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A New Third-Order Iterative Process for Solving Nonlinear Equations

By

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Abstract. In this paper, we build up a modification of the Midpoint method, reducing its operational cost without losing its cubical convergence. Then we obtain a semilocal convergence result for this new iterative process and by means of several examples we compare it with other iterative processes.

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1. Introduction

Several scientific problems can be written in the form

$$F(x) = 0. \tag{1}$$

In order to generalize as much as possible, let F be a nonlinear operator defined in an open convex domain Ω of a Banach space X with values in a Banach space Y. There is a lot of research work concerning the numerical solution of (1) by means of iterative processes, mainly by using Newton's method ([4], [8]).

But there are other iterative processes for solving (1). One of them is the Midpoint method ([2], [8]) given by:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n + F'\left(\frac{x_n + y_n}{2}\right)^{-1} F(x_n). \end{aligned} \tag{2}$$

It is a third order method. In practice, for a given x_n in each iterate of (2), the following steps are needed: First we have to solve a linear system

$$F'(x_n)(y_n - x_n) = -F(x_n),$$

to obtain y_n, and then we need to solve another linear system given by

$$F'\left(\frac{x_n + y_n}{2}\right)(x_{n+1} - x_n) = -F(x_n).$$

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Table 1. Operational cost

Method	Eval. of F	Eval. of F'	Decomp. LR
Midpoint M.	1	2	2
Midpoint modif.	2	1	1
Newton M.	1	1	1

We discuss the high computational cost of this method. We need to evaluate F' at two points and to solve two linear equations systems with their LR decompositions.

In this paper, we consider a modification of the Midpoint method such that the cubical convergence is preserved but we avoid the double evaluation of F' and the realization of two LR decompositions. For that we consider an iterative process in the form:

$$F'(x_n)(y_n - x_n) = -F(x_n),$$
$$F'(x_n)(x_{n+1} - y_n) = -\alpha F(x_n) - \beta F\left(\frac{x_n + y_n}{2}\right), \quad \alpha, \beta \in \mathbb{R}.$$

So, we have changed an evaluation of F' in $\frac{x_n + y_n}{2}$ by an evaluation of F at the same point. Moreover we only need one LR decomposition for their application. In this way, the operational cost has been considerably reduced.

Observe that the method obtained has an operational cost similar to the Newton method and for a particular choice of α and β we can reach cubical convergence. Then we will prove that iteration given by

$$F'(x_n)(y_n - x_n) = -F(x_n), F'(x_n)(x_{n+1} - y_n) = 2F(x_n) - 4F\left(\frac{x_n + y_n}{2}\right), \tag{3}$$

has cubical convergence and a result of semilocal convergence for this new iterative process is given. The local and global behaviors for this algorithm are not the aim of this paper. For the study of these situations, it is very interesting the paper of Wang and Li [9] based on Smale's creative work [7].

Finally, we consider several examples where we compare our results with the results obtained by others methods. Thus, we check the usefulness of the new iterative process defined in this paper.

2. Convergence Analysis

To prove the convergence of the iterative process (3) we write it in the form

$$y_n = x_n - \Gamma_n F(x_n),$$
$$x_{n+1} = y_n + 2\Gamma_n H(x_n, y_n), \tag{4}$$

where

$$\Gamma_n = F'(x_n)^{-1}, \quad H(x_n, y_n) = F(x_n) - 2F\left(\frac{x_n + y_n}{2}\right).$$

We denote $\overline{B(x,r)} = \{y \in X; \|y - x\| \leqslant r\}$ and $B(x,r) = \{y \in X; \|y - x\| < r\}$.

We analyse the convergence of (4) to a solution x^* of $F(x) = 0$. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists at some $x_0 \in \Omega$, where $\mathscr{L}(Y,X)$ is the set of bounded linear operators from Y into X.

We assume the following conditions:

(c_1) $\|\Gamma_0\| \leqslant \beta$,
(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$,
(c_3) $\|F''(x)\| \leqslant M, \quad x \in \Omega, \quad M \geqslant 0$,
(c_4) $\|F''(x) - F''(y)\| \leqslant K\|x - y\|, \quad x, y \in \Omega, \quad K \geqslant 0$.

