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Generalized differentiability conditions for Newton's method

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The use of majorizing sequences is the usual way to prove the convergence of Newton's method. An alternative technique to majorizing sequences is provided in this paper, in which three scalar sequences are used, so that the analysis of convergence is simplified when the traditional convergence condition is relaxed. An application to a nonlinear integral equation is also given, which is also solved and the solution approximated by a discretization process.

Keywords: nonlinear equations in Banach spaces; integral equation; Newton's method; convergence theorem; recurrence relations.

1. Introduction

One of the more common problems in mathematics is the solution of a nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is some nonlinear operator in a Banach space X. This problem is not always easy to solve. Frequently, we cannot obtain an exact solution to the previous equation, so we look for an approximation to a solution. In this case, we use approximation methods, which are generally iterative ones. The best known iteration to solve nonlinear equations is the Newton method:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geqslant 0, \quad \text{given } x_0, \tag{2}$$

provided that $F'(x_n)^{-1}$ exists for all $n \geqslant 0$.

For further reference, we recall some notations. Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear once Fréchet differentiable operator in an open convex domain Ω. Let $x_0 \in \Omega$ and suppose that $F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Most authors have studied the convergence of (2) to a solution x^* of (1) under the original conditions of Kantorovich (see Argyros, 1992; Argyros & Szidarovszky, 1993; Kantorovich & Akilov, 1982), where it is supposed that the second Fréchet derivative F'' is continuous and bounded in Ω or that the first Fréchet derivative F' is Lipschitz continuous in Ω.

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Other authors (Argyros, 1992; Keller, 1992; Rokne, 1972) also consider a generalization of these conditons, which is given by

$$\|F'(x) - F'(y)\| \leqslant K\|x - y\|^p, \quad x, y \in \Omega, \quad K \geqslant 0, \quad p \in [0, 1]. \tag{3}$$

Observe that if $p = 1$ in (3), F' is Lipschitz continuous.

In Argyros (1990), for equations defined by differentiable operators, is considered a generalization of condition (3), that is given by

$$\|F'(x) - F'(y)\| \leqslant \omega(\|x - y\|), \quad x, y \in \Omega, \tag{4}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non-decreasing function such that $\omega(0) \geqslant 0$. Obviously, if $\omega(z) = Kz$ (K = constant), F' is Lipschitz continuous, and if $\omega(z) = Kz^p$ (K = constant and $p \in [0, 1]$), F' is (K, p)-Hölder continuous. This uses real majorizing sequences, although these sequences are not the usual ones. In this paper, the difficulty in using majorizing sequences to prove the convergence of Newton's method, when we want to relax the required conditions to F', is laid out.

We provide an alternative technique to majorizing sequences, in which particular real sequences are also constructed, but they are not majorizing ones. The application of this technique is very simple as we can see when the semilocal convergence result is obtained under condition (4). We also obtain the domains of existence and uniqueness of the solution. Moreover, we can generalize the result obtained in Newton–Kantorovich type conditions (Kantorovich & Akilov, 1982). With this technique we improve the results obtained by majorizing sequences when F' satisfies condition (3), see Keller (1992) and Rokne (1972). The convergence conditions are relaxed for specific values of the parameter p, and we establish a result on the uniqueness of the solution, which is not given in Keller (1992) and Rokne (1972). A result about the R-order of convergence of Newton's method is also obtained and some sharp *a priori* error estimates are provided.

Finally, we apply our semilocal convergence results to nonlinear Hammerstein integral equations of the second kind (Polyanin & Manzhirov, 1998), and obtain a result on the existence and uniqueness of solutions for this type of equation. Solutions of particular Hammerstein integral equations are then approximated by a discretization process.

Throughout the paper we denote

$$\overline{B(x, r)} = \{y \in X;\ \|y - x\| \leqslant r\} \quad \text{and} \quad B(x, r) = \{y \in X;\ \|y - x\| < r\}.$$

2. Semilocal convergence of Newton's method

Under certain conditions for the pair (F, x_0), we study the convergence of the Newton method to a unique solution of (1). From some real parameters, a system of three recurrence relations is constructed in which three sequences of positive real numbers are involved. The convergence of Newton's sequence (2) is then guaranteed from it.

2.1 *Recurrence relations*

We suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$. We also assume the following:

(C_1) $\|\Gamma_0\| \leqslant \beta$,
(C_2) $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
(C_3) $\|F'(x) - F'(y)\| \leqslant \omega(\|x - y\|)$, $x, y \in \Omega$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non-decreasing function such that $\omega(0) \geqslant 0$,
(C_4) a continuous and non-decreasing function $h : [0, 1] \to \mathbb{R}_+$ exists, such that $\omega(tz) \leqslant h(t)\omega(z)$, with $t \in [0, 1]$ and $z \in [0, \infty)$.

Note that condition (C_4) does not involve any restriction, since h always exists, such that $h(t) = 1$, as a consequence of ω being a non-decreasing function. We can even consider $h(t) = \sup_{z>0} \frac{\omega(tz)}{\omega(z)}$. We use it to sharpen the bounds that we obtain for particular expressions, as we will see later.

We now denote $a_0 = \eta$, $b_0 = \beta\omega(a_0)$ and define the following scalar sequences:

$$t_n = Hb_n f(b_n), \quad n \geqslant 0, \tag{5}$$

$$a_n = t_{n-1}a_{n-1}, \quad n \geqslant 1, \tag{6}$$

$$b_n = h(t_{n-1})b_{n-1} f(b_{n-1}), \quad n \geqslant 1, \tag{7}$$

where $H = \int_0^1 h(t)\,\mathrm{d}t$ and

$$f(x) = \frac{1}{1-x}. \tag{8}$$

Observe that we consider the case $b_0 > 0$, since if $b_0 = 0$, a trivial problem results, as the solution of (1) is x_0.

