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A uniparametric family of iterative processes for solving nondifferentiable equations ☆

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Abstract

In this work we study a class of secant-like iterations for solving nonlinear equations in Banach spaces. We consider a condition for divided differences which generalizes the usual ones, i.e., Lipschitz and Hölder continuous conditions. A semilocal convergence result is obtained for nondifferentiable operators. For that, we use a technique based on a new system of recurrence relations to obtain domains of existence and uniqueness of the solution. Finally, we apply our results to the numerical solution of several examples.

Keywords: Secant method; Nondifferentiable operator

1. Introduction

Many scientific and engineering problems can be brought in the form of a nonlinear equation

$$H(x) = 0, \tag{1}$$

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where H is a nonlinear operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y. Newton's method [6] is the most used iteration to solve (1) as a consequence of its computational efficiency, even though sometimes less speed of convergence is reached. But this method needs the existence of the first Frechet derivative of the operator H. If we are concerned with approximating a solution x^* of the equation

$$H(x) = F(x) + G(x) = 0, \tag{2}$$

where $F, G : \Omega \subseteq X \to Y$, F is a differentiable operator and G is a continuous operator but nondifferentiable, the Newton method cannot be applied.

The study of this situation has been considered by several authors, for example, in [1] and [8] it is considered a modification of Newton's method given by

$$x_{n+1} = x_n - \big(F'(x_n)\big)^{-1}\big(F(x_n) + G(x_n)\big), \quad x_0 \in \Omega,\ n \geqslant 0. \tag{3}$$

In [3], the author considers the iteration

$$x_{n+1} = x_n - \big(A(x_n)\big)^{-1}\big(F(x_n) + G(x_n)\big), \quad x_0 \in \Omega,\ n \geqslant 0, \tag{4}$$

where $A(x_n)$ denotes a linear operator which is an approximation of the Fréchet derivative of F evaluated at $x = x_n$.

There are several studies (see [2,5,7]) where it is considered the secant method, i.e., $A(x_n) = [x_{n-1}, x_n; H]$ is, in (4), a first order divided difference of H on the points $x_{n-1}, x_n \in \Omega$. This method is defined as a iteration which uses new information at two points, therefore is a multipoint method [4].

In the present paper, we propose the following uniparametric family of multipoint iterations:

$$\begin{cases} x_{-1}, x_0 \in \Omega, \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad \lambda \in [0,1], \\ x_{n+1} = x_n - [y_n, x_n; H]^{-1} H(x_n), \end{cases} \tag{5}$$

which can be considered as a combination of the secant method ($\lambda = 0$) and Newton's one ($\lambda = 1$).

We analyse, under mild assumptions, the semilocal convergence of (5) to a unique solution x^* of (2).

To finish, we study two important applications. Firstly, we obtain a semilocal convergence result under mild conditions and we apply this result to a boundary value problem where the first order divided difference associated to its discretization is not Hölder continuous. Secondly, we consider a nondifferentiable system of nonlinear equations and compare (5) with (3) and (4).

2. Preliminaries

It is well known that the classical secant method is superlinear convergent with R-order $(1+\sqrt{5})/2$ (see [5]). The secant-like methods given in (5) can

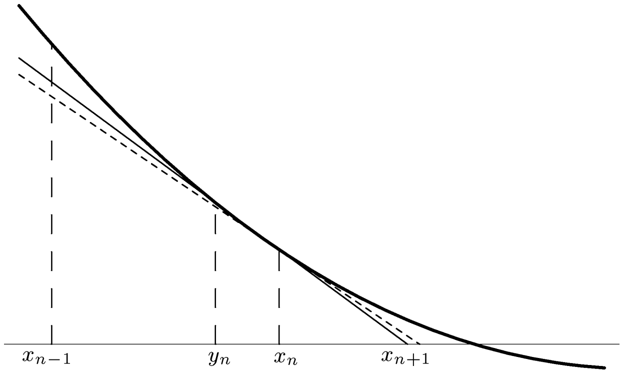

Fig. 1. Secant-like methods.

be considered as generalized secant method since they only use operator values. In the real case, for (5), it is clear that the closer x_n and y_n are, the higher the speed of the convergence is (see Fig. 1).

