\lambda\lambda^* = 1$ ($u = (u^*)^* = \lambda u^* = \lambda\lambda^* u$).

Conversely, if $u^* \in Eu$ then $B \cap B^* = B$ is a maximal subalgebra of A over E and hence it is a maximal $*$-subalgebra of A. $\square$

The final case to be considered is the case of the central simple odd superalgebras.

Theorem 4.9. *Let A be a finite dimensional central simple odd superalgebra over E. Suppose that $A = A_{\bar 0} \oplus A_{\bar 0}u$, with $u \in Z(A)_{\bar 1}$ such that $0 \neq u^2 \in E$. Let B be a maximal subalgebra of A over E. Then $B \cap B^*$ is a maximal $*$-subalgebra of A if and only if either*:

(i) *$B = B_{\bar 0} \oplus B_{\bar 0}u$, with $B_{\bar 0}$ a maximal subalgebra of $A_{\bar 0}$ and $B_{\bar 0} \cap B_{\bar 0}^*$ a maximal $*$-subalgebra of $A_{\bar 0}$.*
(ii) *$B = A_{\bar 0}$.*
(iii) *$A_{\bar 0}$ is a $\mathbb{Z}_2$-graded algebra*: *$A_{\bar 0} = C_{\bar 0} \oplus C_{\bar 1}$, with $C_{\bar 0}^* = C_{\bar 0}$ and $C_{\bar 1}^* = C_{\bar 1}$, and $B = C_{\bar 0} \oplus C_{\bar 1}u$.*

Proof. We recall that $u^* \in Z(A)_{\bar 1}$ and hence $u^* = \lambda u$ with $\lambda \in E$.

If B is of type (i) in Theorem 2.3, $B \cap B^* = (B_{\bar 0} \cap B_{\bar 0}^*) \oplus (B_{\bar 0} \cap B_{\bar 0}^*)u$ and, by maximality, $B_{\bar 0} \cap B_{\bar 0}^*$ is a maximal $*$-subalgebra of $A_{\bar 0}$, with $B_{\bar 0}$ a maximal subalgebra of $A_{\bar 0}$.

Conversely, if $B \cap B^* \subseteq T$, with T a subalgebra of A such that $T^* = T$, then $B_{\bar 0} \cap B_{\bar 0}^* \subseteq T_{\bar 0} = T_{\bar 0}^*$. Hence $B_{\bar 0} \cap B_{\bar 0}^* = T_{\bar 0}$. Since $u \in B \cap B^*$, it follows that $u \in T$ and $T_{\bar 1} = T_{\bar 0}u \subseteq B \cap B^*$. Therefore $B \cap B^*$ is a maximal $*$-subalgebra of A.

If B is of type (ii) in Theorem 2.3, that is, $B = A_{\bar 0}$, then $B \cap B^* = A_{\bar 0}$ is a maximal subalgebra of A and, therefore, a maximal $*$-subalgebra of A.

If B is of type (iii) in Theorem 2.3, then $A_{\bar 0}$ is a $\mathbb{Z}_2$-graded algebra: $A_{\bar 0} = C_{\bar 0} \oplus C_{\bar 1}$, and $B = C_{\bar 0} \oplus C_{\bar 1}u$. Notice that $A = B \oplus Bu$ and, since $u \notin B$, $B \cap B^* \subsetneqq Z(A)(B \cap B^*) = (B \cap B^*) + (B \cap B^*)u$. But $Z(A)(B \cap B^*)$ is $*$-stable, so by maximality $A = B \oplus Bu = (B \cap B^*) + (B \cap B^*)u$ and hence $B = B^*$, that is, $C_{\bar 0}^* = C_{\bar 0}$ and $C_{\bar 1}^* = C_{\bar 1}$.

The converse is clear: $B \cap B^* = B$ is a maximal subalgebra of A and thus it is a maximal subalgebra of $(A, *)$. $\square$

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