Next, we see the following recurrence relations that sequences (2), (6) and (7) verify:

$$\|\Gamma_1\| = \|F'(x_1)^{-1}\| \leqslant f(b_0)\|\Gamma_0\|, \tag{9}$$

$$\|x_2 - x_1\| \leqslant a_1, \tag{10}$$

$$\|\Gamma_1\|\omega(a_1) \leqslant b_1. \tag{11}$$

To do this, we assume that

$$x_1 \in \Omega \quad \text{and} \quad b_0 < 1.$$

As Γ_0 exists, by the Banach lemma, we have that Γ_1 is defined and

$$\|\Gamma_1\| \leqslant \frac{\|\Gamma_0\|}{1 - \|I - \Gamma_0 F'(x_1)\|} \leqslant f(b_0)\|\Gamma_0\|,$$

since

$$\|I - \Gamma_0 F'(x_1)\| \leqslant \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leqslant \beta\omega(a_0) = b_0 < 1.$$

From Taylor's formula and (2), it follows that

$$F(x_1) = \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)\,\mathrm{d}t,$$

and consequently,

$$\|F(x_1)\| = \left(\int_0^1 \omega(ta_0)\,\mathrm{d}t\right)\|x_1 - x_0\| \leqslant H\omega(a_0)\|x_1 - x_0\|.$$

Thus

$$\|x_2 - x_1\| = \|\Gamma_1 F(x_1)\| \leqslant \|\Gamma_1\|\|F(x_1)\| \leqslant Hb_0 f(b_0)\|x_1 - x_0\| \leqslant t_0 a_0 = a_1$$

and

$$\|\Gamma_1\|\omega(a_1) \leqslant f(b_0)\|\Gamma_0\|\omega(t_0 a_0) \leqslant f(b_0)\beta h(t_0)\omega(a_0) = h(t_0)b_0 f(b_0) = b_1.$$

In Theorem 2.2 we will generalize (9)–(11) to every point of sequence (2) and we will show that (2) is a Cauchy sequence. To this purpose we first investigate the scalar sequences $\{t_n\}$, $\{a_n\}$ and $\{b_n\}$ at the beginning of the following section.

2.2 *Analysis of the scalar sequences $\{t_n\}$, $\{a_n\}$ and $\{b_n\}$*

The next goal is to analyse the real sequences (5)–(7) so that the convergence of sequence (2) is guaranteed. To do this, it suffices to see that $x_{n+1} \in \Omega$ and $b_n < 1$, for all $n \geqslant 1$. Next, we generalize the previous recurrence relations, so that we will be able to prove that (2) is a Cauchy sequence. First, we give a technical lemma.

LEMMA 2.1 Let f be the scalar function defined in (8). If $b_0 < m$, where

$$m = \min\left\{1 - h(t_0), \frac{1}{1+H}\right\} \tag{12}$$

and $H = \int_0^1 h(t)\,\mathrm{d}t$, then

(a) sequences $\{t_n\}$, $\{a_n\}$ and $\{b_n\}$ are strictly decreasing,
(b) $t_n < 1$ and $b_n < 1$, for all $n \geqslant 0$.

If $b_0 = 1 - h(t_0) < \frac{1}{1+H}$, then $t_n = t_0 < 1$ and $b_n = b_0 < 1$, for all $n \geqslant 1$.

Proof. Firstly, we consider the case $b_0 < m$, where m is defined in (12). Item (a) is proved by mathematical induction on n. As $b_0 < 1 - h(t_0)$, then $b_1 < b_0$ and $t_1 < t_0$, since f is increasing. Moreover, $a_1 < a_0$ as a consequence of $t_0 < 1$ and $b_0 < m$. Next, we suppose that $b_i < b_{i-1}$, $t_i < t_{i-1}$ and $a_i < a_{i-1}$, for all $i = 1, 2, \ldots, n$. Thus

$$\begin{aligned} b_{n+1} &= h(t_n)b_n f(b_n) < h(t_0)b_n f(b_0) < b_n, \\ t_{n+1} &= Hb_n f(b_n) < Hb_{n-1} f(b_{n-1}) = t_n, \\ a_{n+1} &= t_n a_n < t_0 a_n < a_n, \end{aligned}$$

since f and h are increasing in $[0, 1)$. Consequently, the sequences $\{t_n\}$, $\{a_n\}$ and $\{b_n\}$ are strictly decreasing.

Secondly, to see (b), we have $t_n < t_0 < 1$ and $b_n < b_0 < 1$, for all $n \geqslant 0$, by (a) and $b_0 < m$.

Finally, if $b_0 = 1 - h(t_0)$, it follows that $h(t_0) f(b_0) = 1$, and therefore, $b_n = b_0 = 1 - h(t_0) < 1$, for all $n \geqslant 0$. Moreover, if $b_0 < \frac{1}{1+H}$, then we have $t_n = t_0 < 1$, for all $n \geqslant 0$. □

2.3 *A semilocal convergence result*

We are now ready to prove a semilocal convergence theorem for Newton's method when it is applied to operators that satisfy conditions (C_1)–(C_4).

THEOREM 2.2 Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a once Fréchet differentiable operator in an open convex domain Ω. We suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$ and conditions (C_1)–(C_4) hold. If $b_0 = \beta\omega(\eta) < m$, where m is defined in (12), and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{a_0}{1-t_0}$, then sequence (2), starting from x_0, converges to a solution x^* of (1), the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$.

Proof. Firstly, we prove the following items for sequence (2) and $n \geqslant 1$:

(I) Γ_n exists and $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant f(b_{n-1})\|\Gamma_{n-1}\|$,
(II) $\|x_{n+1} - x_n\| \leqslant a_n$,
(III) $\|\Gamma_n\|\omega(a_n) \leqslant b_n$,
(IV) $x_{n+1} \in \Omega$.

Notice that $x_1 \in \Omega$, since $\eta < R$. Then, from (9)–(11) and

$$\|x_2 - x_0\| \leqslant \|x_2 - x_1\| + \|x_1 - x_0\| \leqslant a_1 + a_0,$$

it follows that (I)–(IV) hold for $n = 1$. If we now suppose that (I)–(IV) are true for some $n = 1, 2, \ldots, i$ we see that (I)–(IV) also hold for $n = i + 1$. We take into account that $t_i < 1$ and $b_i < 1$, for all $i \geqslant 0$.