Moreover, observe that (5) is reduced to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$, since $x_n = y_n$ and $[y_n, x_n; H] = H'(x_n)$.

The use of the secant method is interesting since the calculation of the first derivative H' is not required and the convergence of the successive substitutions method is improved, although it is slower than Newton's one. For this, we consider iteration (5), whose speed of convergence is closed to the one of Newton's method when λ is near 1.

Now, we present some definitions and results that are necessary later.

Let us denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. An operator $[x, y; H] \in \mathcal{L}(X, Y)$ is called a divided difference of first order for the operator H on the points x and y $(x \neq y)$ if the following equality holds:

$$[x, y; H](x - y) = H(x) - H(y). \tag{6}$$

Definition 2.1. We say that the Fréchet-derivative F' is (c, p)-Hölder continuous over the domain Ω if for some $c \geqslant 0$, $p \in [0, 1]$,

$$\left\| F'(x) - F'(y) \right\| \leqslant c\|x - y\|^p, \quad x, y \in \Omega.$$

We then denote $F'(\cdot) \in H_\Omega(c, p)$.

Definition 2.2. Let Ω be a convex open subset of X and we suppose that for each pair of distinct points $x, y \in \Omega$, there exists a first order divided difference of F at these points. If there exists a nonnegative constant k such that

$$\left\| [x, y; F] - [v, w; F] \right\| \leqslant k\left(\|x - v\|^p + \|y - w\|^p\right), \quad p \in [0, 1], \tag{7}$$

for all $x, y, v, w \in \Omega$ with $x \neq y$ and $v \neq w$, we say that F has a Hölder continuous divided difference on Ω. If $p = 1$, we say that F has a Lipschitz continuous divided difference on Ω.

In the previous case, it is known [2] that the Fréchet derivative of F exists in Ω and satisfies

$$[x, x; F] = F'(x), \quad x \in \Omega, \tag{8}$$

and $F'(\cdot) \in H_\Omega(2k, p)$.

In this paper, we relax this requirement and we only assume that the divided difference $[x, y; H]$ satisfies

$$\big\|[x, y; F] - [v, w; F]\big\| \leqslant \omega\big(\|x - v\|, \|y - w\|\big), \quad x, y, v, w \in \Omega, \tag{9}$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in their components.

In the following lemma we will prove that (9) satisfies (8) if $\omega(0, 0) = 0$.

Lemma 2.3. *Let Ω be a convex open subset of X and suppose that, for each pair of points $x, y \in \Omega$, there exists a first order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$ satisfying* (9) *and $\omega(0, 0) = 0$. Then* (8) *is true.*

Proof. Let $\{x_n\} \subseteq \Omega$ be so that $\lim_{n\to\infty} x_n = x$. Let us consider $A_n = [x_n, x; F] \in \mathcal{L}(X, Y)$ and it is verified that

$$\|A_n - A_m\| = \big\|[x_n, x; F] - [x_m, x; F]\big\| \leqslant \omega\big(\|x_n - x_m\|, 0\big).$$

Since $\{x_n\}$ is convergent, it is evident that $\{A_n\}$ is a Cauchy sequence, and therefore there exists $\lim_{n\to\infty} A_n = \tilde{A} \in \mathcal{L}(X, Y)$. So, we can define $[x, x; F] = \tilde{A} = \lim_{n\to\infty} A_n$. Let us check that $\tilde{A} = F'(x)$:

$$\begin{aligned}
&\big\|F(x + \Delta x) - F(x) - [x, x; F](\Delta x)\big\| \\
&\quad = \big\|[x + \Delta x, x; F](\Delta x) - [x, x; F](\Delta x)\big\| \\
&\quad = \big\|\big([x + \Delta x, x; F] - [x, x; F]\big)(\Delta x)\big\| \\
&\quad \leqslant \big\|[x + \Delta x, x; F] - [x, x; F]\big\| \big\|(\Delta x)\big\| \leqslant \omega\big(\|\Delta x\|, 0\big)\|\Delta x\|.
\end{aligned}$$

Then,

$$\begin{aligned}
&\lim_{\|\Delta x\|\to 0} \frac{\big\|F(x + \Delta x) - F(x) - [x, x; F](\Delta x)\big\|}{\|\Delta x\|} \\
&\quad \leqslant \lim_{\|\Delta x\|\to 0} \omega\big(\|\Delta x\|, 0\big) = \omega(0, 0) = 0. \qquad \square
\end{aligned}$$

It is easy to see that condition (9) generalizes condition (7), by only considering $\omega(u_1, u_2) = k(u_1^p + u_2^p)$.

