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Semilocal Convergence of the Secant Method under Mild Convergence Conditions of Differentiability

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Abstract—In this work, we obtain a semilocal convergence result for the secant method in Banach spaces under mild convergence conditions. We consider a condition for divided differences which generalizes those usual ones, i.e., Lipschitz continuous and Hölder continuous conditions. Also, we obtain a result for uniqueness of solutions. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we are concerned with the problem of finding conditions for the semilocal convergence of secant method in Banach spaces. Such a problem is clearly important in numerical analysis, since many applied problems reduce to solve a nonlinear operator equation

$$F(x) = 0, \tag{1}$$

with $F : \Omega \subseteq X \to Y$ and Ω a convex open subset of X and X, Y Banach spaces. To apply the secant method, it is necessary to first consider divided differences. Let us denote by $\mathcal{L}(X, Y)$, the space of bounded linear operators from X to Y. Remember that an operator $[x, y; F] \in \mathcal{L}(X, Y)$ is called a divided difference of first order for the operator F on the points x and y ($x \neq y$) if the following equality holds:

$$[x, y; F](x - y) = F(x) - F(y). \tag{2}$$

Using this definition, Sergeev [1] and Schmidt [2] generalize the secant method to Banach spaces. The secant method is then described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \qquad x_0,\ x_{-1} \text{ given}. \tag{3}$$

The semilocal convergence of the secant method has usually been studied from majorizing sequences [3–5] or by nondiscrete induction [6,7], under Lipschitz or Hölder continuity conditions for divided differences of operator (1).

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It supposes that, for every pair of distinct points $x, y \in \Omega$, there exists a first-order divided difference of F at these points. If there exists a nonnegative constant k such that

$$\|[x,y;F]-[v,w;F]\| \leq k(\|x-v\|^p + \|y-w\|^p), \qquad p\in[0,1], \tag{4}$$

for all $x, y, v, w \in \Omega$ with $x \neq y$ and $v \neq w$, we say that F has a (k,p)-Hölder continuous divided difference on Ω. If $p=1$, we say that F has a Lipschitz continuous divided difference on Ω.

In this paper, we relax condition (4) and consider

$$\|[x,y;F]-[v,w;F]\| \leq \omega(\|x-v\|,\|y-w\|), \qquad x,y,v,w\in\Omega, \tag{5}$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in its two arguments. Applying then, this condition to obtain a semilocal convergence result for the secant method, we consider a new technique by means of using recurrence relations.

Finally, the new semilocal convergence result obtained is applied to approximate the solution of a nonlinear boundary value problem.

2. CONVERGENCE ANALYSIS

Before getting the semilocal convergence result for the secant method under these new conditions, we introduce some notations. Let $x_0, x_{-1} \in \Omega$ and we will take into account the following auxiliary functions:

$$a(u)=\frac{\beta\omega(\alpha,u)}{1-\beta\omega(\alpha,u)}, \qquad b(u)=\frac{\beta\omega(u,2u)}{1-\beta\omega(u+\alpha,u)}, \qquad c(u)=\frac{\beta\omega(2u,2u)}{1-\beta\omega(u+\alpha,u)},$$

where $\alpha=\|x_0-x_{-1}\|$, $\beta=\|[x_{-1},x_0;F]^{-1}\|$.

THEOREM 2.1. *Assume that, for every pair of points $x, y \in \Omega$, there exists a first-order divided difference $[x,y;F] \in \mathcal{L}(X,Y)$ such that (5) holds. Assume the following.*

- *The linear operator $L_0=[x_{-1},x_0;F]$ is invertible and $\|{L_0}^{-1}F(x_0)\| \leq \eta$.*
- *The equation*

$$u=\left(\frac{b(u)a(u)}{1-c(u)}+a(u)+1\right)\eta \tag{6}$$

has at least one positive zero. Let R be the minimum positive one.