(I) Observe

$$\begin{aligned}\|I - \Gamma_i F'(x_{i+1})\| &\leqslant \|\Gamma_i\|\omega(\|x_{i+1} - x_i\|) \leqslant f(b_{i-1})\|\Gamma_{i-1}\|\omega(a_i)\\ &= f(b_{i-1})\|\Gamma_{i-1}\|\omega(t_{i-1}a_{i-1}) \leqslant f(b_{i-1})\|\Gamma_{i-1}\|h(t_{i-1})\omega(a_{i-1})\\ &\leqslant h(t_{i-1})b_{i-1}f(b_{i-1}) = b_i < 1,\end{aligned}$$

since $\{b_n\}$ is decreasing, $b_0 < \frac{1}{1+H}$ and $t_{i-1} < 1$. Then, by the Banach lemma, Γ_{i+1} is defined and

$$\|\Gamma_{i+1}\| \leqslant \frac{\|\Gamma_i\|}{1 - b_i} = f(b_i)\|\Gamma_i\|.$$

(II) By Taylor's formula and (2) it follows, as for (10), that

$$\begin{aligned}\|F(x_{i+1})\| &= \left\| \int_0^1 [F'(x_i + t(x_{i+1} - x_i)) - F'(x_i)](x_{i+1} - x_i)\,\mathrm{d}t \right\|\\ &\leqslant \left(\int_0^1 \omega(t\|x_{i+1} - x_i\|)\,\mathrm{d}t \right)\|x_{i+1} - x_i\| \leqslant H\omega(a_i)a_i.\end{aligned}$$

Therefore

$$\|x_{i+2} - x_{i+1}\| \leqslant f(b_i)\|\Gamma_i\|H\omega(a_i)a_i \leqslant Hb_i f(b_i)a_i = t_i a_i = a_{i+1}.$$

(III) The inequality

$$\|\Gamma_{i+1}\|\omega(a_{i+1}) \leqslant b_{i+1}$$

follows immediately.

(IV)

$$\|x_{i+2}-x_0\| \leqslant \|x_{i+2}-x_{i+1}\| + \|x_{i+1}-x_0\| \leqslant a_{i+1} + \sum_{j=0}^{i} a_j = a_0\left(1+\sum_{j=0}^{i}\left(\prod_{k=0}^{j} t_j\right)\right)$$

$$\overset{(\{t_n\}\searrow)}{<} a_0\left(1+\sum_{j=0}^{i} t_0^{j+1}\right) = \frac{1-t_0^{i+2}}{1-t_0}a_0 < \frac{a_0}{1-t_0} = R.$$

In consequence, $x_{i+2} \in B(x_0, R) \subseteq \Omega$. This completes the induction.

Secondly, we prove that (2) is a Cauchy sequence. To do this, we have, for $m \geqslant 1$ and $n \geqslant 1$,

$$\|x_{n+m}-x_n\| \leqslant \|x_{n+m}-x_{n+m-1}\| + \|x_{n+m-1}-x_{n+m-2}\| + \cdots + \|x_{n+1}-x_n\|$$

$$\overset{(\mathrm{II})}{\leqslant} \sum_{i=n}^{n+m-1} a_i \leqslant a_0 t_0^n \frac{1-t_0^m}{1-t_0}.$$

Thus (2) is a Cauchy sequence.

Thirdly, we show that x^* is a solution of (1). As $\|\Gamma_n F(x_n)\| \to 0$ when $n \to \infty$, if we take into account that

$$\|F(x_n)\| \leqslant \|F'(x_n)\|\|\Gamma_n F(x_n)\| = \|F'(x_n)\|\|x_{n+1}-x_n\|$$

and $\{\|F'(x_n)\|\}$ is bounded, since

$$\|F'(x_n)\| \overset{(\mathrm{C}_3)}{\leqslant} \|F'(x_0)\| + \omega(\|x_n - x_0\|) < \|F'(x_0)\| + \omega(R),$$

it follows that $\|F(x_n)\| \to 0$ when $n \to \infty$. In consequence, we obtain $F(x^*) = 0$ by the continuity of F in $\overline{B(x_0, R)}$. □

REMARK 1 If $b_0 = 1 - h(t_0) < \frac{1}{1+H}$, it follows, similarly to the previous theorem, that Newton's sequence is convergent.

2.4 *Uniqueness of the solution*

Now we provide a result about the uniqueness of the solution x^* of (1).

THEOREM 2.3 Let us suppose that conditions (C_1)–(C_4) hold. Assume that there exists a positive root of the equation

$$2\beta\omega(R+r)\int_{1/2}^{1} h(t)\,\mathrm{d}t = 1. \tag{13}$$

Then, the solution x^* of (1) is unique in $\Omega_0 = B(x_0, r) \cap \Omega$.

Proof. To prove the uniqueness of solution x^*, we assume that z^* is another solution of (1) in $\Omega_0 = B(x_0, r) \cap \Omega$. Then, from the approximation

$$0 = \Gamma_0[F(z^*) - F(x^*)] = \left[\int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\,\mathrm{d}t\right](z^* - x^*) = P(z^* - x^*),$$

we have to prove that the operator $P = \int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\,\mathrm{d}t$ is invertible; then $z^* = x^*$. By the Banach lemma, we only have to note that $\|I - P\| < 1$. Indeed,

$$\begin{aligned}
\|I - P\| &\leqslant \|\Gamma_0\| \int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\|\,\mathrm{d}t \\
&\leqslant \beta \int_0^1 \omega(\|x_0 - x^* - t(z^* - x^*)\|)\,\mathrm{d}t \\
&\leqslant \beta \int_0^1 \omega(\|(1-t)(x_0 - x^*) - t(z^* - x_0)\|)\,\mathrm{d}t \\
&\leqslant \beta \int_0^1 \omega((1-t)\|x^* - x_0\| + t\|z^* - x_0\|)\,\mathrm{d}t \\
&\leqslant \beta \int_0^{1/2} \omega((1-t)(\|x^* - x_0\| + \|z^* - x_0\|))\,\mathrm{d}t \\
&\quad + \beta \int_{1/2}^1 \omega(t(\|x^* - x_0\| + \|z^* - x_0\|))\,\mathrm{d}t \\
&< \beta \int_0^{1/2} h(1-t)\omega(R + r)\,\mathrm{d}t + \beta \int_{1/2}^1 h(t)\omega(R + r)\,\mathrm{d}t \\
&= 2\beta\omega(R + r) \int_{1/2}^1 h(t)\,\mathrm{d}t = 1.
\end{aligned} \tag{14}$$

This completes the proof. □

Observe that the previous r, which satisfies (13), exists if

$$\omega(R) < \frac{1}{2\beta \int_{1/2}^1 h(t)\,\mathrm{d}t},$$

since ω is a non-decreasing function. Moreover, r is unique. If this condition is not satisfied, r does not exist.

