Proceedings of the Edinburgh Mathematical Society (1992) 35, 397–403 ©

THE FRATTINI SUBALGEBRA OF A BERNSTEIN ALGEBRA*

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(Received 8th May 1990)

Let A be a finite-dimensional Bernstein algebra over a field K with characteristic not 2. Maximal subalgebras of A are studied, and they are determined if A is a genetic algebra. It is also proved that the intersection of all maximal subalgebras of A (the Frattini subalgebra of A) is always an ideal. Finally the structure of Bernstein algebras with Frattini subalgebra equal to zero is described.

1980 *Mathematics subject classification* (1985 Revision). 17DXX.

Introduction

A finite-dimensional commutative algebra over a field K is called *baric* is there exists a non trivial algebra homomorphism $w: A \to K$.

A baric algebra is said to be a *Bernstein* algebra if

$$x^2.x^2-(w(x))^2.x^2=0 \quad \text{for every } x \text{ in } A. \tag{1}$$

Bernstein algebras have connections with genetics (see [**2**, **3**, and **8**]).

The homomorphism w is called the *weight homomorphism* of A. In [**8**] it is shown that in a Bernstein algebra this homomorphism is unique.

In a Bernstein algebra A there exists a nonzero idempotent e and A has a decomposition as a direct sum of vector subspaces (see [**8**]):

$$A=K.e\oplus U_e\oplus V_e,$$

with $U_e=\{x\in \operatorname{Ker} w/ex=(1/2)x\}$ and $V_e=\{x\in \operatorname{Ker} w/ex=0\}$. This decomposition is called the *Peirce decomposition* of A. If we express the relation "A is a vector subspace of B" by $A\leqq B$, the vector subspaces U_e, V_e have the following properties:

$$U_e^2\leqq V_e \quad U_eV_e\leqq U_e \quad V_e^2\leqq U_e \quad V_e^2U_e=0$$

and using (1) it is possible to prove that for all $u\in U_e$ and $v\in V_e$

$$u^3=0 \qquad u(uv)=0$$

*Partially supported by D.G.A. CB 1/88.

$$u^2(uv)=0 \qquad (uv)^2=0$$

$$u^2v^2=0. \tag{2}$$

In the above situation, the set of idempotents in A is $\{e+u+u^2/u\in U_e\}$. If $e'=e+u+u^2$ is another idempotent in A, we have $A=K.e'+U_{e'}+V_{e'}$ and then $U_{e'}=\{u_1+2uu_1/u_1\in U_e\}$ and $V_{e'}=\{-2(u+u^2)v_1+v_1/v_1\in V_e\}$.

A Bernstein algebra is called *genetic* if Ker w is nilpotent, that is, if there exists a nonzero positive integer n such that the principal product of every set of n elements from Ker w is zero.

Let A be an algebra and M a subalgebra of A. M is said to be a maximal subalgebra of A if for every subalgebra B of A, such that $M\leqq B\leqq A$, we have either $M=B$ or $B=A$. The intersection of all maximal subalgebras of A is known as the *Frattini subalgebra* $F(A)$ of A (see [**6**]). It has the following properties:

(P1) Let C be a subalgebra of A and B an ideal of A such that $B\leqq F(C)$. Then $B\leqq F(A)$.

(P2) (i) If B is an ideal of A we have $(F(A)+B)/B\leqq F(A/B)$.

(ii) Let B be in ideal of A such that $B\leqq F(A)$. Then $F(A)/B=F(A/B)$.

(P3) If B is an ideal of A such that $B^2=0$ and $B\cap\phi(A)=0$, with $\phi(A)$ the largest ideal of A contained in $F(A)$, then there exists a subalgebra C of A such that $A=B\oplus C$. That is, A is the direct sum of the vector subspaces B and C.

(P4) If A is a nilpotent finite-dimensional algebra, then $F(A)=A^2$.

In the following A will always be a Bernstein algebra with $1<\dim_K A<\infty$, over a field K, with characteristic not 2. The weight homomorphism of A will be denoted by w. If X is a subset of A, we denote by (X) the vector subspace of A generated by X and $\langle X\rangle$ the subalgebra of A generated by X. Sometimes if X has only one element, a, we also write $K.a$ instead of (a).

