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Solving a special case of conservative problems by Secant-like methods

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Abstract

We study a class of Secant-like iterations for solving nonlinear equations in Banach spaces. A semilocal convergence result is obtained, where the first order divided difference of the nonlinear operator is Hölder continuous. For that, we use a technique based on a new system of recurrence relations to obtain existence-uniqueness domains of the solution and a priori error bounds. These results are applied to solve a special case of conservative problems.

Keywords: The Secant method; Convergence analysis; Recurrence relations; A priori error bounds; Boundary value problems

1. Introduction

Many scientific and engineering problems can be brought in the form of a nonlinear equation

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$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset $\mathscr{D}$ of a Banach space X with values in another Banach space Y. In general, if the operator F is nonlinear, iterative methods are used for solving (1). They are numerical methods which provide, from a starting approximation x_0, a sequence of values $\{x_n\}$ which approximate the solution of (1). There are numerous methods for solving (1), but Newton's method is the most famous as a consequence of its computational efficiency, even though sometimes less speed of convergence is reached.

The most inconvenient of Newton's iteration is the evaluation of the first derivative of the operator F at each steep. The Secant method, which uses divided differences, is usually applied when we want to solve the previous inconvenient. On the contrary, speed of convergence is reduced.

In this paper, from the geometrical interpretation of the both previous methods, a one-parametric class of iterative processes is considered. A feature of this class is that these methods do not use the first derivative of the operator as in the Secant method. However we can reach the speed of convergence of Newton's method by varying the values of the parameter.

In Section 2, the class of iterations is presented and, in Section 3, a semilocal convergence result is obtained when the operator F is Hölder (c,p) continuous, $c \geqslant 0$, $p \in [0,1]$.

In Section 4, we illustrate the previous result. An existence-uniqueness result of the solution for a special case of conservative problems is provided. Moreover the solution of a particular conservative problem is located in a convex domain. Then, using a discretization process, we approximate the solution of the corresponding system of equations by some methods of the class. Finally, by interpolation, the solution of the conservative problem is approximated.

2. Secant-like methods

We denote the set of linear and bounded operators from X to Y by $\mathscr{L}(X,Y)$. Then if there exists an operator $[x,y;F] \in \mathscr{L}(X,Y)$ such that the condition

$$[x,y;F](x-y) = F(x) - F(y) \tag{2}$$

is satisfied, this is called a divided difference of F at the points x and y (see [11]). Condition (2) does not determine uniquely the divided difference, with the exception of the case when X is one-dimensional. For the existence of divided differences in linear spaces (see [5]).

Well known methods for solving (1) are the Secant method ([2,13,14]):

$$x_{n+1} = x_n - [x_{n-1},x_n;F]^{-1}F(x_n); \quad x_0, x_{-1} \quad \text{pre-chosen},$$

and Newton-like methods ([3,4,10]):

$$x_{n+1} = x_n - L_n^{-1}F(x_n); \quad x_0 \quad \text{pre-chosen}, \tag{3}$$

where $\{L_n\}$ denotes a sequence of invertible linear operators. These methods are very used due to its high efficiency, since the speed of convergence is acceptable and the operational cost is reduced. The study of the convergence of methods (3) can be found in [6], where the basic assumption made is that F' is Lipschitz continuous in some ball around the initial iterate. Argyros [2] relaxes this requirement to operators that are only Hölder (c,p) continuous, $c \geqslant 0$, $0 \leqslant p \leqslant 1$. Moreover the Secant method is examined as a particular case of (3).

To improve the speed of convergence of the Secant method, we consider the following modification:

$$\begin{cases} x_{-1},\ x_0 \quad \text{pre-chosen}, \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad \lambda \in [0,1], \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1}F(x_n). \end{cases} \tag{4}$$

In the real case, it is clear that the more close x_n and y_n are, the higher the speed of convergence is. Moreover, observe that (4) is reduced to the Secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$, since $y_n = x_n$ and $[y_n, x_n; F] = F'(x_n)$ (see [11]).

The study of the convergence of the Secant method is usually made by means of majorizing sequences [2,6,10,11,13]. In this paper, we analyse the convergence of (4) by using a technique that consists of a new system of recurrence relations, in the way that Gutiérrez and Hernández analyse the convergence of the Chebyshev method in [7]. We then obtain an existence-uniqueness result of the solution of (1) and a priori error bounds. Note that the application of this result is much easier than Argyros' one (see [2]), since there are few hypotheses and they are simpler.

3. Convergence analysis whatever the operator

From now on we assume that F is once Fréchet-differentiable at every point $x \in X$ and note that $F'(x) \in \mathscr{L}(X, Y)$.

Definition 3.1. We say that the Fréchet-derivative F' is Hölder (c,p) continuous over the domain $\mathscr{D}$ if for some $c \geqslant 0$, $p \in [0,1]$,

$$\|F'(x) - F'(y)\| \leqslant c\ \|x - y\|^p; \quad x, y \in \mathscr{D}.$$

We then say that $F'(\cdot) \in H_{\mathscr{D}}(c,p)$ (see [2]).