Let us denote $a_0 = M\beta\eta, b_0 = K\beta\eta^2$ and define sequences

$$a_{n+1} = a_n f(a_n, b_n)^2 g(a_n, b_n), \quad b_{n+1} = b_n f(a_n, b_n)^3 g(a_n, b_n)^2, \tag{5}$$

where

$$f(x,y) = \frac{6}{6 - 3x^2 - 10xy - 6x}, \tag{6}$$

and

$$g(x,y) = \frac{x^3}{8} + \frac{25}{18}xy^2 + \frac{5}{6}x^2 y + \frac{1}{4}x^2 + \frac{5}{6}xy + \frac{181}{108}y. \tag{7}$$

First, we observe that

$$\begin{aligned} H(x_n, y_n) &= F(x_n) - 2F\Big(\frac{x_n + y_n}{2}\Big) \\ &= -\frac{1}{2}\int_0^1 F''\Big(y_n + t\Big(\frac{x_n - y_n}{2}\Big)\Big)(1-t)dt(y_n - x_n)^2 \\ &\quad - 2\int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)](1-t)dt(y_n - x_n)^2 \\ &\quad - \int_0^1 [F''(x_n) - F''(x_n + t(y_n - x_n))]dt(y_n - x_n)^2, \end{aligned} \tag{8}$$

then taking into account initial hypotheses (c_1)–(c_4) and assuming that $y_0 \in \Omega$, the iterate x_1 is well defined, and by (8)

$$\|x_1 - y_0\| \leqslant 2\|\Gamma_0\|\|H(x_0, y_0)\| \leqslant \Big[\frac{1}{2}a_0 + \frac{5}{3}b_0\Big]\|x_0 - y_0\|$$

and

$$\|x_1 - x_0\| \leqslant \|x_1 - y_0\| + \|y_0 - x_0\| \leqslant \Big[\frac{1}{2}a_0 + \frac{5}{3}b_0 + 1\Big]\|x_0 - y_0\|$$

Next we prove the following items hold for all $n \geqslant 1$ by mathematical induction:

[I_n] $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant f(a_{n-1}, b_{n-1})\|\Gamma_{n-1}\|$,
[II_n] $\|y_n - x_n\| \leqslant f(a_{n-1}, b_{n-1})g(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\|$,
[III_n] $\|x_{n+1} - x_n\| \leqslant \big[\frac{1}{2}a_n + \frac{5}{3}b_n + 1\big]\|y_n - x_n\|$.

We now assume $x_n, y_n \in \Omega$ for all $n \geqslant 0$. The proof of this is given in Theorem 2.4. If we suppose

$$b_n < \frac{3}{5}\left[\frac{1}{a_n} - 1 - \frac{a_n}{2}\right], \quad n \geqslant 0, \tag{9}$$

then we have

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\leqslant \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leqslant M\|\Gamma_0\|\|x_1 - x_0\|\\ &\leqslant a_0\left[\frac{1}{2}a_0 + \frac{5}{3}b_0 + 1\right] < 1,\end{aligned}$$

by (9). Hence Γ_1 is defined and $\|\Gamma_1\| \leqslant f(a_0, b_0)\|\Gamma_0\|$.

Using the equalities

$$F(x_{n+1}) = \int_{y_n}^{x_{n+1}} F''(x)(x_{n+1} - x)dx + F(y_n) + F'(y_n)(x_{n+1} - y_n),$$

$$\begin{aligned}F\Big(\frac{x_n + y_n}{2}\Big) &= \int_{y_n}^{\frac{x_n+y_n}{2}} F''(x)\Big(\frac{x_n + y_n}{2} - x\Big)dx + \int_{x_n}^{y_n} F''(x)(y_n - x)dx\\ &\quad + \frac{1}{2}\int_{y_n}^{x_n} F''(x)dx\Gamma_n F(x_n) + \frac{1}{2}F(x_n),\end{aligned}$$