1. Maximal subalgebras of a Bernstein algebra

From Theorem 1 in [**1**] we can deduce that every maximal subalgebra of a n-dimensional Bernstein algebra has dimension equal to $n-1$. This result will be very important in the following discussion.

Lemma 1. *Let A be a Bernstein algebra, e a nonzero idempotent in A such that $A=\mathrm{Ke}\oplus U_e\oplus V_e$, and M a maximal subalgebra of A. Then*

(i) $U_e^2\leqq M$,

(ii) *if* $e\in M$, $V_e^2\leqq M$.

Proof. We suppose that U_e^2 is not contained in M. Thus, let $x\in U_e^2-M$. Since $\dim M+1=\dim A$ we have $M+K.x=A$ with $K.x\leqq V_e$. Therefore, $e=m+\tau x$ with $m\in M$ and $\tau\in K$. But since $A=\mathrm{Ke}\oplus U_e\oplus V_e$ it follows that $m=e-\tau x$, and hence

$m^2=e\in M$. Now if $u\in U_e$, we have as before $u=m'+\lambda x$ with $m'=u-\lambda x\in M$ and $1/2u\in M$. That is $u\in M$ for every $u\in U_e$. But this contradicts the fact that U_e^2 is not contained in M.

Now we suppose that V_e^2 is not contained in M. We consider $y\in V_e^2-M$ and thus $A=M\oplus K.y$ with $K.y\leqq U_e$. But if $e\in M$ we have $M=K.e\oplus U'_e\oplus V'_e$ with $U'_e\leqq U_e$ and $V'_e\leqq V_e$. Therefore $V_e\leqq M$. That is, V_e^2 is contained in M, which is a contradiction.

If A is a Bernstein algebra and B is a subalgebra of A such that B is not contained in Ker w, then B has a nonzero idempotent e, because B is also a Bernstein algebra.

Proposition 2. *Let A be a Bernstein algebra, and e a nonzero idempotent in A such that $A=\mathrm{Ke}\oplus U_e\oplus V_e$. Then a vector subspace of A, M, is a maximal subalgebra if and only if M is one of the following subalgebras:*

(i) $M=\mathrm{Ker}\ w$,

(ii) $M=K.e\oplus U_e\oplus V'_e$ *with* $V'_e\leqq V_e$ *such that* $\dim V_e+1=\dim V_e$ *and* $U_e^2\leqq V'_e$. *In this case M is an ideal.*

(iii) $M=K.e\oplus U'_e\oplus V_e$ *with* $U'_e\leqq U_e$, $\dim U'_e+1=\dim U_e$, $U'_eV_e+V_e^2\leqq U'_e$,

(iv) $M=(e+u)\oplus U'_e\oplus V_e$ *with* $U'_e\leqq U_e$, $\dim U'_e+1=\dim U_e$, $U'_eV_e+V_e^2\leqq U'_e$, $u\in U_e-U'_e$,

(v) $M=K.e_M\oplus U'_{e_M}\oplus V_{e_M}$ *with* $e_M=e+u+u^2$, $u\notin M$, V_e *not contained in* M *and* $U'_{e_M}\leqq U_{e_M}$ *such that* $\dim U'_{e_M}+1=\dim U_{e_M}$ *and* $U'_{e_M}V_{e_M}+U^2_{e_M}\leqq U'_{e_M}$.

Proof. We suppose $M\neq\mathrm{Ker}\ w$. Thus M contains an idempotent and $w|_M$ is a nonzero homomorphism from M onto K. Therefore $M/\mathrm{Ker}\ w|_M\cong K$ and $\dim \mathrm{Ker}\ w|_M=\dim M-1$. Let $B=\mathrm{Ker}\ w\cap M=\mathrm{Ker}\ w|_M$. We know that the set of idempotents in A is $\{e+u+u^2/u\in U_e\}$. Let $e+u+u^2\in M$ with $u\in U_e$. Since $U_e^2\leqq M$ because of Proposition 1, we have $e+u\in M$.

If $U_e\leqq M$, then $M=K.e\oplus U_e\oplus V'_e$ with $V'_e\leqq V_e$ such that $\dim V'_e+1=\dim V_e$. Then M contains every idempotent of A, and M is an ideal because $A.M=(K.e\oplus U_e\oplus V_e).(K.e\oplus U_e\oplus V'_e)=K.e\oplus U_e\oplus U_e^2\leqq M$. Thus we obtain (ii).