Throughout the next section we suppose that there exists a first order divided difference of F at every point $x, y \in \mathscr{D}$ and a nonnegative constant k such that

$$\|[x,y;F] - F'(z)\| \leqslant k\ (\|x-z\|^p + \|y-z\|^p), \quad p \in [0,1], \tag{5}$$

for all $x, y, z \in \mathscr{D}$. Obviously, in this case, $F' \cdot \in H_{\mathscr{D}}(2k, p)$.

3.1. Recurrence relations

We establish the recurrence relations from which the convergence of (4) is proved later. Let $x_0, x_{-1} \in \mathscr{D}$ and assume

(I) $\|x_0 - x_{-1}\| = \alpha$;
(II) there exists $L_0^{-1} = [y_0, x_0; F]^{-1}$ such that $\|[y_0, x_0; F]^{-1}\| \leqslant \beta$;
(III) $\|L_0^{-1} F(x_0)\| \leqslant \eta$.

We denote

$$a_{-1} = \frac{\eta}{\alpha + \eta}, \quad b_{-1} = k\beta\alpha^p$$

and define the sequences

$$a_n = g(a_{n-1}) b_{n-1}, \quad b_n = f(a_n) f(a_{n-1})^p a_{n-1}^p b_{n-1}, \quad n \geqslant 0, \tag{6}$$

where

$$f(x) = \frac{1}{1-x}, \quad g(x) = (1-\lambda)^p + \frac{2}{p+1}(1+\lambda^p) f(x)^p x^p. \tag{7}$$

Note that f and g are increasing, and on the other hand $f(x) > 1$ in (0, 1).

As L_0^{-1} exists, then x_1 is well defined and, from the initial hypotheses, it follows that

$$\begin{aligned} &\|x_1 - x_0\| \leqslant \eta = f(a_{-1}) a_{-1} \|x_0 - x_{-1}\|, \\ &k\|L_0^{-1}\| \|x_0 - x_{-1}\|^p \leqslant k\beta\alpha^p = b_{-1}. \end{aligned} \tag{8}$$

Then, by induction on n, the following items are shown for $n \geqslant 1$:

$$(i_n) \quad \exists L_n^{-1} = [y_n, x_n; F]^{-1} \quad \text{such that} \quad \|L_n^{-1}\| \leqslant f(a_{n-1}) \|L_{n-1}^{-1}\|,$$

$$(ii_n) \quad \|x_{n+1} - x_n\| \leqslant f(a_{n-1}) a_{n-1} \|x_n - x_{n-1}\|,$$

$$(iii_n) \quad k\|L_n^{-1}\| \|x_n - x_{n-1}\|^p \leqslant b_{n-1}.$$

Assuming that $a_0 < 1$ and $x_1 \in \mathscr{D}$, by (5) and (8), we obtain

$$\begin{aligned} \|I - L_0^{-1} L_1\| &\leqslant \|L_0^{-1}\| \|L_0 - F'(x_0) + F'(x_0) - L_1\| \\ &\leqslant k\|L_0^{-1}\| [(1-\lambda)^p \|x_0 - x_{-1}\|^p + (1+\lambda^p) \|x_1 - x_0\|^p] \\ &\leqslant \left[(1-\lambda)^p + \frac{2}{p+1}(1+\lambda^p) f(a_{-1})^p a_{-1}^p\right] b_{-1} = a_0 < 1 \end{aligned}$$

and, by the Banach lemma, L_1^{-1} exists and

$$\|L_1^{-1}\| \leqslant f(a_0)\|L_0^{-1}\|.$$

Then (i_1) is hold.

By Taylor's formula, we have

$$F(x_1) = (F'(x_0) - L_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t.$$

As $[x, x; F] = F'(x)$, then, by (5),

$$\begin{aligned}\|F(x_1)\| &\leqslant k\|x_0 - y_0\|^p\|x_1 - x_0\| + \frac{2k}{p+1}\|x_1 - x_0\|^{p+1}\\ &\leqslant k\left((1-\lambda)^p + (1+\lambda^p)\frac{2}{p+1}f(a_{-1})^p a_{-1}^p\right)\|x_0 - x_{-1}\|^p\|x_1 - x_0\|,\end{aligned}$$

and consequently

$$\|x_2 - x_1\| \leqslant f(a_0)\|L_0^{-1}\|\|F(x_1)\| \leqslant f(a_0)a_0\|x_1 - x_0\|,$$

since x_2 is well defined and L_1^{-1} exists.

Finally, from (8) and (i_1), we get

$$k\|L_1^{-1}\|\|x_1 - x_0\|^p \leqslant f(a_0)f(a_{-1})^p a_{-1}^p b_{-1} = b_0.$$

Now if we suppose that $a_n < 1$, $x_n \in \mathscr{D}$ and (i_n)–(iii_n) are true for a fixed $n \geqslant 1$; we analogously prove (i_{n+1})–(iii_{n+1}).

3.2. Convergence study

In this section, we study the real sequences defined in (6) in order to obtain the convergence of sequence (4) in Banach spaces. It will be sufficient that $a_n < 1$ $(n \geqslant 0)$ and $\{x_n\}$ is a Cauchy sequence. Firstly, we provide the following two lemmas on the real sequences given in (6).