3. A semilocal convergence result

If the operator H is nondifferentiable, we cannot apply Newton's method to approximate the solutions of $H(x) = 0$. However, the last is possible if divided differences are used. Therefore, the condition $\omega(0,0) = 0$ will not required.

So, let us assume that

(I) $\|x_{-1} - x_0\| = \alpha$,
(II) there exists $L_0^{-1} = [y_0, x_0; H]^{-1}$ such that $\|L_0{}^{-1}\| \leqslant \beta$,
(III) $\|L_0{}^{-1}H(x_0)\| \leqslant \eta$,
(IV) $\|[x, y; H] - [v, w; H]\| \leqslant \omega(\|x - v\|, \|y - w\|)$, $x, y, v, w \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in its two arguments.

Now we can already give a semilocal convergence result.

Theorem 3.1. *Under conditions* (I)–(IV), *we assume that, for every pair of distinct points $x, y \in \Omega$, there exists a first order divided difference $[x, y; H] \in \mathcal{L}(X, Y)$. We denote by $m = \max\{\beta\omega((1-\lambda)\alpha, \eta), \beta\omega((1-\lambda)\eta, \eta)\}$ and assume that the equation*

$$u\left(1 - \frac{m}{1 - \beta\omega(u + (1-\lambda)\alpha, u)}\right) - \eta = 0 \tag{10}$$

has at least one positive zero. Let R be the minimum positive one. If

$$\beta\omega\big(R + (1-\lambda)\alpha, R\big) < 1, \qquad M = \frac{m}{1 - \beta\omega(R + (1-\lambda)\alpha, R)} < 1$$

and $\overline{B(x_0, R)} \subset \Omega$, then the sequence $\{x_n\}$, given by (5)*, is well defined, remains in $\overline{B(x_0, R)}$ and converges to the unique solution x^* of Eq.* (2) *in $\overline{B(x_0, R)}$.*

Proof. To simplify the notation, we denote $[y_n, x_n; H] = L_n$. Firstly, we prove, by mathematical induction, that the sequence given in (5) is well defined; namely, iterative procedure (5) makes sense if, at each step, the operator $[y_n, x_n; H]$ is invertible and the point x_{n+1} lies in Ω.

From the initial hypotheses, it follows that x_1 is well defined and $\|x_1 - x_0\| \leqslant \eta < R$. Therefore, $x_1 \in B(x_0, R) \subseteq \Omega$.

Now, using (IV) and assuming that ω is nondecreasing, we obtain

$$\begin{aligned}
\|I - L_0^{-1}L_1\| &\leqslant \|L_0^{-1}\|\|L_0 - L_1\| \leqslant \|L_0^{-1}\|\omega\big(\|y_1 - y_0\|, \|x_1 - x_0\|\big) \\
&\leqslant \|L_0^{-1}\|\omega\big(\lambda\|x_1 - x_0\| + (1-\lambda)\|x_0 - x_{-1}\|, \|x_1 - x_0\|\big) \\
&\leqslant \beta\omega\big(\lambda\eta + (1-\lambda)\alpha, \eta\big) \leqslant \beta\omega\big(R + (1-\lambda)\alpha, R\big) < 1,
\end{aligned}$$

and, by the Banach lemma, L_1^{-1} exists and

$$\|L_1^{-1}\| \leqslant \frac{\beta}{1-\beta\omega(R+(1-\lambda)\alpha, R)}.$$

By (5) and (6), we get

$$H(x_1) = H(x_0) - [x_0, x_1; H](x_0 - x_1) = \big(L_0 - [x_0, x_1; H]\big)(x_0 - x_1).$$

