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ON AN APPLICATION OF NEWTON'S METHOD TO NONLINEAR OPERATORS WITH w-CONDITIONED SECOND DERIVATIVE *

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Abstract.

We present a new approach to study the convergence of Newton's method in Banach spaces, which relax the conditions appearing in the usual studies. The approach is based on the bound required for the second derivative of the operator involved. An application to a nonlinear integral equation is presented.

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1 Introduction.

Consider a nonlinear operator equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear twice Fréchet differentiable operator defined on an open convex domain Ω of a Banach space X with values in a Banach space Y. Under certain conditions, Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geq 0, \quad \text{given } x_0, \tag{1.2}$$

produces a sequence $\{x_n\}$ which converges to a solution x^* of (1.1). In (1.2), $F'(x_n)$ denotes the first Fréchet derivative of the nonlinear operator F at the point x_n. The first convergence theorem for Newton's method in Banach spaces is due to Kantorovich [3]. The Kantorovich theorem gives sufficient conditions under which (1.1) has a unique solution x^* in a certain neighborhood of x_0. The main condition required is

$$\|F''(x)\| \leq K \text{ in some closed ball } \overline{B(x_0, \rho)}. \tag{1.3}$$

Other approaches in studying the convergence of Newton's method were considered by different authors. For example, in [5], Ortega changes condition (1.3) to

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$$\|F'(x) - F'(y)\| \le K\|x - y\|, \quad x, y \in \Omega. \tag{1.4}$$

Others [1, 2, 4, 6] consider a generalization of the last conditions, which is given by

$$\|F'(x) - F'(y)\| \le K\|x - y\|^p, \quad x, y \in \Omega, \quad p \in [0, 1]. \tag{1.5}$$

In practice, the verification of these types of conditions are difficult for some problems, since several technical difficulties are encountered.

In this paper, we use different conditions to the previous ones to study the convergence of Newton's method to a solution of (1.1). We relax these usual conditions to facilitate the convergence of Newton's method. So, the main condition required is

$$\|F''(x)\| \le \omega(\|x\|), \quad x, y \in \Omega,$$

where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ is a continuous real function such that $\omega(0) \ge 0$ and ω is monotonous, i.e. non-decreasing or non-increasing. When this condition is satisfied, we say that F'' is w-conditioned.

From this study, we obtain domains of existence and uniqueness of solution for (1.1) and an application to a nonlinear integral equation is provided, where the advantage of this study is shown.

Throughout the paper we denote

$$\overline{B(x,r)} = \{y \in X; \|y - x\| \le r\} \quad \text{and} \quad B(x,r) = \{y \in X; \|y - x\| < r\}.$$

2 The main results.

Under certain conditions for the pair (F, x_0), we study the convergence of the Newton method to a unique solution of equation (1.1). From some real parameters, a system of four recurrence relations is constructed in which one sequence of positive real numbers are involved. The convergence of Newton's sequence (1.2) is then guaranteed from it.

2.1 Semilocal convergence result.

Let $x_0 \in \Omega$ and suppose that $F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X. We also assume the following:

(C$_1$) $\|\Gamma_0\| \le \beta$,

(C$_2$) $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \le \eta$,

(C$_3$) $\|F''(x)\| \le \omega(\|x\|), x, y \in \Omega$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ is a continuous real function such that $\omega(0) \ge 0$ and ω is monotonous, i. e. non-decreasing or non-increasing,

(C$_4$) The equation

$$3\beta\eta\varphi(t)t - 2\beta\eta^2\varphi(t) - 2t + 2\eta = 0 \tag{2.1}$$

has at least one positive root, where

$$\varphi(t) = \begin{cases} \omega(\|x_0\| + t) & \text{if } \omega \text{ is non-decreasing,} \\ \omega(\|x_0\| - t) & \text{if } \omega \text{ is non-increasing.} \end{cases}$$

We denote the smallest positive root of this equation by R. Note that the R root must be less than $\|x_0\|$ if ω is non-increasing.