and (4), we get

$$\begin{aligned}F(x_{n+1}) &= 4\int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t)dt[\Gamma_n H(x_n, y_n)]^2\\ &\quad + \int_0^1 \Big[F''(x_n + t(y_n - x_n)) - F''\Big(y_n + t\Big(\frac{x_n + y_n}{2}\Big)\Big)\Big](1 - t)dt(y_n - x_n)^2\\ &\quad - 4\int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)](1 - t)dt(y_n - x_n)^2\\ &\quad - 2\int_0^1 [F''(x_n) - F''(x_n + t(y_n - x_n))]dt(y_n - x_n)^2\\ &\quad + 2\int_0^1 F''(x_n + t(y_n - x_n))(1 - t)dt(y_n - x_n)\Gamma_n H(x_n, y_n).\end{aligned} \tag{10}$$

Then, for $n = 0$, we obtain

$$\begin{aligned}\|F(x_1)\| &\leqslant 2M\beta^2\left[\frac{M^2\eta^2}{16} + \frac{25}{36}K^2\eta^4 + \frac{5}{12}MK\eta^3\right]\eta^2\\ &\quad + \frac{181}{108}K\eta^3 + M\eta^2\beta\left[\frac{1}{4}M\eta + \frac{5}{6}K\eta^2\right].\end{aligned}$$

So,

$$\|y_1 - x_1\| \leqslant f(a_0, b_0)g(a_0, b_0)\|y_0 - x_0\|$$

and $[\mathrm{II}_1]$ holds. To show $[\mathrm{III}_1]$, we note that

$$\|H(x_1, y_1)\| \leqslant \left[\frac{1}{4}Mf(a_0, b_0)g(a_0, b_0)\eta + \frac{5}{6}Kf(a_0, b_0)^2 g(a_0, b_0)^2\eta^2\right]\|y_1 - x_1\|.$$

Finally, we easily deduce

$$\|x_2 - y_1\| \leqslant \left[\frac{1}{2}a_1 + \frac{5}{6}b_1\right] \|y_1 - x_1\|,$$

$$\|x_2 - x_1\| \leqslant \left[\frac{1}{2}a_1 + \frac{5}{3}b_1 + 1\right] \|y_1 - x_1\|.$$

Now if we suppose that $[\mathrm{I}_n]$–$[\mathrm{III}_n]$ hold for a fixed $n \geqslant 1$, we can prove $[\mathrm{I}_{n+1}]$–$[\mathrm{III}_{n+1}]$ by induction.

Our next goal is to analyse the real sequences (5) to obtain the convergence of sequence (4) defined in Banach spaces. To obtain the convergence of (4), we only have to prove (4) is a Cauchy sequence under assumption (9). First, we provide a technical lemma whose proof is trivial.

Lemma 2.1. *Let f and g two real functions given in (6) and (7) respectively. Then*

(i) *f is increasing in both variables x and y and $f(x,y) > 1$ for $x, y \geqslant 0$,*
(ii) *g is increasing in both variables x and y,*
(iii) *$f(\gamma x, \gamma y) < f(x,y)$ and $g(\gamma x, \gamma y) < \gamma g(x,y)$ for $\gamma \in (0,1)$.*

It is convenient to introduce auxiliary functions:

$$y_1(x) = \frac{181 + 450x - 270x^2 - 180x^3 - \sqrt{32761 + 227700x - 154440x^2 - 65160x^3}}{300x(2x-1)},$$

$$y_2(x) = \frac{181 + 450x - 270x^2 - 180x^3 + \sqrt{32761 + 227700x - 154440x^2 - 65160x^3}}{300x(2x-1)},$$

and

$$y_3(x) = \frac{3}{5}\left(\frac{1}{x} - 1 - \frac{x}{2}\right),$$

whose graphs can be seen in Figures 1, 2 and 3.

Some properties for sequences $\{a_n\}$ and $\{b_n\}$ given by (5) are shown below.