From (14), it is easy to see that the uniqueness of the solution is guaranteed in $B(x_0, R)$ if $\omega(R) = 1/\beta$.

3. Application to a nonlinear integral equation of Hammerstein type

An interesting possibility arising from the study of the convergence of iterative methods for solving equations is to obtain results of existence and uniqueness of solutions for different types of equations. In this section, we provide some results of this type for a nonlinear

Hammerstein integral equation of the second kind (Polyanin & Manzhirov, 1998):

$$x(s) = l(s) + \int_a^b G(s,t)\Phi(t, x(t))\,\mathrm{d}t, \quad s \in [a,b],$$

for $x \in C[a,b]$, where $G(s,t)$ is the kernel of a linear integral operator in $C[a,b]$ and $\Phi(t,u)$ is a continuous function for $t \in [a,b]$ and $-\infty < u < +\infty$.

In this study, we consider

$$x(s) = l(s) + \int_a^b G(s,t)[x(t)^{1+p} + \lambda x(t)^2]\,\mathrm{d}t, \quad p \in [0,1], \quad \lambda \in \mathbb{R}, \tag{15}$$

where l is a continuous function such that $l(s) > 0$, $s \in [a,b]$, and the kernel G is continuous and non-negative in $[a,b] \times [a,b]$.

Note that if $G(s,t)$ is the Green function (Stakgold, 1998)

$$G(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & t \leqslant s, \\ \dfrac{(s-a)(b-t)}{b-a}, & s \leqslant t, \end{cases} \tag{16}$$

equation (15) is equivalent to the following boundary value problem:

$$\begin{cases} x'' = -x^{1+p} - \lambda x^2 \\ x(a) = \upsilon(a), \ x(b) = \upsilon(b). \end{cases}$$

3.1 *Existence and uniqueness of the solution of* (15)

Observe that solving (15) is equivalent to solving (1), where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \quad \Omega = \{x \in C[a,b];\ x(s) > 0,\ s \in [a,b]\}, \tag{17}$$

$$[F(x)](s) = x(s) - l(s) - \int_a^b G(s,t)[x(t)^{1+p} + \lambda x(t)^2]\,\mathrm{d}t, \quad p \in [0,1], \quad \lambda \in \mathbb{R}. \tag{18}$$

We apply the study of the last section to obtain different results on the existence and uniqueness of solutions of (15).

We start by calculating the parameters β and η that appear in the study. Firstly, we have

$$[F'(x)y](s) = y(s) - \int_a^b G(s,t)[(1+p)x(t)^p + 2\lambda x(t)]y(t)\,\mathrm{d}t.$$

Moreover, for fixed $x_0(s)$, we have

$$\|I - F'(x_0)\| \leqslant ((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|)M,$$

where the max-norm is considered and $M = \max_{[a,b]} \int_a^b |G(s,t)|\,\mathrm{d}t$. By the Banach lemma, if $((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|)M < 1$, we obtain

$$\|\Gamma_0\| \leqslant \frac{1}{1 - ((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|)M}.$$

From the definition of the operator F, we have $\|F(x_0)\| \leqslant \|x_0-l\|+(\|x_0^{1+p}\|+|\lambda|\|x_0^2\|)M$, and therefore

$$\|\Gamma_0 F(x_0)\| \leqslant \frac{\|x_0 - l\| + (\|x_0^{1+p}\| + |\lambda|\|x_0^2\|)M}{1 - ((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|)M}.$$

On the other hand,

$$[(F'(x) - F'(y))z](s) = -\int_a^b G(s,t)[(1+p)(x(t)^p - y(t)^p) + 2\lambda(x(t) - y(t))]z(t)\,\mathrm{d}t,$$

and consequently, $\|F'(x) - F'(y)\| \leqslant \omega(\|x - y\|)$, where

$$\omega(z) = ((1+p)z^p + 2|\lambda|z)M. \tag{19}$$

Moreover, $\omega(tz) \leqslant h(t)\omega(z)$, where $h(t) = t^p$, and $H = \int_0^1 h(t)\,\mathrm{d}t = \frac{1}{1+p}$.

Once the parameters β and η are calculated and the function ω is known, we can establish the following result on the existence of the solution for (15) from Theorem 2.2.

THEOREM 3.1 Let F be the operator defined in (17) and (18) and $x_0 \in \Omega$ a point such that $((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|)M < 1$. If $b_0 = \beta\omega(\eta) < m$, where ω is given by (19) and m by (12), $p \in [0, 1]$, and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{(1+p)(1-b_0)}{(1+p)-(2+p)b_0}\eta$, then a solution of (15) exists at least in $\overline{B(x_0, R)}$. Moreover, this solution is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where r is the positive root of

$$2\beta\omega(R + r)(2^{1+p} - 1) = (1+p)2^{1+p}.$$

Observe that Newton's sequence is also convergent if $b_0 = 1 - t_0^p < \frac{1+p}{2+p}$; see Remark 1.

Note also that the bound given for $F(x_0)$ can be improved when the kernel G and the function l are fixed.

3.2 *Example*

If we consider the following particular case of (15):

$$x(s) = 1 + \int_0^1 G(s,t)[1 + x(t)]x(t)\,\mathrm{d}t, \tag{20}$$

where $G(s, t)$ is Green's function defined by (16) and choose $x_0(s) = 1$ for Theorem 3.1, we have

$$\beta = 2, \quad \eta = 1/2, \quad \omega(z) = z/4 \quad \text{and} \quad h(t) = t.$$

Thus, $b_0 = \beta\omega(\eta) = 1/4 \leqslant m = 2/3$, and the hypotheses of Theorem 3.1 are verified. Then (20) has a solution x^* in $\{u \in C[0, 1];\ \|u - 1\| \leqslant 3/5\}$ and it is unique in $\{u \in C[0, 1];\ \|u - 1\| < 31/15\} \cap \Omega$.

4. A particular case

We now consider the particular case in which F' is (K, p)-Hölder continuous, namely F' verifies (3).

A semilocal convergence result is given under condition (3) for the Newton method. Domains of existence and uniqueness for the solution x^* of (1) are also provided. Moreover, we study the R-order of convergence (Potra & Pták, 1984) of Newton's iteration under condition (3) and *a priori* error bounds are obtained.