Then, by (IV), we have

$$\begin{aligned}\|H(x_1)\| &\leqslant \big\|[x_0, x_1; H] - L_0\big\| \|x_1 - x_0\| \\ &\leqslant \omega\big(\|x_0 - y_0\|, \|x_1 - x_0\|\big)\|x_1 - x_0\| \\ &\leqslant \omega\big((1-\lambda)\alpha, \eta\big)\|x_1 - x_0\| \leqslant \omega\big(R+(1-\lambda)\alpha, R\big)\|x_1 - x_0\|,\end{aligned}$$

and, consequently, the iterate x_2 is well defined. Moreover,

$$\begin{aligned}\|x_2 - x_1\| &\leqslant \|L_1^{-1}\| \|H(x_1)\| \leqslant \frac{m}{1-\beta\omega(R+(1-\lambda)\alpha, R)}\|x_1 - x_0\| \\ &= M\|x_1 - x_0\| < \eta.\end{aligned}$$

On the other hand, if we take into account that R is a solution of (10), then

$$\begin{aligned}\|x_2 - x_0\| &\leqslant \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leqslant (M+1)\|x_1 - x_0\| \leqslant (M+1)\eta < R\end{aligned}$$

and $x_2 \in B(x_0, R)$.

Then, by induction on n, the following items can be shown for $n \geqslant 1$:

(i$_n$) $\exists L_n^{-1} = [y_n, x_n; H]^{-1}$ such that

$$\|L_n^{-1}\| \leqslant \frac{\beta}{1-\beta\omega(R+(1-\lambda)\alpha, R)}.$$

(ii$_n$) $\|x_{n+1} - x_n\| \leqslant M\|x_n - x_{n-1}\| \leqslant M^n\|x_1 - x_0\| \leqslant \eta$.

Assuming that the linear operators L_j are invertible and $x_{j+1} \in B(x_0, R) \subseteq \Omega$ for all $j = 1, \ldots, n-1$, we obtain

$$\begin{aligned}\|I - L_0^{-1}L_n\| &\leqslant \|L_0^{-1}\| \|L_0 - L_n\| \leqslant \beta\omega\big(\|y_n - y_0\|, \|x_n - x_0\|\big) \\ &\leqslant \beta\omega\big(\|y_n - x_0\| + \|x_0 - y_0\|, \|x_n - x_0\|\big) \\ &\leqslant \beta\omega\big(R+(1-\lambda)\alpha, R\big) < 1\end{aligned}$$

and

$$\|L_n^{-1}\| \leqslant \frac{\beta}{1-\beta\omega(R+(1-\lambda)\alpha, R)}.$$

From the definition of the first divided difference and (5), we can obtain

$$\begin{aligned}H(x_n) &= H(x_{n-1}) - [x_{n-1}, x_n; H](x_{n-1} - x_n)\\ &= \big(L_{n-1} - [x_{n-1}, x_n; H]\big)L_{n-1}^{-1}H(x_{n-1})\\ &= \big(L_{n-1} - [x_{n-1}, x_n; H]\big)(x_{n-1} - x_n).\end{aligned}$$

Taking norms in the above equality and (IV), we obtain

$$\begin{aligned}\big\|H(x_n)\big\| &\leqslant \big\|[x_{n-1}, x_n; H] - L_{n-1}\big\|\|x_n - x_{n-1}\|\\ &\leqslant \omega\big((1-\lambda)\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|\big)\|x_n - x_{n-1}\|\\ &\leqslant \omega\big((1-\lambda)\eta, \eta\big)\|x_n - x_{n-1}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\big\|x_{n+1} - x_n\big\| \leqslant \big\|L_n^{-1}\big\|\big\|H(x_n)\big\| &\leqslant \frac{m}{1 - \beta\omega(R + (1-\lambda)\alpha, R)}\|x_n - x_{n-1}\|\\ &= M\|x_n - x_{n-1}\| \leqslant M^n\|x_1 - x_0\| < \eta.\end{aligned}$$

Consequently, from (10) and (ii), it follows

$$\begin{aligned}&\|x_{n+1} - x_0\|\\ &\quad\leqslant \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_2 - x_1\| + \|x_1 - x_0\|\\ &\quad\leqslant [M^n + M^{n-1} + \cdots + 1]\|x_1 - x_0\| \leqslant \left[\frac{1 - M^{n+1}}{1 - M}\right]\|x_1 - x_0\|\\ &\quad< \frac{1}{1-M}\eta = R.\end{aligned}$$

So, $x_{n+1} \in B(x_0, R)$ and the induction is complete.