If $\beta\omega(R+\alpha,R)<1$, $c(R)<1$, and $\overline{B(x_0,R)}\subset\Omega$, then sequence $\{x_n\}$ given by (3) is well defined, remains in $\overline{B(x_0,R)}$, and converges to a unique solution x^* of equation (1) in $\overline{B(x_0,R)}$.

PROOF. To simplify the notation, we denote $a(R)=a$, $b(R)=b$, $c(R)=c$, and $[x_{n-1},x_n;F]=L_n$. First, we prove, by mathematical induction, that the sequence given in (3) is well defined, namely, iterative procedure (3) is well defined if, at each step, the operator $[x_{n-1},x_n;F]$ is invertible and the point x_{n+1} lies in Ω.

From the initial hypotheses, it follows that x_1 is well defined and $\|x_1-x_0\|\leq\eta<R$. Therefore, $x_1\in B(x_0,R)\subseteq\Omega$.

Now, using (5) and assuming that ω is nondecreasing, we obtain

$$\begin{aligned}\left\|I-{L_0}^{-1}L_1\right\| &\leq \left\|{L_0}^{-1}\right\|\,\|L_0-L_1\| \leq \left\|{L_0}^{-1}\right\|\omega\left(\|x_0-x_{-1}\|,\|x_1-x_0\|\right)\\ &\leq \beta\omega(\alpha,R)\leq\beta\omega(R+\alpha,R)<1,\end{aligned}$$

and, by the Banach lemma, ${L_1}^{-1}$ exists and

$$\left\|{L_1}^{-1}\right\| \leq \frac{\beta}{1-\beta\omega(\alpha,R)}.$$

By (2) and (3), we get

$$F(x_1) = F(x_0) - [x_0, x_1; F](x_0 - x_1) = (L_0 - L_1)(x_0 - x_1).$$

Then, by (5), we have

$$\begin{aligned}\|F(x_1)\| &\le \|L_1 - L_0\| \, \|x_1 - x_0\| \le \omega(\|x_0 - x_{-1}\|, \|x_1 - x_0\|) \|x_1 - x_0\| \\ &\le \omega(\alpha, \eta) \|x_1 - x_0\| \le \omega(\alpha, R) \|x_1 - x_0\|,\end{aligned}$$

and consequently, iterate x_2 is well defined. Moreover,

$$\|x_2 - x_1\| \le \|L_1{}^{-1}\| \, \|F(x_1)\| \le \frac{\beta\omega(\alpha, R)}{1 - \beta\omega(\alpha, R)} \|x_1 - x_0\| = a \|x_1 - x_0\|. \tag{7}$$

On the other hand, if we take into account that R is a solution of (6), then

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le \left[\frac{\beta\omega(\alpha, R)}{1 - \beta\omega(\alpha, R)} + 1\right] \|x_1 - x_0\| \le (a+1)\eta < R \tag{8}$$

and $x_2 \in B(x_0, R)$.

Next, by analogy,

$$\begin{aligned}\|I - L_0{}^{-1}L_2\| &\le \|L_0{}^{-1}\| \, \|L_0 - L_2\| \le \beta\omega(\|x_1 - x_{-1}\|, \|x_2 - x_0\|) \\ &\le \beta\omega(\|x_1 - x_0\| + \|x_0 - x_{-1}\|, \|x_2 - x_0\|) \le \beta\omega(R + \alpha, R) < 1,\end{aligned}$$

and consequently, $L_2{}^{-1}$ exists and

$$\|L_2{}^{-1}\| \le \frac{\beta}{1 - \beta\omega(R + \alpha, R)}.$$

Moreover,

$$\begin{aligned}\|F(x_2)\| &\le \|L_2 - L_1\| \, \|x_2 - x_1\| \le \omega(\|x_1 - x_0\|, \|x_2 - x_1\|) \|x_2 - x_1\| \\ &\le \omega(\|x_1 - x_0\|, \|x_2 - x_0\| + \|x_0 - x_1\|) \|x_2 - x_1\| \le \omega(R, 2R) \|x_2 - x_1\|\end{aligned}$$

and

$$\|x_3 - x_2\| \le \|L_2{}^{-1}\| \, \|F(x_2)\| \le \frac{\beta\omega(R, 2R)}{1 - \beta\omega(R + \alpha, R)} \|x_2 - x_1\| = b \|x_2 - x_1\|. \tag{9}$$