Rokne is one of the authors that analysed the semilocal convergence of the Newton method for operators F such that F' satisfies condition (3), see Rokne (1972). But he obtains neither the domain of uniqueness of solutions nor the R-order of convergence. We see that we can improve the *a priori* error bounds given in Rokne (1972) by the technique presented here.

Keller (1992) also obtained a semilocal convergence result for the Newton process under condition (3), but did not arrive at the R-order of convergence and the uniqueness domain of the solution x^* of (1). We compare Keller's result and the one presented here. We see that the assumptions required in this paper are milder than Keller's for certain values of $p \in [0, 1]$.

Finally, we apply our semilocal convergence result to a nonlinear Hammerstein integral equation of the second kind (Polyanin & Manzhirov, 1998), and obtain a result on the existence and uniqueness of solutions for this type of equation. The solution of a particular Hammerstein integral equation is then approximated by a discretization process.

4.1 *Semilocal convergence results*

Obviously, if $\omega(z) = Kz$ (K = constant), F' is Lipschitz continuous, and if $\omega(z) = Kz^p$ (K = constant and $p \in [0, 1]$), F' is (K, p)-Hölder continuous. In the first case, the conditions required in Theorem 2.2 reduce to the ones appearing in the Newton–Kantorovich theorem.

COROLLARY 4.1 Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a once Fréchet differentiable operator in an open convex domain Ω. We suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$. Suppose, in addition, that

(a) $\|\Gamma_0\| \leqslant \beta$,
(b) $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
(c) $\|F'(x) - F'(y)\| \leqslant K\|x - y\|, \quad x, y \in \Omega, \quad K \geqslant 0.$

Then, provided $b_0 = \beta\omega(\eta) = \beta K\eta \leqslant 1/2$ and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{2(1-b_0)}{2-3b_0}\eta$, equation (1) has a solution x^* and Newton's process converges to this solution, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$.

REMARK 2 Notice that the solution x^* is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where $r = \frac{6b_0^2-18b_0+8}{3b_0(2-3b_0)}\eta$. See Theorem 2.3.

REMARK 3 In addition, inequalities (I)–(III) appearing in the proof of Theorem 2.2 are reduced to equalities for the polynomial

$$F(x) = \frac{K}{2}x^2 - \frac{x}{\beta} + \frac{\eta}{\beta},$$

so that (I)–(III) are optimal for it, namely (I)–(III) can be written with equalities. Taking this into account, we can improve the *a priori* error bounds given by other authors. Observe that this polynomial is just the one used to construct the majorizing sequences in the usual study of the convergence of Newton's method, see Kantorovich & Akilov (1982).

If we now consider $\omega(z) = Kz^p$, $p \in [0, 1]$, Theorem 2.2 reduces to the following result when $p \in (0, 1]$.

COROLLARY 4.2 Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a once Fréchet differentiable operator in an open convex domain Ω. We suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$. Suppose, in addition, that

(a) $\|\Gamma_0\| \leqslant \beta$,
(b) $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
(c) $\|F'(x) - F'(y)\| \leqslant K\|x - y\|^p, \quad x, y \in \Omega, \quad K \geqslant 0, \quad p \in (0, 1]$.

Then, provided $b_0 = \beta\omega(\eta) = \beta K\eta^p \in (0, \tau]$, where τ is the unique zero of the function

$$\phi(x) = (1 + p)^p(1 - x)^{1+p} - x^p, \quad p \in (0, 1], \tag{21}$$

in the interval $(0, 1/2]$, $p \in (0, 1]$ and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{(1+p)(1-b_0)}{(1+p)-(2+p)b_0}\eta$, the sequence (2), starting at x_0, converges to a solution x^* of (1), and the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$.

REMARK 4 Note, see Theorem 2.3, that the solution x^* is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where

$$r = \left(\frac{1 + p}{2\beta K(1 - 2^{-(1+p)})}\right)^{1/p} - R. \tag{22}$$

REMARK 5 Observe that in the previous corollary we have not included the value $p = 0$ in condition (c). For this value, condition (c) is reduced to

$$\|F'(x) - F'(y)\| \leqslant K, \quad x, y \in \Omega, \quad K \geqslant 0,$$

and the semilocal convergence result for (2) is now:

> Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a Fréchet differentiable operator in an open convex domain Ω. It is supposed that $F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$ and the conditions
>
> (a) $\|\Gamma_0\| \leqslant \beta$,
> (b) $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
> (c) $\|F'(x) - F'(y)\| \leqslant K, \quad x, y \in \Omega, \quad K \geqslant 0$
>
> hold. If $b_0 = \beta K \in (0, 1/2)$ and $\overline{B(x_0, R)} \subseteq \Omega$ with $R = \frac{1-b_0}{1-2b_0}\eta$, the Newton sequence converges, starting at x_0, to a solution x^* of (1). Moreover, x^* and x_n belong to $\overline{B(x_0, R)}$.

REMARK 6 We compare the assumptions required for the convergence of the Newton iteration in Corollary 4.2 and the ones appearing in Theorem 4 of Keller (1992).

Under the same general conditions (a)–(c) appearing in Corollary 4.2, Keller's Theorem 4 requires that

$$b_0 \leqslant \frac{1}{2+p}\left(\frac{p}{1+p}\right)^p,$$

and in Corollary 4.2,

$$b_0 \leqslant \tau,$$

where τ is the unique zero of function (21) in $(0, 1/2]$. Note that the former condition for b_0 is more restrictive than the latter one if $p \in (0{\cdot}2856\ldots, 1]$, since

$$\frac{1}{2+p}\left(\frac{p}{1+p}\right)^p < \tau.$$

In consequence, the chances of finding starting points in the application of Newton's iteration for operators with (K, p)-Hölder continuous first Fréchet derivative and $p \in (0{\cdot}2856\ldots, 1]$, are greater if Corollary 4.2 is applied.

4.2 *Error estimates and R-order of convergence*

In the following, we obtain *a priori* error bounds for the Newton method when it converges to a solution x^* of (1). For that, we use some properties of the sequence $\{b_n\}$ that are provided in the following lemma.

LEMMA 4.3 Let f be the scalar function defined in (8). If $b_0 \in (0, \tau)$, we define $\gamma = b_1/b_0$, and then

(a) $b_n < \gamma^{(1+p)^{n-1}} b_{n-1}$ and $b_n < \gamma^{\frac{(1+p)^n-1}{p}} b_0$, for all $n \geqslant 2$,

(b) $b_n f(b_n) < \gamma^{\frac{(1+p)^n-1}{p}} b_0 f(b_0) = \gamma^{\frac{(1+p)^n}{p}} \frac{1+p}{f(b_0)^{1/p}}$, for all $n \geqslant 1$.