(C$_5$) $B(x_0, R) \subset \Omega$.

We now denote $\alpha_0 = \beta\eta\varphi(R)$ and define the following scalar sequence:

$$\alpha_n = \frac{\alpha_{n-1}^2}{2} f(\alpha_{n-1})^2, \quad n \geq 1, \tag{2.2}$$

where

$$f(x) = (1 - x)^{-1}. \tag{2.3}$$

Observe that we consider the case $\alpha_0 > 0$, since if $\alpha_0 = 0$, a trivial problem results, as the solution of equation (1.1) is x_0.

Our first goal is to analyse real sequence (2.2) to obtain the convergence of sequence (1.2) defined in Banach spaces. To obtain the convergence of (1.2), we have to prove that it is a Cauchy sequence.

First of all, some properties of the sequence $\{\alpha_n\}$ are shown.

LEMMA 2.1. *Let f be the scalar function defined in* (2.3)*.*

(a) *If $\alpha_0 < 1/2$, the sequence $\{\alpha_n\}$ is strictly decreasing and $\alpha_n < 1/2$, for all $n \geq 0$.*

(b) *If $\alpha_0 = 1/2$, then $\alpha_n = \alpha_0 = 1/2$, for all $n \geq 1$.*

PROOF. We first consider the case $\alpha_0 < 1/2$. Item (a) is proved by mathematical induction on n. As $\alpha_0 < 1/2$, from (2.2), $\alpha_1 < \alpha_0$. Next, we suppose that $\alpha_i < \alpha_{i-1}$, for all $i = 1, 2, \ldots, n$. Thus

$$\alpha_{n+1} = \frac{\alpha_n^2}{2} f(\alpha_n)^2 < \frac{\alpha_{n-1}^2}{2} f(\alpha_{n-1})^2 = \alpha_n,$$

since the function f, given by (2.3), is increasing in $[0, 1)$. Consequently, the sequence $\{\alpha_n\}$ is strictly decreasing, and then (a) holds.

Secondly, if $\alpha_0 = 1/2$, **(b)** follows immediately. □

Next, we give two technical lemmas.

LEMMA 2.2. *Let us consider* **(C$_1$)**–**(C$_5$)** *and* (2.3)*. Then*

$$\sum_{i=0}^{n} \left(\frac{\alpha_0}{2} f(\alpha_0)\right)^i \eta = \frac{1 - \left(\frac{\alpha_0}{2} f(\alpha_0)\right)^{n+1}}{1 - \frac{\alpha_0}{2} f(\alpha_0)} \eta < \frac{2(1 - \alpha_0)}{2 - 3\alpha_0} \eta = R.$$

LEMMA 2.3. *Under the conditions of the previous lemma, we have:*

(a) *There exists* Γ_n *and* $\|\Gamma_n\| \leq f(\alpha_{n-1})\|\Gamma_{n-1}\|$, $n \geq 1$,

(b) $\|x_{n+1} - x_n\| < \dfrac{\alpha_{n-1}}{2} f(\alpha_{n-1})\|x_n - x_{n-1}\|$, $n \geq 1$,

(c) $\varphi(R)\|\Gamma_n\|\|x_{n+1} - x_n\| < \alpha_n$, $n \geq 1$,

(d) $\|x_{n+1} - x_0\| < \left[1 + \displaystyle\sum_{i=0}^{n} \left(\prod_{j=0}^{i} \left(\frac{\alpha_j}{2} f(\alpha_j) \right) \right) \right] \eta < R$, $n \geq 1$.