If U_e is not contained in M but $u\in M$, then $e\in M$ and $M=K.e\oplus B=K.e\oplus U'_e\oplus V'_e$ with $U'_e\leqq U_e$ and $V'_e\leqq V_e$. Since $\dim B+1=\dim \mathrm{Ker}\ w$, we have $M=K.e\oplus U'_e\oplus V_e$. Also $U'_eV_e+V_e^2\leqq U'_e$ because M is a subalgebra, and thus we obtain (iii).

If $V_e^2\leqq M$ and $u\notin M$ we will prove that $V_e\leqq M$. We have that $\mathrm{Ker}\ w=B\oplus K.u$. Let $v\in V_e$. Then $v=b+\lambda u$ with $b\in B$ and $\lambda\in K$. Thus $(e+u)(v-\lambda u)=uv-\lambda/2u-\lambda u^2\in B$ and $b^2=(v-\lambda u)^2=v^2+\lambda^2u^2-2\lambda uv\in B$. Hence, since U_e^2, $V_e^2\leqq M$ because of Lemma 1 and the hypothesis, it follows that $\lambda u\in B$ and therefore $v\in M$. Thus $M=(e+u)\oplus U'_e\oplus V_e$ with $U'_e\leqq U_e$ such that $\dim U'_e+1=\dim U_e$, $V_e^2\leqq U'_e$ and $0\neq u\in U_e-U'_e$. Since M is a subalgebra, it follows also that $U'_eV_e\leqq U'_e$ and we have (iv).

Now we suppose $u\notin M$ and V_e is not contained in M. Then if $e_M=e+u+u^2$, it follows that $M=K.e_M\oplus U'_{e_M}\oplus V'_{e_M}$ with either $U'_{e_M}=U_{e_M}$ or $V'_{e_M}=V_{e_M}$. But if $U'_{e_M}=U_{e_M}$ we have shown that M contains every idempotent of A, that is $u\in M$ that contradicts the hypothesis. Therefore $M=K.e_M\oplus U'_{e_M}\oplus V_{e_M}$ and as in (iii) it follows that $\dim U'_{e_M}+1=\dim U_{e_M}$ and $U'_{e_M}V_{e_M}+U^2_{e_M}\leqq U'_{e_M}$.

Lemma 3. *Let A be a genetic Bernstein algebra. Then* $(\mathrm{Ker}\ w)^2$ *is contained in every maximal subalgebra of M.*

Proof. Since A is genetic, $\mathrm{Ker}\ w$ is nilpotent and thus from (P4) we have $F(\mathrm{Ker}\ w)=(\mathrm{Ker}\ w)^2$. Using that $(\mathrm{Ker}\ w)^2=U_e^2+U_eV_e+V_e^2$ we have that $(\mathrm{Ker}\ w)^2$ is an ideal. Hence from (P1) we obtain $(\mathrm{Ker}\ w)^2\leqq F(A)$. That is, $(\mathrm{Ker}\ w)^2\leqq M$ for maximal subalgebra M of A.

The result of Lemma 3 is not true if A is only a Bernstein algebra. For instance the commutative algebra $A=(e,u,v,z)$ such that $eu=1/2u$, $uv=u$, $e^2=e$ and the other products equals to zero is a Bernstein algebra, but the maximal subalgebra (e,v,z) does not contain $(\mathrm{Ker}\ w)^2=(u,v,z)^2=(u)$.

However there are Bernstein algebras which are not genetic and for which $(\mathrm{Ker}\ w)^2\leqq M$, for every maximal subalgebra M of A. For example the commutative algebra $A=(e,u,v,z)$ with $e^2=e$, $eu=1/2u$, $uv=uz=vz=u$ and the other products zero is a Bernstein algebra such that $(\mathrm{Ker}\ w)^2=(u)$ is contained in every maximal subalgebra.