Lemma 3.2. *Let f and g be the two real functions given in* (7). *If* $a_1/a_0 \leqslant b_1/b_0 < 1$, *then*

(*a*) *both sequences given in* (6) *are decreasing for* $n \geqslant 0$;

(*b*) $a_n < \gamma^{\alpha_n} a_{n-1}$ *and* $b_n < \gamma^{\alpha_{n+1}} b_{n-1}$, *for* $n \geqslant 1$, *where* $\gamma = b_1/b_0 \in (0,1)$ *and* $\{\alpha_n\}$ *is the Fibonacci generalized sequence*:

$$\alpha_1 = \alpha_2 = 1, \quad \alpha_{n+2} = \alpha_{n+1} + p\alpha_n, \quad n \geqslant 1, \tag{9}$$

(*c*) $a_n < \gamma^{s_n} a_0$, *for* $n \geqslant 1$, *where* $s_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Proof. To prove (a), we use mathematical induction on n. From the hypotheses it follows that $a_1 < a_0$ and $b_1 < b_0$. If we assume $a_n < a_{n-1}$ and $b_n < b_{n-1}$, we have that $a_{n+1} < a_n$ and $b_{n+1} < b_n$, since f and g are increasing.

Part (b) is again proved by induction. By hypotheses, (b) is true for $n = 1$ and we assume $b_j < \gamma^{\alpha_{j+1}} b_{j-1}$ and $a_j < \gamma^{\alpha_j} a_{j-1}$, for $j = 1, 2, \dots, n$, are also. Then

$$a_{n+1} < g(a_{n-1})\gamma^{\alpha_{n+1}} b_{n-1} = \gamma^{\alpha_{n+1}} a_n,$$

$$b_{n+1} < f(a_n) f(a_{n-1})^p (\gamma^{\alpha_n} a_{n-1})^p \gamma^{\alpha_{n+1}} b_{n-1} < \gamma^{\alpha_{n+2}} b_n$$

implie (b). Furthermore,

$$a_n < \gamma^{\alpha_n} a_{n-1} < \gamma^{\alpha_n}\gamma^{\alpha_{n-1}} \dots \gamma^{\alpha_2}\gamma^{\alpha_1} a_0 = \gamma^{s_n} a_0.$$

This completes the proof. □

Next, we provide some properties of (9), whose proofs are trivial by applying induction.

Lemma 3.3. *Let $\{\alpha_n\}$ be the sequence defined in (9). Then*

(*a*) $$\alpha_n = \frac{1}{\sqrt{1+4p}}\left[\left(\frac{1+\sqrt{1+4p}}{2}\right)^n - \left(\frac{1-\sqrt{1+4p}}{2}\right)^n\right] \quad \text{and}$$
$$\alpha_n \geqslant \frac{1}{\sqrt{1+4p}}\left(\frac{1+\sqrt{1+4p}}{2}\right)^{n-1},$$

(*b*) $s_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ *is such that* $s_n = (\alpha_{n+2} - 1)/p$ *and* $s_1 + s_2 + \cdots + s_n = [\alpha_{n+4} - p(n+2) - 1]/p^2, \quad n \geqslant 1$.

Denote $\overline{B(x,r)} = \{y \in X; \|y - x\| \leqslant r\}$ and $B(x,r) = \{y \in X; \|y - x\| < r\}$.

Theorem 3.4. *Let $x_{-1}, x_0 \in \mathscr{D}$ and $\lambda \in [0,1]$. Let us suppose (5), (I)–(III) and the hypotheses of Lemma 3.2 are satisfied. If $a_0 < 1/2$ and $\overline{B(x_0, r_0)} \subseteq \mathscr{D}$, where $r_0 = \frac{1-a_0}{1-2a_0}\eta$, then the sequence $\{x_n\}$ given by (4) is well defined and converges to a solution x^* of (1) with at least R-order of convergence $(1+\sqrt{1+4p})/2$. Moreover $x_n, x^* \in \overline{B(x_0, r_0)}$. Furthermore,*

$$\|x^* - x_n\| < \frac{\Delta^n}{1-\Delta}\eta\gamma^{\beta_{n-1}}, \tag{10}$$

where $\Delta = \frac{a_0}{1-a_0}$, $\beta_{-1} = 0 = \beta_0$ and $\beta_n = s_1 + s_2 + \cdots + s_n$, $n \geqslant 1$.