Secondly, we prove that $\{x_n\}$ is a Cauchy sequence. For $k \geqslant 1$ we obtain

$$\begin{aligned}&\|x_{n+k} - x_n\|\\ &\quad\leqslant \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \cdots + \|x_{n+1} - x_n\|\\ &\quad\leqslant [M^{k-1} + M^{k-2} + \cdots + 1]\|x_{n+1} - x_n\|\\ &\quad\leqslant \frac{1 - M^k}{1 - M}\|x_{n+1} - x_n\| < \frac{1}{1-M}M^n\|x_1 - x_0\|.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0, R)}$.

Finally, we see that x^* is a zero of H. Since

$$\big\|H(x_n)\big\| \leqslant \omega\big((1-\lambda)\eta, \eta\big)\|x_n - x_{n-1}\|,$$

and $\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$, we obtain $H(x^*) = 0$.

To show the uniqueness, we assume that there exists a second solution $y^* \in \overline{B(x_0, R)}$ and consider the operator $A = [y^*, x^*; H]$. Since $A(y^* - x^*) = H(y^*) - H(x^*)$, if operator A is invertible then $x^* = y^*$. Indeed,

$$\begin{aligned}
&\|L_0^{-1}A - I\| \\
&\quad \leqslant \|L_0^{-1}\| \|A - L_0\| \leqslant \|L_0^{-1}\| \|[y^*, x^*; H] - [y_0, x_0; H]\| \\
&\quad \leqslant \beta\omega\big(\|y^* - y_0\|, \|x^* - x_0\|\big) \leqslant \beta\omega\big(\|y^* - x_0\| + \|x_0 - y_0\|, \|x^* - x_0\|\big) \\
&\quad \leqslant \beta\omega\big(R + (1-\lambda)\alpha, R\big) < 1
\end{aligned}$$

and the operator A^{-1} exists. □

Remark. Note that the operator H is differentiable when the divided differences are Lipschitz or (k, p)-Hölder continuous. But, under condition (IV), H is differentiable if $\omega(0,0) = 0$. Therefore, if $\omega(0,0) \neq 0$, Theorem 3.1 is true for nondifferentiable operators.

4. Applications

We present two types of applications. The first one is theoretical and practical for differentiable operators, where it is proved the convergence for divided differences that are not Lipschitz and Hölder continuous. Moreover, this applications is not usually studied by other authors. The second one is practical for nondifferentiable operators and we compare the methods presented in the paper with other ones given by several authors.

In the first example a differentiable operator is considered, i.e., $H = F$, $G(x) = 0$. We remark that the semilocal convergence conditions required are mild.

4.1. *Example 1*

Now we apply the semilocal convergence result given above to the following boundary value problem:

$$\begin{cases} x'' + x^{1+p} + x^2 = 0, & p \in [0,1], \\ x(0) = x(1) = 0. \end{cases} \tag{11}$$

To solve this problem by finite differences, we start drawing the usual grid line with the grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Note that x_0 and x_n are given by the boundary conditions, then $x_0 = 0 = x_n$. We first approximate the second derivative $x''(t)$ by

$$\begin{aligned}
&x''(t) \approx \big[x(t+h) - 2x(t) + x(t-h)\big]\big/h^2, \\
&x''(t_i) = (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \ldots, n-1.
\end{aligned}$$

Substituting this expression into the differential equation, we have the following system of nonlinear equations:

$$\begin{cases} 2x_1 - h^2x_1^{1+p} - h^2x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - h^2x_i^{1+p} - h^2x_i^2 - x_{i+1} = 0, \quad i = 2, 3, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2x_{n-1}^{1+p} - h^2x_{n-1}^2 = 0. \end{cases} \tag{12}$$