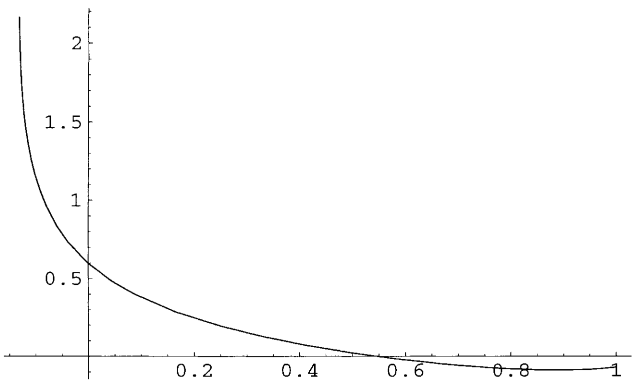

Figure 1. Graph $y_1(x)$

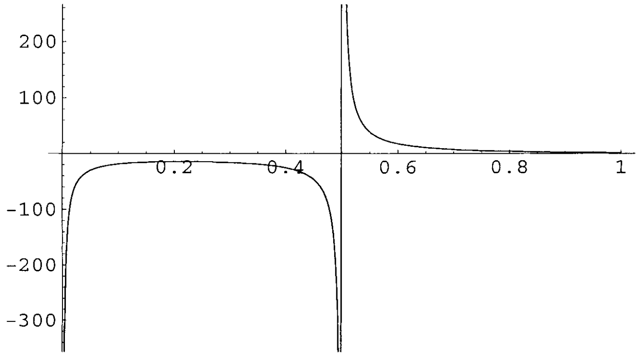

Figure 2. Graph $y_2(x)$

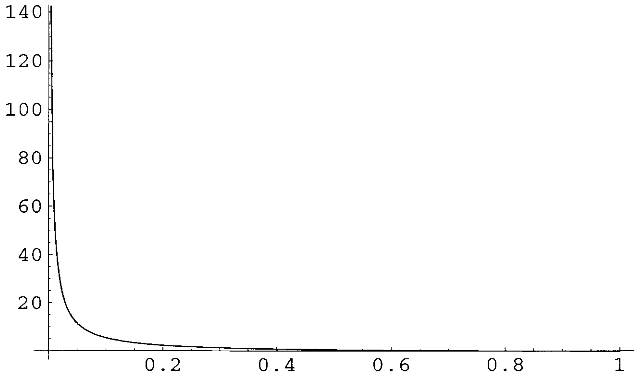

Figure 3. Graph $y_3(x)$

Lemma 2.2. *Let f and g two real functions given by (6) and (7) respectively. If* $a_0 \in (0, 0.544582\ldots)$, *then*

(i) $f(a_0, b_0)^2 g(a_0, b_0) < 1$,

(ii) *sequences* $\{a_n\}$ *and* $\{b_n\}$ *are decreasing*,

(iii) $a_n\left(\frac{a_n}{2} + \frac{5}{3}b_n + 1\right) < 1$ *for all* $n \geqslant 0$.

Proof. Notice that condition $f(a_0, b_0)^2 g(a_0, b_0) < 1$ is equivalent to $u(a_0, b_0) < 0$, where

$$u(x,y) = -216 + 432x + 54x^2 - 189x^3 + (362 + 900x - 540x^2 - 360x^3)y + (300x - 600x^2)y^2.$$

The roots of $u(x,y) = 0$ are functions $y_1(x)$ and $y_2(x)$. Moreover, condition $a_0\left(\frac{a_0}{2} + \frac{5}{3}b_0 + 1\right) < 1$ is equivalent to $v(a_0, b_0) < 0$, where

$$v(x,y) = x\left(\frac{x}{2} + \frac{5}{3}y + 1\right) - 1,$$

$v(x,y) = 0$ has a root, called $y_3(x)$.