Proof. It is clear, from $a_0 < 1/2$, that $a_n < 1/2 (n \geqslant 1)$. We then prove that $x_n \in \mathscr{D}$, for $n \geqslant 1$, and $\{x_n\}$ is a Cauchy sequence. Thus, for arbitrary positive integers m and n, we consider

$$\begin{aligned}\|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leqslant f(a_{n+m-2})a_{n+m-2} \ldots f(a_{n+1})a_{n+1} f(a_n)a_n \|x_{n+1} - x_n\| \\ &\quad + f(a_{n+m-3})a_{n+m-3} \ldots f(a_{n+1})a_{n+1} f(a_n)a_n \|x_{n+1} - x_n\| \\ &\quad + \cdots + f(a_n)a_n \|x_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\ &= \left[\prod_{j=n}^{n+m-2} f(a_j)a_j + \prod_{j=n}^{n+m-3} f(a_j)a_j + \cdots + f(a_n)a_n + 1\right] \\ &\quad \times \|x_{n+1} - x_n\|. \end{aligned} \tag{11}$$

Here we have used the recurrence relation (ii_1) for $i = n + m - 1, n + m - 2, \ldots, n + 1$. Now, by Lemma 2.2 since f is increasing, $\{a_j\}$ is decreasing and $a_j < \gamma^{s_j} a_0 (\gamma < 1)$, we have for $n \geqslant 2$:

$$\begin{aligned}\|x_{n+m} - x_n\| &< \Delta^n [\Delta^{m-1} + \Delta^{m-2} + \cdots + 1] \left(\prod_{j=1}^{n-1} \gamma^{s_j}\right) \|x_1 - x_0\| \\ &= \gamma^{s_1 + s_2 + \cdots + s_{n-1}} \frac{\Delta^n (1 - \Delta^m)}{1 - \Delta} \|x_1 - x_0\|. \end{aligned} \tag{12}$$

where $\Delta = \frac{a_0}{1 - a_0} < 1$.

If $n = 1$, by (11),

$$\|x_{m+1} - x_1\| < \frac{\Delta(1 - \Delta^m)}{1 - \Delta} \|x_1 - x_0\|. \tag{13}$$

and if $n = 0$,

$$\|x_m - x_0\| < \frac{1 - \Delta^m}{1 - \Delta} \|x_1 - x_0\| < \frac{\eta}{1 - \Delta} = r_0. \tag{14}$$

Consequently, x_n is in $\mathscr{D}$ and the sequence $\{x_n\}$ is well defined. Secondly, $\{x_n\}$ is a Cauchy sequence and has a limit, say x^*, in $\overline{B(x_0, r_0)}$. Since

$$\|F(x_n)\| \leqslant k \|x_n - x_{n-1}\| \left(\|x_{n-1} - x_{n-2}\|^p + \frac{2}{p+1} \|x_n - x_{n-1}\|^p \right),$$

the sequence $\{\|F(x_n)\|\}$ has the limit zero. Hence $F(x^*) = 0$. Thirdly, by letting $m \to \infty$ in (12)–(14), we see (10).

Finally, by Lemma 3.3 and (10),

$$\beta_{n-1} > \frac{1}{p^2} \left[\frac{1}{\sqrt{1 + 4p}} \left(\frac{1 + \sqrt{1 + 4p}}{2} \right)^{n+2} - p(n+1) - 1 \right]$$

and

$$\gamma^{\beta_{n-1}} < \frac{\left(\gamma^{\frac{(1+\sqrt{1+4p})^2}{4p^2\sqrt{1+4p}}}\right)^{\left(\frac{1+\sqrt{1+4p}}{2}\right)^n}}{\gamma^{\frac{p(n+1)+1}{p^2}}} < \frac{\gamma^{\left(\frac{1+\sqrt{1+4p}}{2}\right)^n}}{\gamma^{\frac{n}{p}}\gamma^{\frac{p+1}{p^2}}}.$$

Thus, if $C = \gamma^{-\frac{p+1}{p^2}}$,

$$\|x^* - x_n\| < \left(\frac{\Delta}{\gamma^{1/p}}\right)^n \frac{\eta}{\gamma^{\frac{p+1}{p^2}}(1-\Delta)}\gamma^{\left(\frac{1+\sqrt{1+4p}}{2}\right)^n} < \frac{\eta}{1-\Delta}C\gamma^{\left(\frac{1+\sqrt{1+4p}}{2}\right)^n},$$

since $\Delta\gamma^{1/p} < 1$. Method (4) therefore has R-order $(1+\sqrt{1+4p})/2$ at least. □

Remark 1. Observe that, under the conditions of the last theorem, if $a_0 < (3-\sqrt{5})/2$, the solution x^* of (1) is unique in $B(x_0,\tau)\cap\mathscr{D}$, where $\tau = (\frac{p+1}{2}[\frac{1}{\beta k} - (1-\lambda)^p\alpha^p] - r_0^p)^{1/p}$. Indeed, let z^* be another root in $B(x_0,\tau)\cap\mathscr{D}$ and consider

$$0 = F(z^*) - F(x^*) = \int_{x^*}^{z^*} F'(x),\, \mathrm{d}x = \int_0^1 F'(x^* + t(z^* - x^*))(z^* - x^*)\,\mathrm{d}t.$$