We therefore have an operator $F:\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(x) = L(x) - h^2 f(x)$, where

$$f(x) = \left(x_1^{1+p} + x_1^2, x_2^{1+p} + x_2^2, \ldots, x_{n-1}^{1+p} + x_{n-1}^2\right)^t$$

and

$$L = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix}.$$

Thus

$$F'(x) = L - h^2(1+p)\begin{pmatrix} x_1^p & 0 & \ldots & 0 \\ 0 & x_2^p & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n-1}^p \end{pmatrix} - 2h^2\begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n-1} \end{pmatrix}.$$

Let $x \in \mathbb{R}^{n-1}$ and choose the norm $\|x\| = \max_{1\leqslant i\leqslant n-1} |x_i|$. The corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1\leqslant i\leqslant n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [7]) that F has a divided difference at the points $x, y \in \mathbb{R}^{n-1}$, which is defined by the matrix, whose entries are

$$[x, y; F]_{ij} = \frac{1}{x_j - y_j}\big(F_i(x_1, \ldots, x_j, y_{j+1}, \ldots, y_{n-1}) - F_i(x_1, \ldots, x_{j-1}, y_j, \ldots, y_{n-1})\big).$$

Consequently,

$$[x, y; F] = L - h^2 \begin{pmatrix} \frac{x_1^{1+p} - y_1^{1+p} + x_1^2 - y_1^2}{x_1 - y_1} & 0 & \cdots & 0 \\ 0 & \frac{x_2^{1+p} - y_2^{1+p} + x_2^2 - y_2^2}{x_2 - y_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{x_{n-1}^{1+p} - y_{n-1}^{1+p} + x_{n-1}^2 - y_{n-1}^2}{x_{n-1} - y_{n-1}} \end{pmatrix}.$$

In this case, we have that $[x, y; F] = \int_0^1 F'(x + t(y - x))\,dt$. So we study the value $\|F'(x) - F'(v)\|$ to obtain a bound for $\|[x, y; F] - [v, w; F]\|$.

For all $x, v \in \mathbb{R}^{n-1}$ with $|x_i| > 0$, $|v_i| > 0$ $(i = 1, 2, \ldots, n-1)$, and taking into account the max-norm it follows

$$\begin{aligned} \|F'(x) - F'(v)\| &= \left\|\operatorname{diag}\left\{h^2(1+p)\left(v_i^p - x_i^p\right) + 2h^2(v_i - x_i)\right\}\right\| \\ &= \max_{1 \leqslant i \leqslant n-1} \left|h^2(1+p)\left(v_i^p - x_i^p\right) + 2h^2(v_i - x_i)\right| \\ &\leqslant (1+p)h^2 \max_{1 \leqslant i \leqslant n-1} \left|v_i^p - x_i^p\right| + 2h^2 \max_{1 \leqslant i \leqslant n-1} |v_i - x_i| \\ &\leqslant (1+p)h^2 \Big[\max_{1 \leqslant i \leqslant n-1} |v_i - x_i|\Big]^p + 2h^2\|v - x\| \\ &= (1+p)h^2\|v - x\|^p + 2h^2\|v - x\|. \end{aligned}$$

Therefore

$$\begin{aligned} &\|[x, y; F] - [v, w; F]\| \\ &\quad\leqslant \int_0^1 \left\|F'\big(x + t(y - x)\big) - F'\big(u + t(w - v)\big)\right\| dt \\ &\quad\leqslant h^2 \int_0^1 \Big((1+p)\left\|(1-t)(x - v) + t(y - w)\right\|^p \\ &\qquad\qquad + 2\left\|(1-t)(x - v) + t(y - w)\right\|\Big)\,dt \\ &\quad\leqslant h^2(1+p) \int_0^1 \big((1-t)^p\|x - v\|^p + t^p\|y - w\|^p\big)\,dt \\ &\qquad + 2h^2 \int_0^1 \big((1-t)\|x - v\| + t\|y - w\|\big)\,dt \\ &\quad= h^2\big(\|x - v\|^p + \|y - w\|^p + \|x - v\| + \|y - w\|\big). \end{aligned}$$

From (IV), we consider the function $\omega(u_1, u_2) = h^2(u_1^p + u_2^p + u_1 + u_2)$.