PROOF. Note that $\|x_1 - x_0\| \leq \eta < R$, then $x_1 \in B(x_0, R)$. Now, as Γ_0 exists, by the Banach lemma, we have that Γ_1 is defined and

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - \|I - \Gamma_0 F'(x_1)\|} \leq f(\alpha_0)\|\Gamma_0\|,$$

since $x_0 + t(x_1 - x_0) \in B(x_0, R)$, for $t \in [0, 1]$, and

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|x_1 - x_0\| \int_0^1 \|F''(x_0 + t(x_1 - x_0))\|\, dt \leq \beta\eta\varphi(R) = \alpha_0 < 1.$$

From Taylor's formula and (1.2), it follows that

$$F(x_1) = \int_0^1 F''(x_0 + t(x_1 - x_0))(1 - t)(x_1 - x_0)^2\, dt,$$

and consequently,

$$\begin{aligned} \|F(x_1)\| &\leq \left(\int_0^1 \|F''(x_0 + t(x_1 - x_0))\|(1 - t)\, dt \right) \|x_1 - x_0\|^2 \\ &\leq \frac{\varphi(\eta)}{2}\|x_1 - x_0\|^2 < \frac{\eta}{2}\varphi(R)\|x_1 - x_0\|, \end{aligned}$$

since $x_1, x_0 + t(x_1 - x_0) \in B(x_0, R)$. Thus

$$\|x_2 - x_1\| \leq \|\Gamma_1\|\|F(x_1)\| < \frac{\alpha_0}{2} f(\alpha_0)\|x_1 - x_0\|,$$

$$\varphi(R)\|\Gamma_1\|\|x_2 - x_1\| < \frac{\alpha_0^2}{2} f(\alpha_0)^2 = \alpha_1,$$

and, by Lemma 2.2,

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| < \left(1 + \frac{\alpha_0}{2} f(\alpha_0)\right) \eta < R.$$

As a result, $x_2 \in B(x_0, R)$ and (b), (c) and (d) hold.

If we now suppose that (a)–(d) are true for some $n = 1, 2, \ldots, i$, we see that (a)–(d) also hold for $n = i + 1$.

Observe that

$$\|I - \Gamma_i F'(x_{i+1})\| \le \varphi(R)\|\Gamma_i\|\|x_{i+1} - x_i\| < \alpha_i \le 1/2.$$

Then, by the Banach lemma, Γ_{i+1} is defined and

$$\|\Gamma_{i+1}\| \le f(\alpha_i)\|\Gamma_i\|.$$

Taking now into account Taylor's formula and (1.2), we have

$$\begin{aligned} \|F(x_{i+1})\| &= \left\| \int_0^1 F''(x_i + t(x_{i+1} - x_i))(1-t)(x_{i+1} - x_i)^2\, dt \right\| \\ &\le \frac{\varphi(R)}{2}\|x_{i+1} - x_i\|^2, \end{aligned} \tag{2.4}$$

since $x_i + t(x_{i+1} - x_i) \in B(x_0, R)$. Therefore

$$\|x_{i+2} - x_{i+1}\| \le \|\Gamma_{i+1}\|\|F(x_{i+1})\| < \frac{\alpha_i}{2} f(\alpha_i)\|x_{i+1} - x_i\|,$$

$$\varphi(R)\|\Gamma_{i+1}\|\|x_{i+2} - x_{i+1}\| < \varphi(R) f(\alpha_i)\|\Gamma_i\|\frac{\alpha_i}{2} f(\alpha_i)\|x_{i+1} - x_i\| < \frac{\alpha_i^2}{2} f(\alpha_i)^2 = \alpha_{i+1}$$

and

$$\begin{aligned} \|x_{i+2} - x_0\| &\le \|x_{i+2} - x_{i+1}\| + \|x_{i+1} - x_0\| \\ &\le \frac{\alpha_i}{2} f(\alpha_i)\|x_{i+1} - x_i\| + \left[1 + \sum_{j=0}^{i-1}\left(\prod_{k=0}^{j}\left(\frac{\alpha_k}{2} f(\alpha_k)\right)\right)\right]\|x_1 - x_0\| \\ &\le \prod_{k=0}^{i}\left(\frac{\alpha_k}{2} f(\alpha_k)\right)\|x_1 - x_0\| + \left[1 + \sum_{j=1}^{i-1}\left(\prod_{k=0}^{j}\left(\frac{\alpha_k}{2} f(\alpha_k)\right)\right)\right]\|x_1 - x_0\| \\ &\le \left[1 + \sum_{j=1}^{i}\left(\prod_{k=0}^{j}\left(\frac{\alpha_k}{2} f(\alpha_k)\right)\right)\right]\eta < \left[1 + \sum_{j=1}^{i}\left(\frac{\alpha_0}{2} f(\alpha_0)\right)^j\right]\eta < R, \end{aligned}$$