It is easy to prove that $y_1 < y_3$ and $y_1 < y_2$ in $[0, 0.544582\ldots]$, so item (i) follows immediately. We show item (ii) by mathematical induction on n. The fact that $0 < a_1 < a_0$ and $0 < b_1 < b_0$ follows by previous item (i) and lemma 2.1 (i) and (ii). Next, it is supposed that $a_j < a_{j-1}$ and $b_j < b_{j-1}$ for $j = 1, 2, \ldots, n$. Then

$$a_{n+1} = a_n f(a_n, b_n)^2 g(a_n, b_n) < a_n f(a_0, b_0)^2 g(a_0, b_0) < a_n$$

and

$$\begin{aligned} b_{n+1} &= b_n f(a_n, b_n)^3 g(a_n, b_n)^2 < b_n f(a_0, b_0)^3 g(a_0, b_0)^2 \\ &< b_n f(a_0, b_0)^4 g(a_0, b_0)^2 < b_n \end{aligned}$$

since f and g are increasing and $f(x, y) > 1$ for $x, y > 0$.

Finally, we have $a_n\left(\frac{a_n}{2} + \frac{5}{3}b_n + 1\right) < a_0\left(\frac{a_0}{2} + \frac{5}{3}b_0 + 1\right)$ for all $n \geqslant 0$, since $\{a_n\}$ and $\{b_n\}$ are decreasing sequences and $a_0 \in (0, 0.544582\ldots)$. □

Lemma 2.3. *Let us suppose the hypotheses of Lemma 2.2 and define* $\gamma = a_1/a_0$. *Then,*

(i$_n$) $a_n < \gamma^{3^{n-1}} a_{n-1} < \gamma^{\frac{3^n-1}{2}} a_0$, *for all* $n \geqslant 2$,

(ii$_n$) $b_n < (\gamma^{3^{n-1}})^2 b_{n-1} < \gamma^{3^n-1} b_0$, *for all* $n \geqslant 2$,

(iii$_n$) $f(a_n, b_n)g(a_n, b_n) < \gamma^{3^n-1} f(a_0, b_0)g(a_0, b_0) = \dfrac{\gamma^{3^n}}{f(a_0, b_0)}$, *for all* $n \geqslant 1$.

Proof. We prove (i$_n$) following an inductive procedure. As $a_1 = \gamma a_0$ and $b_1 = b_0 f(a_0, b_0)^3 g(a_0, b_0)^2 < b_0 f(a_0, b_0)^4 g(a_0, b_0)^2 < \gamma^2 b_0$, we have $a_2 = a_1 f(a_1, b_1)^2 g(a_1, b_1) < \gamma^3 a_1$, and $b_2 < b_1 f(a_1, b_1)^4 g(a_1, b_1)^2 < b_1\left(\frac{a_2}{a_1}\right)^2 < (\gamma^3)^2 b_1$. If we suppose that (i$_n$) hold, then

$$\begin{aligned} a_{n+1} &= a_n f(a_n, b_n)^2 g(a_n, b_n) \\ &< \gamma^{3^{n-1}} a_{n-1} f(\gamma^{3^{n-1}} a_{n-1}, (\gamma^{3^{n-1}})^2 b_{n-1})^2 g(\gamma^{3^{n-1}} a_{n-1}, (\gamma^{3^{n-1}})^2 b_{n-1}) \\ &< \gamma^{3^{n-1}} (\gamma^{3^{n-1}})^2 a_n = \gamma^{3^n} a_n. \end{aligned}$$

Moreover,

$$a_n < \gamma^{3^{n-1}} a_{n-1} < \gamma^{3^{n-1}} \gamma^{3^{n-2}} a_{n-2} < \cdots < \gamma^{\frac{3^n-1}{2}} a_0,$$

Item (ii$_n$) follows in a similar way.

On the other hand, we observe that

$$\begin{aligned} f(a_n, b_n)g(a_n, b_n) &< f(\gamma^{\frac{3^n-1}{2}} a_0, \gamma^{3^{n-1}} b_0) g(\gamma^{\frac{3^n-1}{2}} a_0, \gamma^{3^{n-1}} b_0) \\ &< \gamma^{3^n-1} f(a_0, b_0) g(a_0, b_0) = \frac{\gamma^{3^n}}{f(a_0, b_0)}, \quad n \geqslant 1. \end{aligned}$$

The proof is complete. □

After that we show the following result on the semilocal convergence of sequence (4).