Now we apply the secant method to approximate the solution of $F(x) = 0$. If $n = 10$, then (12) gives nine equations. Since a solution of (11) would vanish at the end points and be positive in the interior, a reasonable choice of the initial approximation seems to be $10 \sin \pi t$. This approximation gives us the following vector y_{-1}:

$$y_{-1} = \begin{pmatrix} 3.090169943749474 \\ 5.877852522924731 \\ 8.090169943749475 \\ 9.51056516295136 \\ 10.00000000000000 \\ 9.51056516295136 \\ 8.090169943749475 \\ 5.877852522924731 \\ 3.090169943749474 \end{pmatrix}.$$

Choose y_0 by setting $y_0(t_i) = y_{-1}(t_i) - 10^{-5}$, $i = 1, 2, \ldots, 9$, and using iteration (5) ($\lambda = 0$), after two iterations, we obtain y_1 and y_2:

$$y_1 = \begin{pmatrix} 2.453176290658909 \\ 4.812704101582601 \\ 6.8481873135861 \\ 8.252997367741953 \\ 8.75737771678512 \\ 8.252997367741953 \\ 6.8481873135861 \\ 4.812704101582601 \\ 2.453176290658909 \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} 2.404324055268407 \\ 4.713971539035271 \\ 6.7003394962933925 \\ 8.066765882171131 \\ 8.556329565792526 \\ 8.066765882171131 \\ 6.7003394962933924 \\ 4.713971539035271 \\ 2.404324055268407 \end{pmatrix}.$$

Taking $x_{-1} = y_1$ and $x_0 = y_2$, we obtain $\alpha = 0.201048$, $\beta = 15.319$, $\eta = 0.0346555$. In this case, the solution of Eq. (10) given in Theorem 3.1 has a minimum positive solution $R = 0.041100361$. Besides, $\beta\omega(\alpha + R, R) = 0.14983 < 1$ and $R = 0.156808 < 1$.

Therefore, the hypotheses of Theorem 3.1 are fulfilled and a unique solution of Eq. (2) exists in $\overline{B(x_0, R)}$.

We obtain the vector x^* as the solution of system (12), after nine iterations:

$$x^* = \begin{pmatrix} 2.394640794786742 \\ 4.694882371216001 \\ 6.672977546934751 \\ 8.033409358893319 \\ 8.520791423704788 \\ 8.033409358893319 \\ 6.67297754693475 \\ 4.694882371216 \\ 2.394640794786742 \end{pmatrix}.$$

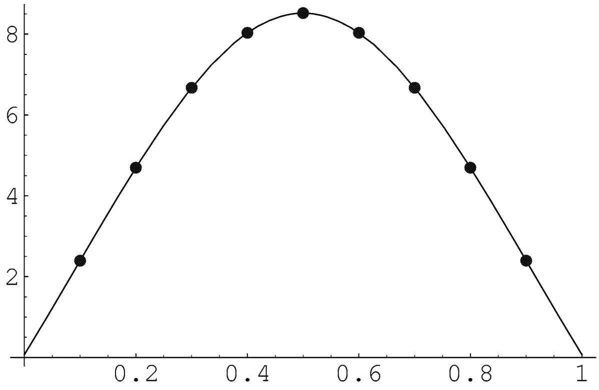

Fig. 2. x^* and the approximate solution $\bar{x}^*$.

If x^* is now interpolated, its approximation $\bar{x}^*$ to the solution of (11) with $p = 1/2$ is the one appearing in Fig. 2.

Note that, in this example, the convergence cannot be guaranteed from classical studies [2,7], where divided differences are Lipschitz or (k, p)-Hölder continuous, whereas we can do it by the technique presented in this paper.

4.2. *Example 2*

Consider the nondifferentiable system of equations

$$\begin{cases} 3x^2y + y^2 - 1 + |x - 1| = 0, \\ x^4 + xy^3 - 1 + |y| = 0. \end{cases} \tag{13}$$

We therefore have an operator $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H = (H_1, H_2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we take $H_1(x_1, x_2) = 3x_1^2x_2 + x_2^2 - 1 + |x_1 - 1|$, $H_2(x_1, x_2) = x_1^4 + x_1x_2^3 - 1 + |x_2|$.