by Lemma 2.2. Consequently, $x_{i+2} \in B(x_0, R)$. This completes the induction. □

We are now ready to prove a semilocal convergence theorem for Newton's method when it is applied to operators that satisfy conditions **(C_1)–(C_5)**.

THEOREM 2.4. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a twice Fréchet differentiable operator in an open convex domain Ω. We suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$ and conditions **(C_1)–(C_5)** hold. If $\alpha_0 = \beta\eta\varphi(R) \in (0, 1/2)$ and $\overline{B(x_0, R)} \subseteq \Omega$, then sequence* (1.2)*, starting from x_0, converges to a solution x^* of equation* (1.1).

PROOF. From Lemma 2.3 it follows that (1.2) is a Cauchy sequence. Indeed, for $m \ge 1$ and $n \ge 1$, we have

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\|$$

$$\leq \sum_{i=n-1}^{n+m-2} \left(\prod_{j=0}^{i} \frac{\alpha_j}{2} f(\alpha_j) \right) \|x_1 - x_0\| < \left(\frac{\alpha_0}{2} f(\alpha_0) \right)^n \frac{1 - \left(\frac{\alpha_0}{2} f(\alpha_0) \right)^m}{1 - \frac{\alpha_0}{2} f(\alpha_0)} \eta,$$

since $\{\alpha_n\}$ is decreasing and $\alpha_0 < 1/2$.

As $\{x_n\}$ is a Cauchy sequence, we have that $\lim_n x_n = x^*$ and, by letting $n \to \infty$ in (2.4), we obtain $F(x^*) = 0$, by the continuity of F in $\overline{B(x_0, R)}$. □

Remark 2.1. If $\alpha_0 = 1/2$, Newton's sequence is also convergent.

Remark 2.2. Observe that if $R < 2\eta$, then $\alpha_0 < 1/2$.

2.2 Uniqueness of the solution.

Now we provide a result about the uniqueness of the solution x^* of (1.1).

Theorem 2.5. *Let us suppose that conditions* $(\mathbf{C}_1)$–$(\mathbf{C}_5)$ *hold. Then, the solution x^* of equation* (1.1) *is unique in $\Omega_0 = B(x_0, r) \cap \Omega$, where r is the smallest positive root of*

$$\beta \int_0^1 \int_0^1 \varphi(s(R + t(r - R)))\, ds\, (R + t(r - R)\, dt = 1. \tag{2.5}$$

Proof. To prove the uniqueness of solution x^*, we assume that z^* is another solution of (1.1) in $\Omega_0 = B(x_0, r) \cap \Omega$. Then, from the approximation

$$0 = \Gamma_0[F(z^*) - F(x^*)] = \left[\int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\, dt \right] (z^* - x^*) = P(z^* - x^*),$$

we have to prove that operator $P = \int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\, dt$ is invertible; then $z^* = x^*$. By the Banach lemma, we only have to note that $\|I - P\| < 1$. Indeed,

$$\begin{aligned}
\|I - P\| &\leq \|\Gamma_0\| \left\| \int_0^1 F'(x^* + t(z^* - x^*)) - F'(x_0)\, dt \right\| \\
&\leq \beta \left\| \int_0^1 \int_0^1 F''(x_0 + s((1-t)(x^* - x_0) + t(z^* - x_0)))\, ds \right. \\
&\qquad \left. \times ((1-t)(x^* - x_0) + t(z^* - x_0))\, dt \right\| \\
&< \beta \int_0^1 \int_0^1 \|F''(x_0 + s((1-t)(x^* - x_0) + t(z^* - x_0))\|\, ds\, (R + t(r - R))\, dt \\
&\leq \beta \int_0^1 \int_0^1 \varphi(s(R + t(r - R)))\, ds\, (R + t(r - R))\, dt = 1. \qquad (2.6)
\end{aligned}$$