Theorem 4. *Let A be a genetic Bernstein algebra and e a nonzero idempotent in A such that $A=\mathrm{K}e\oplus U_e\oplus V_e$. Then a vector subspace M of A, is a maximal subalgebra if and only if M satisfies one of the following conditions:*

(i) $M=\mathrm{Ker}\ w$,

(ii) $M=K.e\oplus U_e\oplus V'_e$ *with* $V'_e\leqq V_e$ *such that* $\dim V'_e+1=\dim V_e$ *and* $U_e^2\leqq V'_e$,

(iii) $M=K.e\oplus U'_e\oplus V_e$ *with* $U'_e\leqq U_e$, $\dim U'_e+1=\dim U_e$, $U_eV_e+V_e^2\leqq U'_e$,

(iv) $M=(e+u)\oplus U'_e\oplus V_e$ *with* $U'_e\leqq U_e$, $\dim U'_e+1=\dim U_e$, $U_eV_e+V_e^2\leqq U'_e$, $u\in U_e-U'_e$.

Proof. From Lemma 3 $(\mathrm{Ker}\ w)^2=U_e^2+U_eV_e+V_e^2$ is contained in every maximal subalgebra, and from Proposition 2 and its proof we have that M is as in (i), (ii), (iii) or (iv).

Corollary 5. *If A is a genetic Bernstein algebra, then* $F(A)=(\mathrm{Ker}\ w)^2$.

2. The Frattini subalgebra

In this paragraph we study the intersection of all maximal subalgebras of a general Bernstein algebra, that is, its Frattini subalgebra. We also describe Bernstein algebras with Frattini subalgebra equal to zero, using the subalgebra spanned by the minimal ideals of the algebra.

Theorem 6. *Let A be a finite dimensional Bernstein algebra. Then $F(A)$ is an ideal.*

Proof. We suppose $F(A)$ is not an ideal. Then there exists $x\in F(A)$ and $y\in A$ such that $xy \notin F(A)$. That is, for some maximal subalgebra M of A, $xy \notin M$. Clearly $M \neq \text{Ker } w$ and therefore M contains a nontrivial idempotent e such that $A = Ke + U_e + V_e$, and $M = Ke + U'_e + V_e$ with $U'_e \leqq U_e$ such that $\dim U'_e + 1 = \dim U_e$ because of Proposition 2.

In [5] it is shown that $F(A) \leqq (\text{Ker } w)^2$ and then $x = u_1 + u_2 + v'$ with $u_1 \in U_e V_e$, $u_2 \in V_e^2$, $v' \in U_e^2$. Since U_e^2 and $V_e^2 \leqq M$ because of Lemma 1, it follows that u_1, u_2, $v' \in M$. On the other hand $y = \lambda e + u + v$ with $\lambda \in K$, $u \in U_e$, $v \in V_e$. That is, λe and $v \in M$.

Thus $xy \notin M$ implies $uv' \notin M$. But we can prove that if $v' \in U_e^2$ and $u \in U_e - M$, then $uv' \in M$. We suppose that $v' = u'u''$, with u', $u'' \in U_e$. Since $A = M + Ku$, then $u' = a + \delta u$ and $u'' = b + \omega u$ with $\delta, \omega \in K$ and $a, b, \in U'_e$. Therefore

$$uv' = u(u'u'') = u((a + \delta u)(b + \omega u)) = u(ab) + \omega u(au) + \delta u(ub) + \delta\omega u^3.$$

But linearizing the first identity in (2) we have

$$u(ab) = -a(ub) - b(ua) \in U'_e U_e^2 \leqq M$$

$$u(au) = -1/2 au^2 \in U'_e U_e^2 \leqq M$$

$$uu^2 = 0.$$

Therefore $uv' \in M$, which is a contradiction, and thus $xy \in F(A)$ and $F(A)$ is an ideal.

Proposition 7. *Let A be a Bernstein agelbra, and e a nonzero idempotent of A such that $A = Ke + U_e + V_e$. Let $N = U_e + U_e^2$. Then $N^2 \leqq F(A) \leqq (\text{Ker } w)^2$.*

Proof. In [5] we proved that $(\text{Ker } w)^2$ contains $F(A)$.

On the other hand from [4] it is known that a Bernstein algebra B with $B^2 = B$ is genetic. It is easy to check that $B = Ke + U_e + U_e^2$ satisfies this condition. Thus B is genetic and from Corollary 5 we have $N^2 = F(B)$. But N is an ideal of A and because of [7] (or checking it directly) N^2 is also an ideal of A. Now we apply (P1) and we have $N^2 \leqq F(A)$.