For $v, w \in \mathbb{R}^2$, we take $[v, w; H] \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ as

$$[v, w; H]_{i1} = \frac{H_i(v_1, w_2) - H_i(w_1, w_2)}{v_1 - w_1},$$
$$[v, w; H]_{i2} = \frac{H_i(v_1, v_2) - H_i(v_1, w_2)}{v_2 - w_2}, \quad i = 1, 2.$$

Now we apply several methods to solve (13). See Table 1 for method (3) with $x_0 = (1, 0)$. Note that the approximated solution used is

$$x^* = (0.8946553733346867, 0.3278265117462974).$$

For the secant method with $x_{-1} = (5, 5)$ and $x_0 = (1, 0)$, see Table 2; for method (5) with $\lambda = 0.5$, $x_{-1} = (5, 5)$ and $x_0 = (1, 0)$, see Table 3; for method (5) with $\lambda = 0.99$, $x_{-1} = (5, 5)$ and $x_0 = (1, 0)$, see Table 4.

Table 1

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^* - x_n\|$
1	1	0.3333333333333333	1.05345×10^{-1}
2	0.9065502183406114	0.3540029112081513	2.61764×10^{-2}
3	0.8853284006634119	0.3380272763613319	1.02008×10^{-2}
4	0.891329556832800	0.3266139765935657	3.32582×10^{-3}
5	0.8952388154638436	0.3264068528436253	1.41967×10^{-3}
6	0.8951546713726346	0.3277303340450432	4.99298×10^{-4}
7	0.8946737434711373	0.3279791543720321	1.52633×10^{-4}
8	0.8945989089774475	0.3278650593487548	5.64644×10^{-5}
9	0.894643228355865	0.3278150392082856	1.2145×10^{-5}
10	0.8946599936156449	0.3278198892648906	6.63248×10^{-6}
11	0.8946576401953287	0.3278267282085600	2.26686×10^{-6}
12	0.8946552195650909	0.3278273518268564	8.30018×10^{-7}
⋮	⋮	⋮	⋮
34	0.8946553733346867	0.3278265217462975	5.55112×10^{-17}

Table 2

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^* - x_n\|$
1	0.9898008742107 82	0.0126274890723652	3.15199×10^{-1}
2	0.9218147654932871	0.3079399161522621	2.71594×10^{-2}
3	0.900073765669214	0.325927010697792	5.41839×10^{-3}
4	0.8949398516241052	0.3277254373962255	2.84478×10^{-4}
5	0.8946584205860127	0.3278253635007827	3.04725×10^{-6}
6	0.8946553750774177	0.3278265210518334	1.74273×10^{-9}
7	0.8946553733346976	0.3278265217462931	1.08802×10^{-14}
8	0.8946553733346867	0.3278265217462976	1.66533×10^{-16}
9	0.8946553733346867	0.3278265217462975	1.11022×10^{-16}

Table 3

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x^* - x_n\|$
1	0.9829778065072182	0.0344753285929756	2.93351×10^{-1}
2	0.9191516755790264	0.3114163466921295	2.44963×10^{-2}
3	0.8976925362896486	0.3267124870002544	3.03037×10^{-3}
4	0.8947380642577267	0.3277957962677528	8.26909×10^{-5}
5	0.8946556314301652	0.3278264207451973	2.58095×10^{-7}
6	0.8946553733563231	0.3278265217375175	2.16364×10^{-11}
7	0.8946553733346867	0.3278265217462975	5.55112×10^{-17}

Therefore the methods included in (5) improve the results given by other authors. Moreover, if the value of the parameter λ is increased, better approximations are obtained.

Table 4

n	$x_n^{(1)}$	$x_n^{(2)}$	$\lVert x^* - x_n \rVert$
1	0.9228095274055251	0.3269365280425139	2.81542×10^{-2}
2	0.8959888360193688	0.3276958684879607	1.33346×10^{-3}
3	0.8946591561955859	0.3278259055081464	3.78286×10^{-6}
4	0.894655373452723	0.3278265217196517	1.18036×10^{-10}
5	0.8946553733346867	0.3278265217462975	1.11022×10^{-16}
6	0.8946553733346867	0.3278265217462975	5.55112×10^{-17}

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