The proof is then complete. □

From (2.6), it is easy to see that the uniqueness of the solution is guaranteed in $B(x_0, R)$ if $\int_0^1 \varphi(sR)\, ds \leq 1/(R\beta)$.

3 Application to a nonlinear integral equation of Hammerstein type.

An interesting possibility arising from the study of the convergence of iterative methods for solving equations is to obtain results of existence and uniqueness of solutions for different types of equations. In this section, we provide some results of this type for a nonlinear Hammerstein integral equation of the second kind:

$$x(s) = \ell(s) + \int_a^b G(s,t)\Phi(t, x(t))\, dt, \quad s \in [a,b],$$

for $x \in C[a,b]$, where $G(s,t)$ is the kernel of a linear integral operator in $C[a,b]$ and $\Phi(t,u)$ is a continuous function for $t \in [a,b]$ and $-\infty < u < +\infty$.

In this study, we consider

$$x(s) = \ell(s) + \int_a^b G(s,t)[x(t)^{1+p} + \lambda x(t)^2]\, dt, \quad p \in [0,1], \quad \lambda \in \mathbb{R}, \tag{3.1}$$

where ℓ is a continuous function such that $\ell(s) > 0$, $s \in [a,b]$, and the kernel G is continuous and non-negative in $[a,b] \times [a,b]$.

Note that if $G(s,t)$ is the Green function

$$G(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & t \le s, \\[2ex] \dfrac{(s-a)(b-t)}{b-a}, & s \le t, \end{cases} \tag{3.2}$$

equation (3.1) is equivalent to the following boundary value problem:

$$\begin{cases} x'' = -x^{1+p} - \lambda x^2, \\ x(a) = \upsilon(a),\ x(b) = \upsilon(b). \end{cases}$$

Observe that solving equation (3.1) is equivalent to solving (1.1), where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \quad \Omega = \{x \in C[a,b];\ x(s) > 0, \quad s \in [a,b]\}, \tag{3.3}$$

$$[F(x)](s) = x(s) - \ell(s) - \int_a^b G(s,t)[x(t)^{1+p} + \lambda x(t)^2]\, dt, \quad p \in [0,1], \quad \lambda \in \mathbb{R}. \tag{3.4}$$

3.1 Existence and uniqueness of the solution of (3.1).

The interest of this type of integral equations is that, given (3.4), the first Fréchet-derivative of this operator is neither Lipschitz continuous nor (K,p)-Hölder continuous and the usual studies for Newton's method are not applicable.

We apply the study of the last section to obtain different results on the existence and uniqueness of solutions of equation (3.1).

We start calculating the parameters β and η that appears in the study. Firstly, we have

$$[F'(x)y](s) = y(s) - \int_a^b G(s,t)[(1+p)x(t)^p + 2\lambda x(t)]y(t)\, dt.$$

Moreover, for fixed $x_0(s)$, we have

$$\|I - F'(x_0)\| \le M\left((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|\right),$$

where the max-norm is considered and $M = \max_{[a,b]} \int_a^b |G(s,t)|\,dt$. By the Banach lemma, if $M\left((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|\right) < 1$, we obtain

$$\|\Gamma_0\| \le \frac{1}{1 - M\left((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|\right)}.$$

From the definition of the operator F, we have

$$\|F(x_0)\| \le \|x_0 - f\| + M\big(\|x_0^{1+p}\| + |\lambda|\|x_0^2\|\big),$$

and therefore

$$\|\Gamma_0 F(x_0)\| \le \frac{\|x_0 - \ell\| + M\big(\|x_0^{1+p}\| + |\lambda|\|x_0^2\|\big)}{1 - M\big((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|\big)}.$$