Theorem 2.4. *Let X, Y be Banach spaces and $F:\Omega\subseteq X\to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain Ω. Let us assume that $\Gamma_0=F'(x_0)^{-1}\in\mathscr{L}(Y,X)$ exists at some $x_0\in\Omega$ and* (c$_1$)–(c$_4$) *are satisfied. Denote by $a_0=M\beta\eta$ and $b_0=K\beta\eta^2$. Under conditions of Lemma 2.2, if $\overline{B(x_0,R\eta)}\subseteq\Omega$, where $R=\left[\frac{1}{2}a_0+\frac{5}{3}b_0+1\right]\frac{1}{1-\gamma\Delta}$, $\gamma=a_1/a_0$ and $\Delta=1/f(a_0,b_0)$, the sequence $\{x_n\}$ defined in (4) has at least R-order of convergence three, and starting at x_0 converges to a solution x^* of $F(x)=0$. The solution x^* and the iterates x_n and y_n belong to $\overline{B(x_0,R\eta)}$. Moreover, the solution x^* is unique in $B\left(x_0,\frac{2}{M\beta}-R\eta\right)\cap\Omega$. Furthermore, the following error bounds hold:*

$$\|x^*-x_n\|\leqslant\left[\frac{1}{2}\left(\gamma^{\frac{3^n-1}{2}}\right)a_0+\frac{5}{3}(\gamma^{3^n-1})b_0+1\right]\gamma^{\frac{3^n-1}{2}}\frac{\Delta^n}{1-\gamma^{3^n}\Delta}\eta,\quad n\geqslant 0. \tag{11}$$

Proof. Firstly, we prove that $\{x_n\}$ is a Cauchy sequence. From [II$_n$], we observe that

$$\begin{aligned}\|x_{n+1}-x_n\|&\leqslant\left[\frac{1}{2}a_n+\frac{5}{3}b_n+1\right]\|y_n-x_n\|\\&\leqslant\left[\frac{1}{2}a_0+\frac{5}{3}b_0+1\right]f(a_{n-1},b_{n-1})g(a_{n-1},b_{n-1})\|y_{n-1}-x_{n-1}\|\\&\leqslant\cdots\leqslant\left[\frac{1}{2}a_0+\frac{5}{3}b_0+1\right]\|y_0-x_0\|\prod_{j=0}^{n-1}f(a_j,b_j)g(a_j,b_j).\end{aligned}$$

We now have, from Lemma 2.3

$$\prod_{j=0}^{n-1}f(a_j)g(a_j)\leqslant\prod_{j=0}^{n-1}(\gamma^{3^j}\Delta)=\gamma^{\frac{3^n-1}{2}}\Delta^n,$$

where $\gamma=a_1/a_0<1$ and $\Delta=1/f(a_0)<1$. So, for $m\geqslant 1$ and $n\geqslant 1$,

$$\begin{aligned}\|x_{n+m}-x_n\|&\leqslant\|x_{n+m}-x_{n+m-1}\|+\|x_{n+m-1}-x_{n+m-2}\|+\cdots+\|x_{n+1}-x_n\|\\&\leqslant\left[\frac{1}{2}a_{n+m-1}+\frac{5}{3}b_{n+m-1}+1\right]\eta\prod_{j=0}^{n+m-2}f(a_j,b_j)g(a_j,b_j)\\&\quad+\cdots+\left[\frac{1}{2}a_n+\frac{5}{3}b_n+1\right]\eta\prod_{j=0}^{n-1}f(a_j,b_j)g(a_j,b_j)\\&\leqslant\left[\frac{1}{2}a_n+\frac{5}{3}b_n+1\right]\left(\gamma^{\frac{3^{n+m-1}-1}{2}}\Delta^{n+m-1}+\cdots+\gamma^{\frac{3^n-1}{2}}\Delta^n\right)\eta\\&\leqslant\left[\frac{1}{2}\left(\gamma^{\frac{3^n-1}{2}}\right)a_0+\frac{5}{3}(\gamma^{3^n-1})b_0+1\right]\gamma^{\frac{3^n-1}{2}}\Delta^n\frac{1-\gamma^{\frac{3^nm}{2}}\Delta^m}{1-\gamma^{3^n}\Delta}\eta,\end{aligned}\tag{12}$$