Denoting $A = \int_0^1 F'(x^* + t(z^* - x^*))\,\mathrm{d}t$, we have

$$\begin{aligned}\|L_0^{-1}A - I\| &\leqslant \|L_0^{-1}\|\|A - L_0\| \\ &\leqslant \|L_0^{-1}\|\int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0) + F'(x_0) - L_0\|\mathrm{d}t \\ &\leqslant \beta\left(\int_0^1 2k\|x^* + t(z^* - x^*) - x_0\|^p\,\mathrm{d}t + \|F'(x_0) - L_0\|\right) \\ &\leqslant \beta\left(\int_0^1 2k((1-t)^p\|x^* - x_0\|^p + t^p\|z^* - x_0\|^p)\,\mathrm{d}t + k(1-\lambda)^p\alpha^p\right) \\ &< \beta k\left(\frac{2}{p+1}(r_0^p + \tau^p) + (1-\lambda)^p\alpha^p\right) = 1,\end{aligned}$$

and the operator A is therefore invertible, and consequently $z^* = x^*$. Moreover, from $a_0 < (3-\sqrt{5})/2$, we have $\tau > 0$, as we can see in the following. Taking into account that

$$\tau^p = \frac{1+p}{2}\left[\frac{1}{k\beta} - (1-\lambda)^p\alpha^p\right] - r_0^p,$$

$$r_0^p = \left(\frac{1-a_0}{1-2a_0}\right)^p\eta^p \leqslant \frac{1-a_0}{1-2a_0}(1+\lambda^p)\eta^p$$

and $a_0 = k\beta[(1-\lambda)^p\alpha^p + \frac{2}{1+p}(1+\lambda^p)\eta^p]$, we have

$$\tau^p > \frac{1+p}{2k\beta(1-2a_0)}[(1-a_0)^2 - a_0 + a_0k\beta(1-\lambda)^p\alpha^p] > 0$$

if $a_0 < (3-\sqrt{5})/2$.

Note that we have obtained the uniqueness of solution x^* for all $\lambda \in [0,1]$.

4. A special case of conservative problems

It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases we assume the law of conservation of energy, namely, that the sum of the kinetic energy and the potential energy is constant. A system of this kind is said to be conservative.

If ρ and σ are arbitrary functions with the property that $\rho(0) = 0$ and $\sigma(0) = 0$, the general equation

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + \sigma\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) + \rho(x) = 0, \tag{15}$$

can be interpreted as the equation of motion of a mass m under the action of a restoring force $-\rho(x)$ and a damping force $-\sigma(\mathrm{d}x/\mathrm{d}t)$. In general these forces are nonlinear, and Eq. (15) can be regarded as the basic equation of nonlinear mechanics. In this paper we shall consider the special case of a nonlinear conservative system described by the equation

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + \rho(x(t)) = 0,$$

in which the damping force is zero and there is consequently no dissipation of energy. Extensive discussions of (15), with applications to a variety of physical problems, can be found in classical references [1] and [15].

In this paper, we study the existence of a unique solution for a special case of a nonlinear conservative system described by the equation

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + \Phi(x(t)) = 0 \tag{16}$$

with the boundary conditions

$$x(0) = 0 = x(1). \tag{17}$$

In order to study the application of (4) for the numerical solution of differential equation problems, we illustrate the theory for the case of particular second-order ordinary differential equation (16) subject to the boundary conditions (17).

It is required to find a solution of problem (16) and (17) in the interval $0 \leqslant t \leqslant 1$. Under suitable restrictions on the function Φ, we will see that a unique solution of (16) and (17)exists. Moreover the method of discretization is used to project the boundary value problem of second order into a finite-dimensional space. The new class of Secant-like methods are applied to this problem to approximate the solution of the corresponding system of equations.

Firstly, we suppose that Φ is once continuously differentiable and Φ' is Hölder (C,p) continuous. So the operator

$$[F(x)](t) = \frac{d^2x(t)}{dt^2} + \Phi(x(t)) \tag{18}$$

is defined from $C^{(2)}[0,1]$ into $C[0,1]$ and it is once differentiable.

4.1. Existence and uniqueness of the solution

In order to see that a unique solution of problem (16) and (17) exists, we apply Theorem 3.4. Then the bounds α, β, η and k, which appear in the previous section, are necessary. The first derivative of F at $x = x(t)$ is

$$F'(x)y(t) = \frac{d^2y(t)}{dt^2} + \Phi'(x(t))y(t),$$

when it is applied to the function $y(t)$. To start the analysis of the convergence of (4) to a solution of problem (16) and (17), from the starting functions $x_{-1}(t)$ and $x_0(t)$, we first prove that $L_0^{-1} = [y_0, x_0; F]^{-1}$ exists. Observe that

$$[F'(x) - F'(y)]u(t) = (\Phi'(x) - \Phi'(y))u(t).$$

Then

$$\|F'(x) - F'(y)\| = C\|x - y\|^p,$$

where C is the Hölder constant for Φ'. Since F' exists and is Lipschitz continuous, it follows that the operator

$$[x, y; F] = \int_0^1 F'(x + \tau(y - x))\, d\tau$$

is a divided difference at the points $x, y \in C^{(2)}[0,1]$. Hence condition (5) is satisfied with $k = C/(1 + p)$.

If x_{-1}, x_0 and $\lambda \in [0,1]$ are now fixed, then $y_0 = \lambda x_0 + (1-\lambda)x_{-1} \in C^{(2)}[0,1]$. Taking into account that

$$L_0u(t) \equiv [y_0, x_0; F]u(t) = \frac{d^2u(t)}{dt^2} + \int_0^1 \Phi'(y_0(t) + \tau(x_0(t) - y_0(t)))u(t)\, d\tau \equiv v(t),$$

it follows that $u(t) = L_0^{-1}v(t)$ if L_0^{-1} exists.