If $b_0 = \tau$, then $b_n f(b_n) = b_0 f(b_0) = \frac{1+p}{f(b_0)^{1/p}}$, for all $n \geqslant 1$.

Proof. Case $b_0 \in (0, \tau)$. The proof of (a) follows by an induction process. If $n = 2$, we have

$$\begin{aligned} b_2 &= h(t_1) b_1 f(b_1) = t_1^p b_1 f(b_1) \\ &= \frac{b_1^{1+p}}{(1+p)^p} f(b_1)^{1+p} = \frac{(\gamma b_0)^{1+p}}{(1+p)^p} < \gamma^{1+p} b_1 = \gamma^{2+p} b_0. \end{aligned}$$

We now suppose that

$$b_{n-1} < \gamma^{(1+p)^{n-2}} b_{n-2} < \gamma^{\frac{(1+p)^{n-1}-1}{p}} b_0.$$

Then, by the same reasoning,

$$\begin{aligned} b_n &= h(t_{n-1})b_{n-1}f(b_{n-1}) = \frac{b_{n-1}^{1+p}}{(1+p)^p} f(b_{n-1})^{1+p} \\ &< \frac{(\gamma^{(1+p)^{n-2}} b_{n-2})^{1+p}}{(1+p)^p} f(\gamma^{(1+p)^{n-2}} b_{n-2})^{1+p} < \gamma^{(1+p)^{n-1}} h(t_{n-2}) b_{n-2} \\ &= \gamma^{(1+p)^{n-1}} b_{n-1} < \gamma^{(1+p)^{n-1}} \gamma^{(1+p)^{n-2}} b_{n-2} < \cdots < \gamma^{\frac{(1+p)^n-1}{p}} b_0. \end{aligned}$$

To prove (b), we observe that

$$\begin{aligned} b_n f(b_n) &< \gamma^{\frac{(1+p)^n-1}{p}} b_0 f(\gamma^{\frac{(1+p)^n-1}{p}} b_0) \\ &< \gamma^{\frac{(1+p)^n-1}{p}} b_0 f(b_0) = \gamma^{\frac{(1+p)^n}{p}} \frac{1+p}{f(b_0)^{1/p}}, \quad n \geqslant 1. \end{aligned}$$

The case $b_0 = \tau$ follows by analogy. □

The recurrence relations (I)–(IV) given in Theorem 2.2 and property (b) of sequence $\{b_n\}$ appearing in the previous lemma are used to obtain the following *a priori* error bounds and the R-order of convergence.

THEOREM 4.4 Under the same conditions as in Corollary 4.2, we have the following *a priori* error estimates:

$$\|x^* - x_n\| \leqslant (\gamma^{\frac{(1+p)^n-1}{p^2}}) \frac{\Delta^n}{1 - \gamma^{\frac{(1+p)^n}{p}} \Delta} \eta, \quad n \geqslant 0, \tag{23}$$

where $\gamma = b_1/b_0$ and $\Delta = (1 - b_0)^{1/p}$. Moreover, sequence (2) has R-order of convergence at least $1 + p$ if $b_0 \in (0, \tau)$ or at least one if $b_0 = \tau$.

Proof. Taking into account that $b_0 \in (0, \tau)$, $\gamma = b_1/b_0 \in (0, 1)$ and $\Delta = 1/f(b_0)^{1/p}$, it follows, for $m \geqslant 1$ and $n \geqslant 1$, that

$$\|x_{n+m} - x_n\| \leqslant a_0 \sum_{i=n-1}^{n+m-2} \left(\prod_{j=0}^{i} t_j \right),$$

see Theorem 2.2. Since $a_0 = \eta$ and $t_j = H b_j f(b_j) = \frac{b_j}{1+p} f(b_j)$, for all $j \geqslant 0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \eta \sum_{i=n-1}^{n+m-2} \left(\prod_{j=0}^{i} \frac{b_j}{1+p} f(b_j) \right) \\ &\overset{\text{lemma 4.3 (b)}}{<} \eta \sum_{i=n-1}^{n+m-2} \left(\prod_{j=0}^{i} \gamma^{\frac{(1+p)^j-1}{p}} \frac{b_0}{1+p} f(b_0) \right) = \eta \sum_{i=n-1}^{n+m-2} \left(\prod_{j=0}^{i} (\gamma^{\frac{(1+p)^j}{p}} \Delta) \right) \\ &= \eta \sum_{i=n-1}^{n+m-2} (\gamma^{\frac{(1+p)^{1+i}-1}{p^2}} \Delta^{1+i}) = \sum_{i=0}^{m-1} (\gamma^{\frac{(1+p)^{n+i}-1}{p^2}} \Delta^{n+i}) \|x_1 - x_0\|. \end{aligned}$$

By Bernoulli's inequality, we have

$$\gamma^{\frac{(1+p)^{n+i}-1}{p^2}} = \gamma^{\frac{(1+p)^n-1}{p^2}} \gamma^{\frac{(1+p)^n}{p^2}((1+p)^i-1)} \leqslant \gamma^{\frac{(1+p)^n-1}{p^2}} \gamma^{\frac{(1+p)^n}{p}i},$$

and consequently,

$$\|x_{n+m} - x_n\| < \left(\sum_{i=0}^{m-1} (\gamma^{\frac{(1+p)^n}{p}i} \Delta^i) \right) \gamma^{\frac{(1+p)^n-1}{p^2}} \Delta^n \eta < \frac{1 - (\gamma^{\frac{(1+p)^n}{p}} \Delta)^m}{1 - \gamma^{\frac{(1+p)^n}{p}} \Delta} \gamma^{\frac{(1+p)^n-1}{p^2}} \Delta^n \eta. \tag{24}$$

By letting $m \to \infty$ in (24), we obtain (23).