by Bernouilli's inequality. Thus $\{x_n\}$ is a Cauchy sequence. For $m \geqslant 1$ and $n = 0$, we obtain

$$\|x_m - x_0\| < \left[\frac{1}{2}a_0 + \frac{5}{3}b_0 + 1\right]\left(1 + \gamma\Delta\frac{1-(\gamma^3\Delta)^{m-1}}{1-\gamma\Delta}\right)\eta < R\eta. \qquad (13)$$

On the other hand, it follows from (13) that $x_m \in B(x_0, R\eta)$ for all $m \geqslant 0$. We similarly have $y_n \in B(x_0, R\eta)$ for all $n \geqslant 0$.

To see that x^* is a solution of $F(x) = 0$, we note that $\|\Gamma_n F(x_n)\| \to 0$ as $n \to \infty$. Taking into account $\|F(x_n)\| \leqslant \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and that the sequence $\{\|F'(x_n)\|\}$ is bounded, we infer that $\|F(x_n)\| \to 0$ as $n \to \infty$. Consequently, we obtain $F(x^*) = 0$ by continuity of F.

To prove uniqueness, let us assume another solution z^* of $F(x) = 0$ exists in $B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$. Using the identity

$$0 = \Gamma_0[F(z^*) - F(x^*)] = \Gamma_0\int_0^1 F'(x^* + t(z^* - x^*))dt(z^* - x^*),$$

it suffices to prove that the operator $\int_0^1 F'(x^* + t(z^* - x^*))dt$ is invertible and then $z^* = x^*$. Indeed, from

$$\begin{aligned} &\|\Gamma_0\|\int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\|dt \\ &\leqslant M\beta\int_0^1 \|x^* + t(z^* - x^*) - x_0\|dt \\ &\leqslant M\beta\int_0^1 ((1-t)\|x^* - x_0\| + t\|z^* - x_0\|)dt < 1, \end{aligned}$$

it follows that $\left[\int_0^1 F'(x^* + t(z^* - x^*))dt\right]^{-1}$ exists.

Finally, by letting $m \to \infty$ in (12) and (13), we obtain (11) for all $n \geqslant 0$. In addition, from (11) it follows that the R-order of convergence [5] of (4) is at least three, since

$$\|x^* - x_n\| \leqslant \left(1 + \frac{a_0}{2}\right)\frac{\eta}{\gamma^{1/2}(1-\gamma\Delta)}(\gamma^{1/2})^{3^n}, \quad n \geqslant 0.$$

The proof is complete. □

3. Applications

Finally we give two examples to illustrate the previous results. We consider two functions used as a test in several papers. In these examples, we compare the error bounds obtained for different third order iterative processes.

Example 1 ([1]). Let $X = C[0,1]$ be the space of continuous functions defined on the interval [0,1], with the max-norm and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = \lambda x(s)\int_0^1 \frac{s}{s+t}x(t)dt - x(s) + 1,$$

with $s \in [0,1], x \in C[0,1]$ and $0 < \lambda \leqslant 2$. Integral equations of this kind (called Chandrasekhar equations) arise in elasticity or neutron transport problems (see [1], [6]).

For $\lambda = 1/4$, and starting at $x_0 = x_0(s) = 1$, we obtain (see [1]), $\|\Gamma_0\| = 1.53039421 = \beta$, $\|\Gamma_0 F(x_0)\| \leqslant 0.2651971 = \eta$, $\|F''(x)\| \leqslant 0.3465735 = M$ and $K = 0$. So $a_0 = M\beta\eta = 0.140659$ and $b_0 = 0$. We give the upper bound $C = 2.307135$ to the number $10^{10}\|x^* - x_2\|$, where x_2 is the second iterate of the new method (4). Taking into account the error estimates given in ([2]) and ([3]), the upper bound for $10^{10}\|x^* - x_2\|$ to the Midpoint method are $C = 5.67272$ and $C = 5.59621$ respectively. We slightly improved both constants.