Remark 8. Since $F(A) \leqq (\text{Ker } w)^2 \leqq U_e + U_e^2$ and $U_e + U_e^2$ is nilpotent, because $B = Ke + U_e + U_e^2$ is a genetic algebra, we have that $F(A)$ is nilpotent. (The author is aware that this result has also been obtained by A. Koulibaly and M. Ouattara).

Now we can consider the algebra $A/F(A)$, which is also a Bernstein algebra. From (P2) this algebra is such that $F(A/F(A)) = F(A)/F(A) = 0$. In the following we study Bernstein algebras such that $F(A) = 0$. First we define two concepts: The *zero socle* of A, denoted $\text{Zsoc}(A)$, which is the sum of all minimal ideals with product zero and the *socle* of A, denoted by $\text{Soc}(A)$, which is the sum of all minimal ideals of A. It is clear that $\text{Zsoc}(A) \leqq \text{Soc}(A)$. In general for an arbitrary algebra $\text{Zsoc}(A) \neq \text{Soc}(A)$, but in nontrivial Bernstein algebras $\text{Soc}(A) = \text{Zsoc}(A)$.

Proposition 9. *Let A be a Bernstein algebra such that if e is nonzero idempotent $U_e \neq 0$. Then* Zsoc $(A) =$ Soc $(A) \leqq$ Ker w.

Proof. We are going to prove that if I is a minimal ideal of A then I has product zero and thus Zsoc $(A) =$ Soc (A). Let I be a minimal ideal of A. If I is not contained in Ker w, then there exists a nonzero idempotent, e, in I such that $A = Ke + U_e + V_e$ and $I = Ke + U'_e + V'_e$ with $U'_e \leqq U_e$ and $V'_e \leqq V_e$. But $U'_e + V'_e$ is an ideal of A and $U'_e + V'_e$ is contained in I and is different from I. Therefore $I \leqq$ Ker w for every minimal ideal I of A. From [7] we know that the product of ideals of A contained in Ker w is also an ideal of A. Thus $I^2 = I$ or $I^2 = 0$. If $I^2 = I$, then $Ke + I = C$ is a Bernstein algebra such that $C^2 = C$. That is, from [4], C is a genetic algebra and therefore I is nilpotent, which is a contradiction.

Theorem 10. *Let A be a Bernstein algebra such that $F(A) = 0$. Then if e is a nonzero idempotent of A such that $A = Ke + U_e + V_e$ we have:*

(i) $U_e^2 = 0$,

(ii) Zsoc $(A) = U'_e + V'_e$ *with* $V'_e \leqq V_e$ *and* $U'_e \leqq U_e$ *such that* $V'_e V_e = V'_e \dot{U_e} = 0$,

(iii) $A =$ Zsoc $(A) + C$ *with C a subalgebra of A,* $C = (e+u) + W$ *with* $W \leqq \{-2uv + v / v \in V_e\}$ *such that* $W^2 = 0$ *and* $u \in U_e$.

Moreover if A is a Bernstein algebra verifying (i), (ii) *and* (iii) *we have that* $F(A) = 0$.

Proof. Because of Proposition 7 we have (i).