Next, we consider the linear boundary value problem

$$\begin{aligned} &\frac{d^2u(t)}{dt^2} + \psi(x_0(t), y_0(t))u(t) = v(t), \\ &u(0) = 0 = u(1), \end{aligned} \tag{19}$$

where $\psi(x_0(t), y_0(t)) = \int_0^1 \Phi'(y_0(t) + \tau(x_0(t) - y_0(t)))\,d\tau$. It is known, see [9], that problem (19) may be written in the form of the second kind Fredholm equation

$$u(t) = -\int_0^1 K(t,s)v(s)\,ds + [P(u)](t), \quad 0 \leqslant t \leqslant 1,$$

where

$$K(t,s) = \begin{cases} s(1-t), & t \geqslant s, \\ t(1-s), & t \leqslant s. \end{cases}$$

and

$$[P(u)](t) = \int_0^1 K(t,s)\psi(x_0(s), y_0(s))u(s)\,ds.$$

Thus

$$[(I-P)(u)](t) = -\int_0^1 K(t,s)v(s)\,ds \equiv (Kv)(t).$$

On the other hand, using the max-norm and denoting $S = \sup_{0\leqslant t\leqslant 1} |\psi(x_0 t), y_0(t))|$, we have $\|P\| \leqslant S/8$. Consequently, by the Banach lemma, $(I-P)^{-1}$ exists if $S < 8$, and then

$$u(t) = (I-P)^{-1}(Kv)(t).$$

Since

$$\|Kv\| \leqslant \left(\sup_{0\leqslant t\leqslant 1} \int_0^1 |K(t,s)|\,ds\right)\|v\| \leqslant \|v\|/8,$$

then L_0^{-1} exists, $\|L_0^{-1}\| \leqslant 1/(8-S)$ and $\|L_0^{-1}F(x_0)\| \leqslant \|F(x_0)\|/(8-S)$.

We can now establish a result on the existence and the uniqueness of the solution of problem (16) and (17), whose proof follows as that in Theorem 3.4.

Theorem 4.1. *Following the previous notation, we consider the operator defined in (18), where* $F: C^{(2)}[0,1] \to C[0,1]$. *Assume that* $x_{-1}, x_0 \in C^{(2)}[0,1]$, $\lambda \in [0,1]$ *fixed,* $S = \sup_{0\leqslant t\leqslant 1} |\psi(x_0(t), y_0(t))| < 8$, $a_0 < 1/2$ *and* $a_1/a_0 \leqslant b_1/b_0 < 1$, *where* a_0, a_1, b_0, b_1 *are defined in the previous section with*

$$\alpha = \|x_0 - x_{-1}\|, \quad \beta = \frac{1}{8-S}, \quad \eta = \frac{\|F(x_0)\|}{8-S}, \quad k = \frac{C}{1+p}$$

and C the Hölder constant for Φ'. Then there exists at least a solution of problem (16) and (17) in $\overline{B(x_0, r_0)}$, where $r_0 = \frac{1-a_0}{1-2a_0}\eta$.

Remark 2. Under the assumptions of the previous theorem, if $a_0 < (3-\sqrt{5})/2$, the solution of problem (16) and (17) is unique in the open ball $B(x_0, \tau)$, where $\tau = \left(\frac{p+1}{2}\left[\frac{1}{\beta k} - (1-\lambda)^p \alpha^p\right] - r_0^p\right)^{1/p}$.

Next, we here show the application of the previous study to the following boundary value problem:

$$\begin{aligned} &\frac{d^2x(t)}{dt^2} + x(t)^{1+p} + Q = 0, \quad p \in [0,1], \quad Q \in \mathbb{R}, \\ &x(0) = 0 = x(1). \end{aligned} \tag{20}$$

in the space $C^{(2)}[0,1]$ of all twice Fréchet-differentiable functions with the max-norm. Now (18) can be written in the form

$$[F(x)](t) = \frac{d^2x(t)}{dt^2} + x(t)^{1+p} + Q. \tag{21}$$

To obtain the existence and the uniqueness of the solution of (20), we first consider

$$\mathscr{D}_0 = \{x, y \in C^{(2)}[0,1] : \|\psi(x,y)\| < 8\} \subseteq C^{(2)}[0,1], \tag{22}$$

where $\psi(x(t), y(t)) = (1+p)\int_0^1 (y(t) + \tau(x(t) - y(t)))^p \, d\tau$, so that $F : \mathscr{D}_0 \to C[0,1]$. Taking into account (16), we have $\Phi(x(t)) = x(t)^{1+p} + Q$. Then, by the Banach lemma, L_0^{-1} exists and $\|L_0^{-1}\| \leqslant 1/(8-S)$, where

$$S = (1+p) \sup_{0 \leqslant t \leqslant 1} \left| \int_0^1 (y_0(t) + \tau(x_0(t) - y_0(t)))^p \, d\tau \right|.$$