Then, by (6)–(9), we have

$$\begin{aligned}\|x_3 - x_0\| &\le \|x_3 - x_2\| + \|x_2 - x_0\| \le b \|x_2 - x_1\| + \|x_2 - x_0\| \\ &\le (ba + a + 1) \|x_1 - x_0\| \le (ba + a + 1)\eta < R.\end{aligned}$$

Therefore, $x_3 \in B(x_0, R)$.

Then, by induction on n, the following items can be shown for $n \ge 3$:

(i_n) $\exists\, L_n{}^{-1} = [x_{n-1}, x_n; F]^{-1}$ such that $\|L_n{}^{-1}\| \le \beta/(1 - \beta\omega(R + \alpha, R))$.

(ii_n) $\|x_{n+1} - x_n\| \le c\|x_n - x_{n-1}\|$.

We have

$$\begin{aligned}\|I - L_0{}^{-1}L_3\| &\le \|L_0{}^{-1}\| \, \|L_0 - L_3\| \le \beta\omega(\|x_2 - x_{-1}\|, \|x_3 - x_0\|) \\ &\le \beta\omega(\|x_2 - x_0\| + \|x_0 - x_{-1}\|, \|x_3 - x_0\|) \le \beta\omega(R + \alpha, R) < 1,\end{aligned}$$

Then, by the Banach lemma, ${L_3}^{-1}$ exists and

$$\left\|{L_3}^{-1}\right\| \leq \frac{\beta}{1-\beta\omega(R+\alpha,R)}.$$

From the definition of the first divided difference and the secant method, we can obtain

$$F(x_3) = F(x_2) - [x_2,x_3;F](x_2-x_3) = (L_2 - [x_2,x_3;F])\, L_2^{-1}F(x_2) = (L_2-L_3)\,(x_2-x_3).$$

Taking norms in the above equality and (5), we obtain

$$\begin{aligned}\|F(x_3)\| &\leq \|L_3-L_2\|\,\|x_3-x_2\| \leq \omega\left(\|x_2-x_1\|,\|x_3-x_2\|\right)\|x_3-x_2\| \\ &\leq \omega\left(\|x_2-x_0\|+\|x_0-x_1\|,\|x_3-x_0\|+\|x_0-x_2\|\right)\|x_3-x_2\| \leq \omega\,(2R,2R)\,\|x_3-x_2\|.\end{aligned}$$

Thus,

$$\|x_4-x_3\| \leq \left\|{L_3}^{-1}\right\|\,\|F(x_3)\| \leq \frac{\beta\omega\,(2R,2R)}{1-\beta\omega(R+\alpha,R)}\,\|x_3-x_2\| = c\,\|x_3-x_2\|. \tag{10}$$

Now, if we suppose that (i_k),(ii_k) hold for all $k=3,\dots,n-1$, we analogously prove (i_{k+1}),(ii_{k+1}).

Consequently, from (6)–(10), it follows

$$\begin{aligned}\|x_{n+1}-x_0\| &\leq \|x_{n+1}-x_n\| + \|x_n-x_{n-1}\| + \cdots + \|x_3-x_2\| + \|x_2-x_0\| \\ &\leq \left[c^{n-2}+c^{n-3}+\cdots+1\right]\|x_3-x_2\| + \|x_2-x_0\| \\ &\leq \left[\frac{1-c^{n-1}}{1-c}ba+a+1\right]\|x_1-x_0\| < \left[\frac{ba}{1-c}+a+1\right]\eta = R.\end{aligned}$$

That is, $x_{n+1} \in B(x_0,R)$.