For the proof of (ii), we consider a minimal ideal I cZsoc(A). Let e be a nonzero idempotent in A. Since $eI \leqq I$, we have that $I = \bar{U}_e + \bar{V}_e$ with $\bar{U}_e \leqq U_e$ and $\bar{V}_e \leqq V_e$. But from (i) $\bar{U}_e . U_e = 0$ and since I is an ideal, $\bar{U}_e . V_e \leqq U_e \cap I = \bar{U}_e$. Thus $\bar{U}_e$ is an ideal of A and it is contained in the minimal ideal I of A. Therefore we have $I = \bar{U}_e$ or $\bar{U}_e = 0$. If $\bar{U}_e = 0$, then $I = \bar{V}_e$. However $\bar{V}_e . V_e$ and $\bar{V}_e . U_e$ are contained in $\bar{U}_e$ and thus $\bar{V}_e . A = 0$. So Zsoc $(A) = U'_e + V'_e$ with $(U'_e)^2 = 0 = V'_e U_e = V_e . V'_e$. We remark that these conclusions follow for every nonzero idempotent using only the hypothesis $U_e^2 = 0$. Now from (P3) we have $A =$ Zsoc $(A) \oplus C$ with C a subalgebra of A. So C contains a nonzero idempotent $e_1 = e + u$ with $u \in U_e$ and $C = Ke_1 \oplus \tilde{U}_{e_1} \oplus W$ such that $\tilde{U}_{e_1} \leqq U_{e_1}$ and $W \leqq V_{e_1}$. We know that $U_{e_1} = \{u_1 + 2uu_1 / u_1 \in U_e\}$ and $V_{e_1} = \{-2(u+u^2)v_1 + v_1 / v_1 \in V_e\}$. Therefore $U_{e_1} = U_e$ and $V_{e_1} = \{-2uv_1 + v_1 / v_1 \in V_e\}$. Since $C \cap$ Zsoc $(A) = 0$, we have that C contains no minimal ideals of A. But $\tilde{U}_{e_1}$ is an ideal because $U_{e_1} = U_e$, $U_e^2 = 0$, C is a subalgebra and $V'_e . U_e = 0$. Therefore $\tilde{U}_e = 0$. Moreover if C is a subalgebra, $W^2 \leqq U_e \cap C = 0$.

Conversely let A be a Bernstein algebra satisfying (i), (ii) and (iii). We know that $F(A) \leqq (\text{Ker } w)^2 = U_e^2 + V_e^2 + U_e V_e$. Thus $F(A) \leqq U_e$ W because of (i), (ii) and (iii). To prove (ii), note that we have shown using only (i), that Zsoc $(A) = U'_e + V'_e$ with $U'_e = \sum_{i \in J} I_i$ and $V'_e = \sum_{b \in B} J_b$ where I_i, J_b are the minimal ideals of A satisfying $I_i \leqq U_e$, and $I_i^2 = 0$ for all $i \in J$, and $J_b \leqq V_e$, $J_b^2 = 0$ for all $b \in B$. That is, the sums $\sum_{i \in J} I_i$ and $\sum_{b \in B} J_b$ are direct sums of algebras. Now we consider

$$A_i=\left(\sum_{s\in J s\neq i} I_s+V'_e\right)\oplus C \quad\text{and}\quad D_b=\left(U'_e+\sum_{s\in B s\neq b} J_b\right)\oplus C.$$

A_i is ideal and $A/A_i\cong I_i$ is a nilpotent algebra. Moreover D_b is also an ideal and $A/D_b\cong J_b$ is a nilpotent algebra. From (P4) we have $F(A/A_i)=(A/A_i)^2\cong I_i^2=0$ and $F(A/D_b)=(A/D_b)^2\cong J_b^2=0$. From (P2) we know that $(F(A)+A_i)/A_i\leqq F(A/A_i)=0$ for all $i\in J$ and $(F(A)+D_b)/D_b\leqq F(A/D_b)=0$ for all b in J. So $F(A)\leqq(\bigcap A_i)\cap(\bigcap D_b)\leqq C$. But we also have $F(A)\leqq U_e.W\leqq U_e$. Therefore $F(A)=0$.

Corollary 11. *Let A be a Bernstein algebra such that $F(A)=0$. Then there exists a nonzero idempotent e such that $A=\text{Zsoc}\,A\oplus C$ with $\text{Zsoc}\,A=U'_e+V'_e$, $C=Ke+\tilde{V}_e$ and V'_e, $\tilde{V}_e\leqq V_e$ verifying $V'_e.\tilde{V}_e=0$.*

Proof. From Theorem 10 and its proof we know that $A=\text{Zsoc}\,A\oplus C$, where C contains a nonzero idempotent e, $C=Ke+\tilde{V}_e$ with $\tilde{V}_e\leqq V_e$ and $\text{Zsoc}\,A=U_e+V'_e$ with $U'_e\leqq U_e$ and $V'_e\leqq V_e$ such that $V'_eV_e=0$. Therefore $U'_e=U_e$ and we have the corollary.

Acknowledgement. The author would like to express his gratitude to Professor Santos González of the University of Zaragoza for the supervision of this paper and the information he gave to him of another recent result about this subject.

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