On the other hand,

$$[F''(x)(yz)](s) = -\int_a^b G(s,t)[(1+p)px(t)^{p-1} + 2\lambda]z(t)y(t)\,dt,$$

and consequently, $\|F''(x)\| \le \omega(\|x\|)$, $x \in \Omega$, where

$$\omega(z) = M\left((1+p)pz^{p-1} + 2|\lambda|\right). \tag{3.5}$$

Observe that the function ω given by (3.5) is non-decreasing, and therefore $\varphi(t) = \omega(\|x_0\| - t)$.

Once parameters β and η are calculated and function ω is known, we can already establish the following result on the existence of the solution for equation (3.1) from Theorem 2.4.

THEOREM 3.1. *Let F be the operator defined in* (3.3) *and* (3.4) *and $x_0 \in \Omega$ a point such that $M\left((1+p)\|x_0^p\| + 2|\lambda|\|x_0\|\right) < 1$. If $\alpha_0 = \beta R\varphi(R) < 1/2$, where $\varphi(t) = \omega(\|x_0\| - t)$ and ω is given by* (3.5)*, $R < \|x_0\|$, $p \in [0,1]$ and $\overline{B(x_0,R)} \subseteq \Omega$, where R is the smallest positive root of equation* (2.1)*, then a solution of* (3.1) *exists at least in $\overline{B(x_0,R)}$.*

Observe that Newton's sequence is also convergent if $\alpha_0 = 1/2$, see Remark 2.1. Note also that the bound given for $F(x_0)$ can be improved when kernel G and function ℓ are fixed.

REMARK 3.1. According to Theorem 2.5, the solution of (3.1) is unique in $\Omega_0 = B(x_0,r) \cap \Omega$, where r is the smallest positive root of

$$M\beta \int_0^1\int_0^1 \Big[(1+p)p\left(\|x_0\| - s(R+t(r-R))\right)^{p-1} + 2|\lambda|\Big]\,ds\,(R+t(r-R))\,dt = 1.$$

3.2 Localization of the solution for a particular case of (3.1).

We study the following particular case of equation (3.1):

$$x(s) = 1 + \int_0^1 G(s,t)[x(t)^{3/2} + \lambda x(t)^2]\,dt, \quad \lambda \in \mathbb{R}, \tag{3.6}$$

where $G(s,t)$ is the Green function (3.2). Our immediate goal is to obtain a result, depending on the parameter λ, for the existence and uniqueness of the solution of equation (3.6). Next, (3.6) is discretized and (1.2) is applied to obtain an approximate solution.

If we run the operations undertaken for (3.1) with $\ell(s) = 1$, $G(s,t)$ the Green function (3.2), $p = 1/2$ and $[a,b] = [0,1]$, we obtain the existence of $\Gamma_0 = F'(x_0)^{-1}$, which is guaranteed by the Banach lemma, since

$$\|[(I - F'(x_0))y](s)\| \leq \frac{1}{8}\left(\frac{3}{2}\left\|x_0^{1/2}\right\| + 2|\lambda|\|x_0\|\right)\|y\|,$$

and $\|I - F'(x_0)\| < 1$ if

$$|\lambda| < \frac{16 - 3\left\|x_0^{1/2}\right\|}{4\|x_0\|}. \tag{3.7}$$

Moreover

$$\|\Gamma_0\| \leq \frac{16}{16 - 3\left\|x_0^{1/2}\right\| - 4|\lambda|\|x_0\|}.$$

We also have

$$\|F(x_0)\| \leq \|x_0 - 1\| + \frac{1}{8}\left(\left\|x_0^{3/2}\right\| + |\lambda|\left\|x_0^2\right\|\right)$$

and

$$\|F''(x)\| \leq \frac{1}{8}\left(\frac{3}{4}\|x\|^{-1/2} + 2|\lambda|\right), \quad x \in \Omega.$$