Second, we prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq 1$ and $n \geq 2$, we obtain

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-x_{n+m-1}\| + \|x_{n+m-1}-x_{n+m-2}\| + \cdots + \|x_{n+1}-x_n\| \\ &\leq \left[c^{m-1}+c^{m-2}+\cdots+1\right]\|x_{n+1}-x_n\| \\ &\leq \frac{1-c^m}{1-c}\,\|x_{n+1}-x_n\| < \frac{1}{1-c}c^{n-2}\,\|x_3-x_2\|.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0,R)}$.

Finally, we see that x^* is a zero of F. Since

$$\|F(x_n)\| \leq \omega\,(2R,2R)\,\|x_n-x_{n-1}\|,$$

and $\|x_n-x_{n-1}\| \to 0$ as $n\to\infty$, we obtain $F(x^*)=0$.

To show uniqueness, we assume that there exists a second solution $y^* \in \overline{B(x_0,R)}$ and consider the operator $A=[y^*,x^*;F]$. Since $A(y^*-x^*) = F(y^*)-F(x^*)$, if operator A is invertible, then $x^*=y^*$. Indeed,

$$\begin{aligned}\left\|{L_0}^{-1}A-I\right\| &\leq \left\|{L_0}^{-1}\right\|\,\|A-L_0\| \leq \left\|{L_0}^{-1}\right\|\,\|[y^*,x^*;F]-[x_{-1},x_0;F]\| \\ &\leq \beta\omega\left(\|y^*-x_{-1}\|,\|x^*-x_0\|\right) \leq \beta\omega\left(\|y^*-x_0\|+\|x_0-x_{-1}\|,\|x^*-x_0\|\right) \\ &\leq \beta\omega\,(R+\alpha,R) < 1\end{aligned}$$

and the operator A^{-1} exists. ∎

REMARK. Note that the operator F is differentiable when the divided differences are Lipschitz or (k,p)-Hölder continuous. But, under condition (5), F is differentiable if $\omega(0,0)=0$. Therefore, if $\omega(0,0)\neq 0$, our semilocal convergence result also can be true for nondifferentiable operators.

3. APPLICATION

Now we apply the semilocal convergence result given above to the following nonlinear boundary value problem:

$$\begin{gathered} y'' + y^{1+p} + \mu y^2 = 0, \qquad \mu \in \mathbb{R}, \quad p \in [0,1], \\ y(0) = y(1) = 0. \end{gathered} \tag{11}$$

We divide interval $[0,1]$ into n subintervals and we let $h = 1/n$. We denote the points of subdivision by $t_i = ih$ and $y(t_i) = y_i$. Notice that y_0 and y_n are given by the boundary conditions, so $y_0 = 0 = y_n$. We first approximate the second derivative $y''(t)$ in the differential equation by

$$\begin{aligned} y''(t) &\approx \frac{[y(t+h) - 2y(t) + y(t-h)]}{h^2}, \\ y''(t_i) &\approx \frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2}, \qquad i = 1, 2, \dots, n-1. \end{aligned}$$

By substituting this expression into the differential equation, we have the following system of nonlinear equations:

$$\begin{aligned} 2y_1 - h^2 y_1^{1+p} - h^2 \mu y_1^2 - y_2 &= 0, \\ -y_{i-1} + 2y_i - h^2 y_i^{1+p} - h^2 \mu y_i^2 - y_{i+1} &= 0, \qquad i = 2, 3, \dots, n-2, \\ -y_{n-2} + 2y_{n-1} - h^2 y_{n-1}^{1+p} - h^2 \mu y_{n-1}^2 &= 0. \end{aligned} \tag{12}$$