In addition

$$\|L_0^{-1} F(x_0)\| \leqslant \frac{\|F(x_0)\|}{8-S}.$$

On the other hand,

$$\|[x, y; F] - F'(z)\| \leqslant \|x - z\|^p + \|y - z\|^p; \quad x, y, z \in \mathscr{D}_0, \quad p \in [0,1].$$

Corollary 4.2. *Let $F : \mathscr{D}_0 \subseteq C^{(2)}[0,1] \to C[0,1]$, where $\mathscr{D}_0$ is defined in (22) and F in (21). Let $x_{-1}, x_0 \in \mathscr{D}_0$, $\lambda \in [0,1]$ fixed. Let us suppose that $a_0 < 1/2$ and $a_1/a_0 \leqslant b_1/b_0 < 1$, where a_0, a_1, b_0, b_1 are defined in the previous section with*

$$\alpha = \|x_0 - x_{-1}\|, \quad \beta = \frac{1}{8-S}, \quad \eta = \frac{\|F(x_0)\|}{8-S}, \quad k = 1$$

and $S = (1+p)\sup_{0\leqslant t\leqslant 1}|\int_0^1 (y_0(t) + \tau(x_0(t) - y_0(t)))^p\, d\tau|$. *If* $\overline{B(x_0, r_0)} \subseteq \mathscr{D}_0$, *where* $r_0 = \frac{1-a_0}{1-2a_0}\eta$, *then a solution of* (20) *exists at least in* $\overline{B(x_0, r_0)}$.

Remark 3. Under the assumptions of the last corollary, if $a_0 < (3-\sqrt{5})/2$, the solution of (20) is unique in $B(x_0, \tau) \cap \mathscr{D}_0$, where $\tau = (\frac{p+1}{2}[\frac{1}{\beta k} - (1-\lambda)^p \alpha^p] - r_0^p)^{1/p}$.

4.2. Location of the solution

To illustrate the previous result, we consider the Secant method and boundary value problem (20), where $Q = 1/4$ and $p = 1/2$. As the solution would vanish at the endpoints and be positive in the interior, a reasonable choice of initial approximation seem to be $x_{-1}(t) = 0.4\sin \pi t$. On the other hand, we choose $x_0(t) = 0$ in order to simplify the existence domain and reduce the operational cost. So

$$S = \sqrt{0.4}, \quad \alpha = 0.4\pi^2, \quad \beta = 1/(8-S), \quad \eta = 1/(4(8-S)) \quad \text{and} \quad k = 1.$$

As a result

$$a_0 = 0.303022 < (3-\sqrt{5})/2 < 1/2,$$

$$a_1/a_0 = 0.222462 \leqslant b_1/b_0 = 0.707029 < 1,$$

and the conditions of corollary 4.2 hold. Then, see Fig. 1, there exists a solution x^* of (20) in $\{w \in C^{(2)}[0,1]; \|w\| \leqslant r_0\}$, where $r_0 = 0.0600328$, and x^* is unique in $\{w \in C^{(2)}[0,1]; \|w\| < 14.3675\} \cap \mathscr{D}_0$.

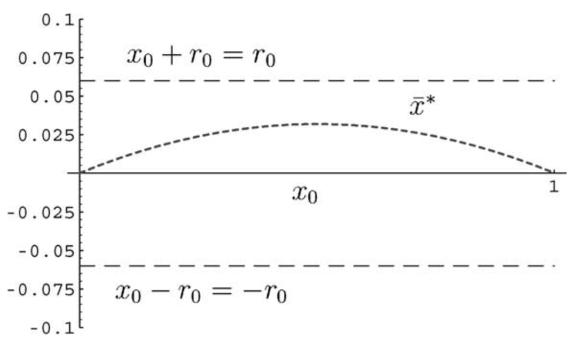

Fig. 1. The existence domain of the solution and the approximated solution.

4.3. Numerical solution of the finite-difference equations

To show how boundary value problem (20) can be changed to a system of algebraic equations, we replace the derivative in the differential equation with their finite-difference approximations. The system of algebraic equations can then be solved numerically by (4) in order to obtain an approximate solution to boundary value problem (20).

To solve this problem by finite differences, we start by drawing the usual grid line with grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Note that x_0 and x_n are given by the boundary conditions, then $x_0 = 0 = x_n$, and our job is to find the other x_i ($i = 1, 2, \ldots, n-1$). To do this, we begin by replacing the second derivative $x''(t)$ in the differential equation with its approximation

$$x''(t) \approx [x(t+h) - 2x(t) + x(t-h)]/h^2,$$
$$x''(t_i) = (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \ldots, n-1.$$

By substituting this expression into the differential equation, we have the following system of nonlinear equations

$$\begin{cases} 2x_1 - h^2 x_1^{1+p} - x_2 - h^2 Q = 0, \\ -x_{i-1} + 2x_i - h^2 x_i^{1+p} - x_{i+1} - h^2 Q = 0, \quad i = 2, 3, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^{1+p} - h^2 Q = 0. \end{cases} \tag{23}$$