Now, from (23), it follows that the R-order of convergence of sequence (2) is at least $1 + p$, since

$$\|x^* - x_n\| \leqslant \frac{\eta}{\gamma^{1/p^2}(1 - \gamma^{1/p}\Delta)} (\gamma^{1/p^2})^{(1+p)^n}, \quad n \geqslant 0.$$

On the other hand, if $b_0 = \tau$, we have that $b_n = b_0 = \tau$, for all $n \geqslant 0$. Following an analogous procedure to the previous one, we obtain the same results, now taking into account that $\gamma = 1$ and $\Delta = \frac{b_0}{1+p} f(b_0) < 1$, except for the R-order of convergence; in this case, it is at least one. □

REMARK 7 Taking into account that estimates regarding consecutive points are optimal to measure $\|x^* - x_n\|$ (see Remark 3), we look for a element x_k $(k > n)$ of the sequence $\{x_n\}$ such that $\|x^* - x_k\|$ is small enough, and $\|x^* - x_n\|$ is measured from the distance between two consecutive points. So,

$$\|x^* - x_n\| \leqslant \|x^* - x_{n+j}\| + \|x_{n+j} - x_{n+j-1}\| + \cdots + \|x_{n+1} - x_n\|, \quad j \geqslant 1, \quad n \geqslant 1,$$

and the error given in (23) is then improved.

4.3 *Application*

We analyse the following particular case of (15):

$$x(s) = l(s) + \int_a^b G(s,t) x(t)^{1+p}\, \mathrm{d}t, \quad p \in [0, 1], \tag{25}$$

where l is a continuous function such that $l(s) > 0$, $s \in [a, b]$, and kernel G is the Green function (16).

Taking into account that

$$[F'(x)y](s) = y(s) - (1 + p) \int_a^b G(s,t) x(t)^p y(t)\, \mathrm{d}t.$$

and

$$[(F'(x) - F'(y))z](s) = -(1 + p) \int_a^b G(s,t)[x(t)^p - y(t)^p] z(t)\, \mathrm{d}t,$$

it follows that

$$\|\Gamma_0\| \leqslant \frac{1}{1-(1+p)M\|x_0^p\|},$$
$$\|\Gamma_0 F(x_0)\| \leqslant \frac{\|x_0 - l\| + M\|x_0^{1+p}\|}{1-(1+p)M\|x_0^p\|}$$

if $(1+p)M\|x_0^p\| < 1$ and

$$\|F'(x) - F'(y)\| \leqslant (1+p)M\|x-y\|^p.$$

Therefore a result of existence and uniqueness of the solution for (25) is obtained once the parameters

$$\beta = \frac{1}{1-(1+p)M\|x_0^p\|}, \quad K = (1+p)M, \quad \eta = \frac{\|x_0 - l\| + M\|x_0^{1+p}\|}{1-(1+p)M\|x_0^p\|}$$

are calculated for Corollary 4.2.

COROLLARY 4.5 Let F be the operator defined in (17) and (18) with $\lambda = 0$, and $x_0 \in \Omega$ a point such that $(1+p)M\|x_0^p\| < 1$. If $b_0 = \beta K \eta^p \in (0, \tau]$, where $p \in (0, 1]$ and τ is the only zero of (21) in $(0, 1/2]$, and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{(1+p)(1-b_0)}{(1+p)-(2+p)b_0}\eta$, then a solution of (25) exists at least in $\overline{B(x_0, R)}$. Moreover, this solution is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where r is defined by (22).

By analogy, if $p = 0$, a similar result can be given, see Remark 5.

Note that the bound given for $F(x_0)$ is improved once the function l is known.

Localization of the solution for a particular case of (25). We now study the following particular case of (25):

$$x(s) = 1 + \int_0^1 G(s,t)x(t)^{3/2}\,\mathrm{d}t, \tag{26}$$

where $G(s, t)$ is Green's function defined in (16). Our immediate goal is to obtain a result for the existence and uniqueness of the solution of (26).

If we run the operations undertaken for (25) with $l(s) = 1$, $G(s, t)$ Green's function given by (16), $p = 1/2$ and $[a, b] = [0, 1]$, we obtain the existence of $\Gamma_0 = F'(x_0)^{-1}$, which is guaranteed by the Banach lemma, since

$$\|[(I - F'(x_0))y](s)\| \leqslant \frac{3}{2}\left|\int_0^1 G(s,t)\,\mathrm{d}t\right| \|x_0^{1/2}\|\|y\|,$$

and $\|I - F'(x_0)\| < 1$ if $\|x_0^{1/2}\| < 16/3$. Moreover

$$\|\Gamma_0\| \leqslant \frac{16}{16 - 3\|x_0^{1/2}\|}.$$

We also have

$$\|F(x_0)\| \leqslant \|x_0 - 1\| + \tfrac{1}{8}\|x_0^{3/2}\|$$

and

$$\|F'(x) - F'(y)\| \leqslant \tfrac{3}{16}\|x - y\|^{1/2}, \quad x, y \in \Omega.$$

As a result, taking into account that

$$\beta = \frac{16}{16 - 3\|x_0^{1/2}\|}, \quad K = 3/16, \quad \eta = \frac{16(\|x_0 - 1\| + \frac{1}{8}\|x_0^{3/2}\|)}{16 - 3\|x_0^{1/2}\|},$$

the following corollary is obtained.

COROLLARY 4.6 With the previous notation, let $F : \Omega \subseteq C[0, 1] \to C[0, 1]$, where

$$\Omega = \{x \in C[0, 1];\ x(s) > 0,\ s \in [0, 1]\},$$

such that

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s, t)x(t)^{3/2}\,\mathrm{d}t.$$

If $x_0 \in \Omega$ satisfies $\|x_0^{1/2}\| < 16/3$, $b_0 = \beta K \eta^{1/2} \leqslant \tau = 0{\cdot}3718\ldots$ and $\overline{B(x_0, R)} \subseteq \Omega$, where $R = \frac{3(1-b_0)}{3-5b_0}\eta$, then a solution of (26) exists at least in $\overline{B(x_0, R)}$, and it is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where

$$r = \left(\frac{4}{\beta(1 - 2^{-3/2})}\right)^2 - R.$$

EXAMPLE If we now choose $x_0(s) = 1$, the previous result is satisfied and we have

$$\beta = 16/13, \quad K = 3/16 \quad \text{and} \quad \eta = 2/13.$$

Therefore $b_0 = \beta K \eta^{1/2} = 0{\cdot}0905\cdots \leqslant \tau = 0{\cdot}3718\ldots$, and the assumptions of Corollary 4.2 hold. Then (26) has a solution x^* in $\{u \in C[0, 1];\ \|u - 1\| \leqslant 0{\cdot}1647\ldots\}$ (see Fig. 1) and it is unique in $\{u \in C[0, 1];\ \|u - 1\| < 25{\cdot}1108\ldots\} \cap \Omega$. Notice that this is an improvement in the domain of existence $\{u \in C[0, 1];\ \|u - 1\| < 0{\cdot}1842\ldots\}$ which Keller would obtain from his Theorem 4.