We, therefore, have an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(y) = H(y) - h^2 g(y) - h^2 \mu f(y)$, where

$$y = (y_1, y_2, \dots, y_{n-1})^t, \qquad g(y) = \left(y_1^{1+p}, y_2^{1+p}, \dots, y_{n-1}^{1+p}\right)^t, \qquad f(y) = \left(y_1^2, y_2^2, \dots, y_{n-1}^2\right)^t,$$

and

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Thus,

$$F'(y) = H - h^2(1+p) \begin{pmatrix} y_1^p & 0 & \cdots & 0 \\ 0 & y_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{n-1}^p \end{pmatrix} - 2h^2 \mu \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{n-1} \end{pmatrix}.$$

Then, we apply Theorem 2.1 to find a solution y^* of the equation

$$F(y) = 0. \tag{13}$$

Let $x \in \mathbb{R}^{n-1}$, and choose the norm $\|x\| = \max_{1 \le i \le n-1} |x_i|$. The corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [7]) that F has a divided difference at the points $v, w \in \mathbb{R}^{n-1}$, which is defined by the matrix whose entries are

$$[v, w; F]_{ij} = \frac{1}{v_j - w_j} \left(F_i \left(v_1, \dots, v_j, w_{j+1}, \dots, w_m\right) - F_i \left(v_1, \dots, v_{j-1}, w_j, \dots, w_m\right)\right), \quad m = n-1.$$

Therefore,

$$[v,w;F]=H-h^2\begin{pmatrix} \frac{v_1^{1+p}-w_1^{1+p}+\mu(v_1^2-w_1^2)}{v_1-w_1} & 0 & \cdots & 0 \\ 0 & \frac{v_2^{1+p}-w_2^{1+p}+\mu(v_2^2-w_2^2)}{v_2-w_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{v_{n-1}^{1+p}-w_{n-1}^{1+p}+\mu(v_{n-1}^2-w_{n-1}^2)}{v_{n-1}-w_{n-1}} \end{pmatrix}.$$

In this case, we have $[v,w;F] = \int_0^1 F'(v+t(w-v))\,dt$. So we study the value $\|F'(x)-F'(v)\|$ to obtain a bound for $\|[x,y;F]-[v,w;F]\|$.

For all $x, v \in \mathbb{R}^{n-1}$ with $|x_i| > 0, |v_i| > 0$, $i = 1,2,\dots,n-1$, and taking into account the max-norm, it follows

$$\begin{aligned} \|F'(x)-F'(v)\| &= \left\|\operatorname{diag}\left\{h^2(1+p)\left(v_i^p-x_i^p\right)+2\mu h^2\left(v_i-x_i\right)\right\}\right\| \\ &= \max_{1\le i\le n-1}\left|h^2(1+p)\left(v_i^p-x_i^p\right)+2\mu h^2\left(v_i-x_i\right)\right| \\ &\le (1+p)h^2\max_{1\le i\le n-1}|v_i^p-x_i^p|+2\mu h^2\max_{1\le i\le n-1}|v_i-x_i| \\ &\le (1+p)h^2\left[\max_{1\le i\le n-1}|v_i-x_i|\right]^p+2\mu h^2\|v-x\| \\ &= (1+p)h^2\|v-x\|^p+2\mu h^2\|v-x\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \|[x,y;F]-[v,w;F]\| &\le \int_0^1 \|F'\left(x+t(y-x)\right)-F'\left(v+t(w-v)\right)\|\,dt \\ &\le h^2\int_0^1 \left((1+p)\|(1-t)(x-v)+t(y-w)\|^p+2\mu\|(1-t)(x-v)+t(y-w)\|\right)dt \\ &\le h^2(1+p)\int_0^1\left((1-t)^p\|x-v\|^p+t^p\|y-w\|^p\right)dt+2\mu h^2\int_0^1\left((1-t)\|x-v\|+t\|y-w\|\right)dt \\ &= h^2\left(\|x-v\|^p+\|y-w\|^p+\mu(\|x-v\|+\|y-w\|)\right). \end{aligned}$$

From (5), we consider the function

$$\omega\left(u_1,u_2\right)=h^2\left(u_1^p+u_2^p+\mu\left(u_1+u_2\right)\right). \tag{14}$$

We now study two situations: $\mu = 0$ and $\mu \neq 0$.