As a result, taking into account that

$$\beta = \frac{16}{16 - 3\left\|x_0^{1/2}\right\| - 4|\lambda|\|x_0\|}, \qquad \eta = 2\frac{8\|x_0 - 1\| + \left\|x_0^{3/2}\right\| + |\lambda|\left\|x_0^2\right\|}{16 - 3\left\|x_0^{1/2}\right\| - 4|\lambda|\|x_0\|},$$

and

$$\omega(z) = \frac{|\lambda|}{4} + \frac{3}{32 z^{1/2}}, \qquad \varphi(t) = \omega(\|x_0\| - t),$$

the following corollary is obtained.

Corollary 3.2. *With the previous notation, let* $F : \Omega \subseteq C[0,1] \to C[0,1]$, *where*

$$\Omega = \{x \in C[0,1];\ x(s) > 0,\ s \in [0,1]\},$$

such that

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s,t)[x(t)^{3/2} + \lambda x(t)^2]\,dt.$$

If $x_0 \in \Omega$, λ *satisfies* (3.7), $\alpha_0 = \beta R\varphi(R) \leq 1/2$, $R < \|x_0\|$ *and* $\overline{B(x_0, R)} \subseteq \Omega$, *where* R *is the smallest positive root of equation* (2.1), *then a solution of* (3.6) *exists at least in* $\overline{B(x_0, R)}$.

Remark 3.2. Following Remark 3.1, the solution of (3.6) is unique in $B(x_0, r) \cap \Omega$, where r is the smallest positive root of

$$\frac{\beta}{8}\int_0^1\int_0^1 \left[\frac{3}{4}\left(\|x_0\| - s(R + t(r-R))\right)^{-1/2} + 2|\lambda|\right] ds\,(R + t(r-R))\,dt = 1.$$

If we now choose $x_0(s) = 1$, the previous result is then satisfied for all λ such that $|\lambda| < 13/4$ and $\alpha_0 \leq 1/2$.

In particular, if we take $\lambda = 1/2$ the last inequalities are verified. For this λ, we have

$$\beta = 16/11, \quad \eta = 3/11 \quad \text{and} \quad \varphi(t) = \omega(1-t) = \frac{1}{8} + \frac{3}{32(1-t)^{1/2}}.$$

Therefore equation (2.1) is now

$$\frac{6}{11} - \frac{254t}{121} - \frac{9t}{121\sqrt{1-t}} + \frac{6t^2}{11} + \frac{9t^2}{22\sqrt{1-t}} = 0$$

and $R = 0.2886\ldots$. Observe that $\|x_0\| = 1 > R = 0.2886\ldots$. Consequently, $\alpha_0 = \beta R\varphi(R) = 0.0991\ldots < 1/2$, and the assumptions of Corollary 3.2 hold. Then (3.6) with $\lambda = 1/2$ has a unique solution x^* in $\{u \in C[0,1];\ \|u - 1\| \leq 0.2886\ldots\}$ (see Figure 3.1).

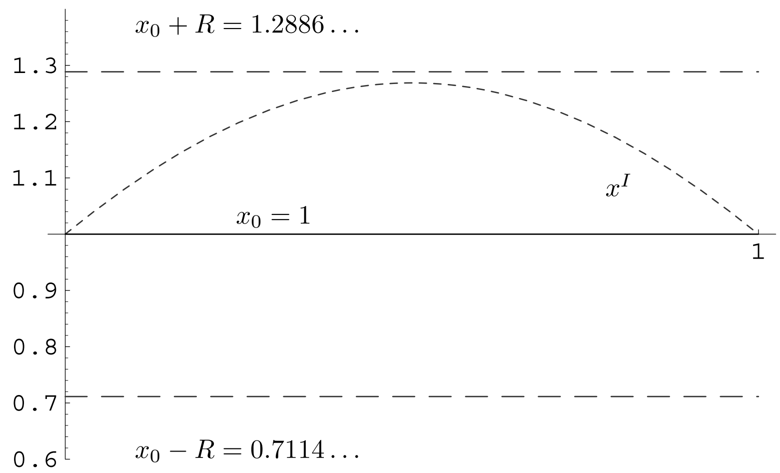

Figure 3.1: Approximate solution of equation (3.6).