Example 2. Let us consider the system of equations $F(x,y) = 0$ where $F : [5,6.5] \times [5,6.5] \to \mathbb{R}^2$ and

$$F(x,y) = (x^2 - y - 19, y^3/6 - x^2 + y - 17).$$

Then we have

$$F'(x,y)^{-1} = \frac{1}{y^2}\begin{pmatrix} \dfrac{1+y^2/2}{x} & 1/x \\ 2 & 2 \end{pmatrix}$$

if (x,y) does not belong to the lines $x = 0$ or $y = 0$. The second derivative is a bilinear operator on $\mathbb{R}^2$ given by

$$F''(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ \hline -2 & 0 \\ 0 & y \end{pmatrix}.$$

We take the max-norm in $\mathbb{R}^2$ and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

As in [6] we define the norm of a bilinear operator B on $\mathbb{R}^2$ by

$$\|B\| = \sup_{\|x\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} x_k \right|,$$

where $x = (x_1, x_2)$ and

$$B = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ \hline b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix}.$$

If we choose $\mathbf{x_0} = (5.5, 6.5)$, then the parameters appearing in theorem 2.4 are

$$\beta = 0.0995159, \quad \eta = 0.473866, \quad M = 6.5 \quad \text{and} \quad K = 1.$$

Table 2. Error Estimates by the Newton Method

n	$x^* - x_n$	$y^* - y_n$
0	0.500000000000000	0.500000000000000
1	0.026134122287968	0.037475345167652
2	0.000091036676663	0.000232132746151
3	$1.726783414298796 \times 10^{-9}$	$8.980472046340010 \times 10^{-9}$
4	$1.642326062324995 \times 10^{-18}$	$1.344147966902442 \times 10^{-17}$
5	$3.280946417712476 \times 10^{-36}$	$3.011222928213283 \times 10^{-35}$
6	$1.618890014525233 \times 10^{-71}$	$1.511243920566230 \times 10^{-70}$

Table 3. Error estimates by the Midpoint method

n	$x^* - x_n$	$y^* - y_n$
0	0.500000000000000	0.500000000000000
1	0.001499751488195	0.002719524170867
2	$1.033469592474757 \times 10^{-10}$	$5.116958907032497 \times 10^{-10}$
3	$3.972869118699638 \times 10^{-31}$	$3.411494897979901 \times 10^{-30}$
4	1.09431×10^{-91}	1.010981×10^{-90}
5	0.0×10^{-115}	0.0×10^{-115}

Table 4. Error estimates by modification of the Midpoint method

n	$x^* - x_n$	$y^* - y_n$
0	0.500000000000000	0.500000000000000
1	0.002764057990316	0.004953507953267
2	$1.264338970403490 \times 10^{-9}$	$6.173583846519567 \times 10^{-9}$
3	$1.399308445753599 \times 10^{-27}$	$1.198259788110184 \times 10^{-26}$
4	$9.48629093642443 \times 10^{-81}$	$8.76177083110573107 \times 10^{-80}$
5	-0.0×10^{-95}	0.0×10^{-98}
6	-0.0×10^{-115}	0.0×10^{-118}

Thus, $a_0 = 0.306522$, and $b_0 = 0.0223462$. Therefore the hypotheses of theorem 2.4 are verified and we obtain the existence domain $B(\mathbf{x_0}, 1.29556)$. Moreover this solution is unique in $B(\mathbf{x_0}, 2.47797)$.

In Tables 4, 5 and 6, we see, under an operational cost similar to the one of Newton's method, that the speed of convergence is increased using iteration (4). A similar speed of convergence close to the one of the Midpoint method iteration is obtained.

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