3.1. $\mu = 0$

This example has been also considered by other authors in [3,8]. Problem (11) is now

$$\begin{gathered} y''+y^{1+p}=0, \qquad p\in[0,1], \\ y(0)=y(1)=0. \end{gathered} \tag{15}$$

In this case, by (14), the divided difference is (k,p)-Hölder continuous with $k = h^2$.

Now we apply the secant method to approximate the solution of $F(y) = 0$. We choose $p = 1/2$ and if $n = 10$, then (12) gives nine equations. Since a solution of (15) would vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be

$135 \sin \pi t$. This approximate gives us the following vector:

$$z_{-1} = \begin{pmatrix} 41.7172942406179 \\ 79.35100905948387 \\ 109.2172942406179 \\ 128.3926296998458 \\ 135.0000000000000 \\ 128.3926296998458 \\ 109.2172942406179 \\ 79.35100905948387 \\ 41.7172942406179 \end{pmatrix}.$$

We choose z_0 by setting $z_0(t_i) = z_{-1}(t_i) - 10^{-5}$, $i = 1, 2, \ldots, 9$. Using iteration (3), after three iterations, we obtain

$$z_2 = \begin{pmatrix} 33.64838334335734 \\ 65.34766285832966 \\ 91.77113354118937 \\ 109.4133887062593 \\ 115.6232519796117 \\ 109.4133887062593 \\ 91.77113354118937 \\ 65.34766285832964 \\ 33.64838334335733 \end{pmatrix} \quad \text{and} \quad z_3 = \begin{pmatrix} 33.57498274928053 \\ 65.20452867501265 \\ 91.56893412724006 \\ 109.1710943553677 \\ 115.3666988182897 \\ 109.1710943553677 \\ 91.56893412724006 \\ 65.20452867501265 \\ 33.57498274928053 \end{pmatrix}.$$

Then we take $y_{-1} = z_2$ and $y_0 = z_3$. With the notation of Theorem 2.1, we can easily obtain the following results:

$$\alpha = 0.256553, \qquad \beta = 26.5446, \qquad \eta = 0.00365901.$$

Since $h^2 = 0.01$, in this particular case, the solution of equation (6) is $R = 0.0043494$. Besides, $\beta\omega(R + \alpha, R) = 0.153092 < 1$ and $c(R) = 0.0584655 < 1$. Therefore, the hypotheses of Theorem 2.1 are fulfilled, which ensures that a unique solution of equation (13) exists in $\overline{B(y_0, R)}$.

We obtain the vector y^* as the solution of system (12), after seven iterations:

$$y^* = \begin{pmatrix} 33.5739120483378 \\ 65.20245092365437 \\ 91.5660200355396 \\ 109.1676242966424 \\ 115.3630336377466 \\ 109.1676242966424 \\ 91.5660200355396 \\ 65.20245092365437 \\ 33.5739120483378 \end{pmatrix}.$$

If y^* is now interpolated, its approximation $\bar{y}^*$ to the solution of (15) with $p = 1/2$ is that appearing in Figure 1.

Note that the study made in [3] by Argyros is not applicable if we take as starting points y_{-1} and y_0, since the requirements considered in that study are not fulfilled. Therefore, by Argyros' study, the convergence of the secant method is not guaranteed.