Table 3.1: Nodes and weights for the Gauss–Legendre formula.

i	t_i	w_i	i	t_i	w_i
1	0.019855	0.050614	5	0.591717	0.181342
2	0.101667	0.111191	6	0.762766	0.156853
3	0.237234	0.156853	7	0.898333	0.111191
4	0.408283	0.181342	8	0.980145	0.050614

3.3 An arithmetic model for (3.6).

Finally, we discretize (3.6) to transform it into a finite dimensional problem and we apply (1.2) to obtain an approximated solution. This procedure consists of approximating the integral appearing in (3.6) by a numerical quadrature formula. To obtain a numerical solution, we use the Gauss–Legendre formula to approximate an integral

$$\int_0^1 v(t)\,dt \simeq \sum_{i=1}^{m} w_i v(t_i),$$

where the nodes t_i and the weights w_i are determined; in particular, see Table 3.1 for $m = 8$.

If we denote the approximation of $x(t_j)$ by x_j $(j = 1, 2, \dots, m)$, (3.6) is now equivalent to the following nonlinear system of equations:

$$x_j = 1 + \sum_{k=1}^{m} \alpha_{jk} \left(x_k^{3/2} + x_k^2/2 \right), \quad j = 1, 2, \dots, m, \tag{3.8}$$

where

$$\alpha_{jk} = \begin{cases} w_k t_k (1 - t_j) & \text{if } k \le j, \\ w_k t_j (1 - t_k) & \text{if } k < j. \end{cases} \tag{3.9}$$

System (3.8) can be written in the form

$$x = \mathbf{1} + A \left(x^{3/2} + x^2/2 \right), \quad F(x) \equiv x - \mathbf{1} - A \left(x^{3/2} + x^2/2 \right) = 0,$$

where

$$x = (x_1, x_2, \dots, x_m)^T, \quad \mathbf{1} = (1, 1, \dots, 1)^T, \quad A = (\alpha_{jk})_{j,k=1}^{m},$$
$$x^{3/2} + x^2/2 = (x_1^{3/2} + x_1^2/2, x_2^{3/2} + x_2^2/2, \dots, x_m^{3/2} + x_m^2/2)^T.$$

Moreover,

$$F'(x) = I - \frac{1}{2} A \cdot \operatorname{diag} \left\{ 3x_1^{1/2} + 2x_1, 3x_2^{1/2} + 2x_2, \dots, 3x_m^{1/2} + 2x_m \right\}.$$

Starting at $x^{(0)}$, the iterations of Newton's method are calculated as follows:

1. solve: $F'(x^{(k)}) y^{(k)} = -F(x^{(k)})$;
2. define: $x^{(k+1)} = y^{(k)} + x^{(k)}$.

Table 3.2: Numerical solution of (3.6).

i	x_i	i	x_i
1	1.01960	5	1.25935
2	1.09396	6	1.19091
3	1.19091	7	1.09396
4	1.25935	8	1.01960

For $m = 8$ and taking into account that we have previously considered the starting function $x_0(s) = 1$, we now choose the vector $x^{(0)} = (1, 1, \ldots, 1)^T$ as the initial iterate. We then obtain the numerical solution appearing in Table 3.2.

We now interpolate the points of Table 3.2. Taking into account that the solution of (3.6) satisfies $x(0) = 1 = x(1)$, an approximation x^I of the numerical solution is obtained; see Figure 3.1. Notice that the interpolated approximation x^I lies within the existence domain of the solutions obtained in Corollary 3.2.

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