We therefore have an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(x) = H(x) - h^2 \varphi(x)$, where

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}, \quad \varphi(x) = \begin{pmatrix} x_1^{1+p} + 1/4 \\ x_2^{1+p} + 1/4 \\ \vdots \\ x_{n-1}^{1+p} + 1/4 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

Thus

$$F'(x) = H - h^2(1+p)\,\mathrm{diag}\,\{x_1^p, x_2^p, \ldots, x_{n-1}^p\}.$$

Let $x \in \mathbb{R}^{n-1}$ then our norm will be $\|x\| = \max_{1 \leqslant i \leqslant n-1} |x_i|$. The corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1 \leqslant i \leqslant n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [2,11]) that F has a Hölder continuous divided difference at the points $x, y \in \mathbb{R}^{n-1}$, which is defined by the matrix whose entries are

Table 1

i	x_i^*
1	0.01141648508671
2	0.02032077189279
3	0.02669609124653
4	0.03052779202087
5	0.03180615401355
6	0.03052779202087
7	0.02669609124653
8	0.02032077189279
9	0.01141648508671

$$[x,y;F]_{ij} = \frac{1}{x_j - y_j}\left(F_i\left(x_1,\ldots,x_j,y_{j+1},\ldots,y_{n-1}\right) - F_i\left(x_1,\ldots,x_{j-1},y_j,\ldots,y_{n-1}\right)\right).$$

If $n = 10$ then (23) gives 9 equations. Taking into account the values considered in remark 4, the data for the initial iterate are $x_{-1}(t_i) = 0.4\sin\pi t_i$ and $x_0(t_i) = 0$ for $i = 1, 2, \ldots, 9$. After five iterates, we obtain the vector x^* (see Table 1) as the solution of system (23).

If x^* is now interpolated, the present approximation $\bar{x}^*$ to the solution of (20) with $p = 1/2$ is that appearing in Fig. 1. Notice that the interpolated approximation $\bar{x}^*$ lies within the existence domain of solutions mentioned above.

5. Final remark

Finally, we analyse two things. Firstly, we study the domain of the starting points and, secondly, we analyse the speed of convergence of the class of iterative methods given by (4). If we now choose $Q = 0$ in (20), the corresponding boundary value problem has already been used by other authors as a test problem (see [2,8,12]).

We again start using the method of discretization to project this boundary value problem into a finite-dimensional space. Let $n = 10$ and $x_{-1}(t_i) = 135\sin\pi t_i$ ($i = 1, 2, \ldots, 9$) be the initial approximation. We choose, as in [2], x_0 by setting $x_0(t_i) = x_{-1}(t_i) - 10^{-5}$, $i = 1, 2, \ldots, 9$. If we apply the secant method ($\lambda = 0$) to the previous points, after two and three iterations we obtain two points x_2 and x_3 in which the conditions required in this paper are satisfied for the secant method, but the ones required by Argyros in [2] are not. Consequently, we can take x_2 and x_3 as the true starting points.

On the other hand, we obtain the errors $\|x_n - x^*\|_\infty$, which appear in Table 2, for the iterates x_n generated by (4) for different values of the parameter $\lambda \in [0, 1]$ and starting at x_{-1} and x_0.

Table 2

n	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.99$
1	2.24748	2.24748	2.24748	2.24748	2.24748
2	2.60218×10^{-1}	2.08533×10^{-1}	1.53355×10^{-1}	9.43008×10^{-2}	3.35497×10^{-2}
3	3.66518×10^{-3}	2.28181×10^{-3}	1.16525×10^{-3}	3.79963×10^{-4}	1.20966×10^{-5}
4	6.16651×10^{-6}	2.31808×10^{-6}	5.83199×10^{-7}	5.88006×10^{-8}	2.73843×10^{-11}
5	1.47125×10^{-10}	2.58780×10^{-11}	2.20268×10^{-12}	0.0	0.0

Table 3
$x_{-1}(t_i) = 135 \sin \pi t_i$, $x_0(t_i) = 0$ $(i = 0, 1, \ldots, 10)$

n	$\|\bar{x} - x_n\|_\infty$
1	2.24749
2	3.09256×10^{-2}
3	2.84217×10^{-13}
4	0.0

The numerical results, using 14 significative decimal figures, indicate that the Secant method is not optimal for approximating the solution x^* of $F(x) = 0$. Moreover, iteration (4) converges faster to x^* for increasing values of the parameter $\lambda \in [0, 1]$.

Next, observe that (4), where λ is near one, gives similar approximations, without using F', to the solution x^* of $F(x) = 0$ to Newton's method (see Table 3).

Note that iteration (4) and Newton's method have R-order of convergence at least $(1 + \sqrt{1 + 4p})/2$ and $1 + p$ respectively, under the same general convergence conditions. As we have $\mu(0,p) \leqslant \mu(\lambda,p)$, if iteration (4) has R-order of convergence $\mu(\lambda,p)$ such that $\mu(0,p) = (1 + \sqrt{1 + 4p})/2$ and $\mu(1,p) = 1 + p$, it is an open problem for future to determinate exactly $\mu(\lambda,p)$ as a function of λ.

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