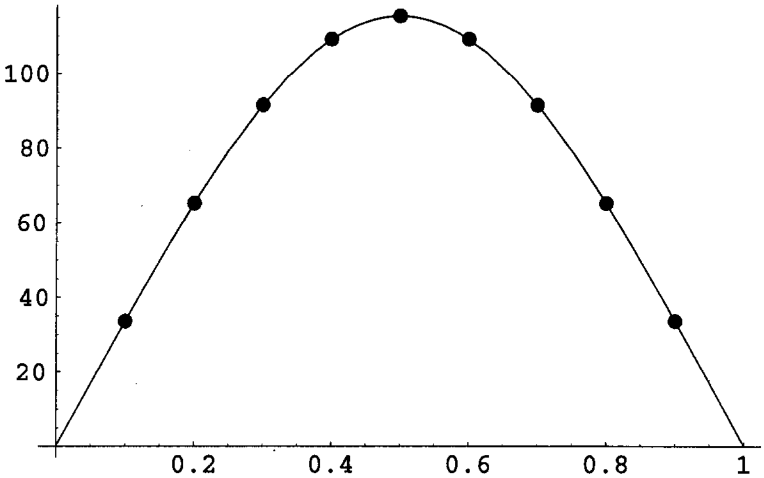

Figure 1. y^* and the approximate solution $\bar{y}^*$.

3.2. $\mu \neq 0$

We consider, for example, $\mu = 1$. From (14), we consider the function $\omega(u_1, u_2) = h^2(u_1^p + u_2^p + u_1 + u_2)$. Then, as in the previous example, we choose $p = 1/2$, $n = 10$ and the initial approximation seems to be $10 \sin \pi t$.

Now we apply the secant method to approximate the solution of (13). This approximate gives us the following vector:

$$z'_{-1} = \begin{pmatrix} 3.090169943749474 \\ 5.877852522924731 \\ 8.090169943749475 \\ 9.51056516295136 \\ 10.00000000000000 \\ 9.51056516295136 \\ 8.090169943749475 \\ 5.877852522924731 \\ 3.090169943749474 \end{pmatrix}.$$

Choose z'_0 by setting $z'_0(t_i) = z'_{-1}(t_i) - 10^{-5}$, $i = 1, 2, \ldots, 9$. Using iteration (3), after two iterations, we obtain

$$z'_1 = \begin{pmatrix} 2.453176290658909 \\ 4.812704101582601 \\ 6.8481873135861 \\ 8.252997367741953 \\ 8.75737771678512 \\ 8.252997367741953 \\ 6.8481873135861 \\ 4.812704101582601 \\ 2.453176290658909 \end{pmatrix} \quad \text{and} \quad z'_2 = \begin{pmatrix} 2.404324055268407 \\ 4.713971539035271 \\ 6.7003394962933925 \\ 8.066765882171131 \\ 8.556329565792526 \\ 8.066765882171131 \\ 6.7003394962933924 \\ 4.713971539035271 \\ 2.404324055268407 \end{pmatrix}.$$

Taking $y_{-1} = z'_1$ and $y_0 = z'_2$, we obtain $\alpha = 0.201048$, $\beta = 15.319$, $\eta = 0.0346555$. In this case, equation (6) given in Theorem 2.1 has a minimum positive solution $R = 0.0408385$. Besides, $\beta\omega(R + \alpha, R) = 0.14961 < 1$ and $c(R) = 0.132392 < 1$. Therefore, we obtain by Theorem 2.1 that sequence $\{y_n\}$ given by the secant method converges to a unique solution y^* in $\overline{B(x_0, R)}$ of equation $F(y) = 0$.

We obtain the following vector $\bar{y}$ as the solution of system (12), after 11 iterations:

$$\bar{y} = \begin{pmatrix} 2.394640794786742 \\ 4.694882371216001 \\ 6.672977546934751 \\ 8.033409358893319 \\ 8.520791423704788 \\ 8.033409358893319 \\ 6.67297754693475 \\ 4.694882371216 \\ 2.394640794786742 \end{pmatrix}.$$

Note that, in this example, the convergence cannot be guaranteed by classical studies, where divided differences used are Lipschitz or (k, p)-Hölder continuous, whereas we can achieve this by the technique presented in this paper.

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