Error estimates for (26) *by Newton's method.* We can also use Theorem 4.4 to obtain the *a priori* error bounds (23) for (26), which improve the ones obtained by Rokne's technique (Rokne, 1972). See Table 1.

Moreover, taking into account Remark 7, if we consider

$$\|x^* - x_n\| \leqslant \|x^* - x_{n+1}\| + \|x_{n+1} - x_n\|, \quad n \geqslant 1,$$

we obtain better bounds than by (23), see Table 2.

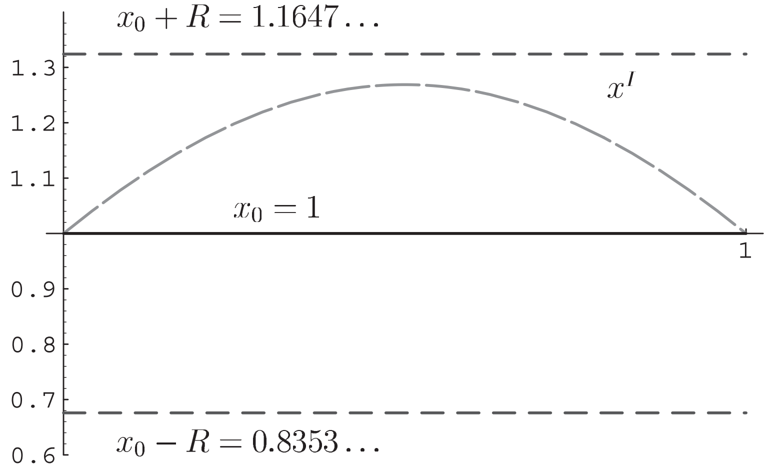

FIG. 1. Approximated solution of equation (26).

TABLE 1 *Error bounds* $\|x^* - x_n\|$ *for* (26)

n	Bounds (23)	Rokne's bounds
1	0·010 403	0·010 973 9
2	0·000 192 358	0·000 730 658
3	$5{\cdot}433\,61 \times 10^{-7}$	0·000 048 6482
4	$9{\cdot}003\,89 \times 10^{-11}$	$3{\cdot}239\,06 \times 10^{-6}$
5	$2{\cdot}112\,29 \times 10^{-16}$	$2{\cdot}156\,61 \times 10^{-7}$

TABLE 2 *New error bounds for* (26)

n	$\|x^* - x_n\|$
1	0·010 399 9
2	0·000 179 584
3	$4{\cdot}174\,52 \times 10^{-7}$
4	$4{\cdot}698\,19 \times 10^{-11}$
5	$5{\cdot}611\,15 \times 10^{-17}$

An arithmetic model to approximate the solution of (26). Finally, we discretize (26) to transform it into a finite-dimensional problem and we apply (2) to obtain an approximated solution. This procedure consists of approximating the integral appearing in (26) by a numerical quadrature formula. To obtain a numerical solution, we use the Gauss–Legendre formula to approximate an integral

$$\int_0^1 v(t)\,\mathrm{d}t \simeq \sum_{i=1}^{m} w_i v(t_i),$$

where the nodes t_i and the weights w_i are determined; in particular, see Table 3 for $m = 8$.

TABLE 3 *Nodes and weights for the Gauss–Legendre formula*

i	t_i	w_i	i	t_i	w_i
1	0·019 855	0·050 614	5	0·591 717	0·181 342
2	0·101 667	0·111 191	6	0·762 766	0·156 853
3	0·237 234	0·156 853	7	0·898 333	0·111 191
4	0·408 283	0·181 342	8	0·980 145	0·050 614

If we denote the approximation of $x(t_j)$ by x_j $(j = 1, 2, \ldots, m)$, (26) is now equivalent to the following nonlinear system of equations:

$$x_j = 1 + \sum_{k=1}^{m} \alpha_{jk} x_k^{3/2}, \quad j = 1, 2, \ldots, m, \tag{27}$$

where

$$\alpha_{jk} = \begin{cases} w_k t_k (1 - t_j) & \text{if } k \leqslant j, \\ w_k t_j (1 - t_k) & \text{if } k < j. \end{cases} \tag{28}$$

System (27) can be written in the form

$$x = \mathbf{1} + Ax^{3/2}, \quad (F(x) \equiv x - \mathbf{1} - Ax^{3/2} = 0),$$

where

$$x = (x_1, x_2, \ldots, x_m)^T, \quad \mathbf{1} = (1, 1, \ldots, 1)^T,$$
$$A = (\alpha_{jk})_{j,k=1}^{m}, \quad x^{3/2} = (x_1^{3/2}, x_2^{3/2}, \ldots, x_m^{3/2})^T.$$

Moreover,

$$F'(x) = I - \tfrac{3}{2} A \cdot \operatorname{diag}\{x_1^{1/2}, x_2^{1/2}, \ldots, x_m^{1/2}\}.$$

Starting at $x^{(0)}$, the iterations of Newton's method are calculated as follows:

(1) solve: $F'(x^{(k)})y^{(k)} = -F(x^{(k)})$;
(2) define: $x^{(k+1)} = y^{(k)} + x^{(k)}$.

For $m = 8$ and taking into account that we have previously considered the starting function $x_0(s) = 1$, we now choose the vector $x^{(0)} = (1, 1, \ldots, 1)^T$ as the initial iterate. We then obtain the numerical solution appearing in Table 4.

We now interpolate the points of Table 4. Taking into account that the solution of (26) satisfies $x(0) = 1 = x(1)$, an approximation x^I of the numerical solution is obtained, see Fig. 1. Notice that the interpolated approximation x^I lies within the existence domain of the solutions obtained in Corollary 4.6.

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TABLE 4 *Numerical solution of* (26)

i	x_i	i	x_i
1	1·011 48	5	1·147 81
2	1·054 58	6	1·109 66
3	1·109 66	7	1·054 58
4	1·147 81	8	1·011 48

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