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Two-step Newton methods

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ABSTRACT

We present sufficient convergence conditions for two-step Newton methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. The advantages of our approach over other studies such as Argyros et al. (2010) [5], Chen et al. (2010) [11], Ezquerro et al. (2000) [16], Ezquerro et al. (2009) [15], Hernández and Romero (2005) [18], Kantorovich and Akilov (1982) [19], Parida and Gupta (2007) [21], Potra (1982) [23], Proinov (2010) [25], Traub (1964) [26] for the semilocal convergence case are: weaker sufficient convergence conditions, more precise error bounds on the distances involved and at least as precise information on the location of the solution. In the local convergence case more precise error estimates are presented. These advantages are obtained under the same computational cost as in the earlier stated studies. Numerical examples involving Hammerstein nonlinear integral equations where the older convergence conditions are not satisfied but the new conditions are satisfied are also presented in this study for the semilocal convergence case. In the local case, numerical examples and a larger convergence ball are obtained.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where, F is a Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$. Many problems in Applied Sciences reduce to solving an equation in the form (1.1). These solutions can be rarely found in closed form. That is why the most solution methods for these equations are iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of convergence balls from the information around a solution.

The Newton method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0 is an initial point, is undoubtedly the most popular iterative method for generating a sequence approximating $x^\star$. The Newton method is quadratically convergent if x_0 is chosen sufficiently close to the solution $x^\star$. There is a plethora of local as well as semilocal convergence results for the Newton method. We refer the reader to [1–26] (and the references there in) for the history and recent results on the Newton method. In order to increase the convergence order higher convergence order iterative methods have also been used [1,3,5–7,9,11,14–18,21,22,26,27]. The convergence domain usually gets smaller as the order of convergence of the method increases. That is why it is important to enlarge the convergence domain as much as possible using the same conditions and constants as before. This is our main motivation for this paper. In particular, we revisit the two-step Newton methods defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(y_n). \end{aligned} \tag{1.4}$$

Two-step Newton methods (1.3) and (1.4) are of convergence order four and three, respectively [1,3,6,7,15,18]. It is well known that if the Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le L\|x - y\| \quad \text{for each } x \text{ and } y \in \mathcal{D} \tag{1.5}$$

as well as

$$\|F'(x_0)^{-1}F(x_0)\| \le \nu \tag{1.6}$$

holds for some $L > 0$ and $\nu > 0$, then the sufficient semilocal convergence condition for both the Newton method (1.2) and the two-step Newton method (1.3) is given by the famous, for its simplicity and clarity, Newton–Kantorovich hypothesis [19]:

$$h = L\nu \le \frac{1}{2}. \tag{1.7}$$

Hypothesis (1.7) is only sufficient for the convergence of the Newton method. That is why we challenged it in a series of papers [1–8] by introducing the center-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_0\| \quad \text{for each } x \in \mathcal{D}. \tag{1.8}$$

Notice that

$$L_0 \le L \tag{1.9}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2,3,6,8]. Our sufficient convergence conditions are given by

$$h_1 = L_1\nu \le \frac{1}{2}, \tag{1.10}$$

$$h_2 = L_2\nu \le \frac{1}{2}, \tag{1.11}$$

and

$$h_3 = L_3 \nu \leq \frac{1}{2}, \tag{1.12}$$

where

$$\begin{aligned} L_1 &= \frac{L_0 + L}{2}, \\ L_2 &= \frac{1}{8}\left(L + 4L_0 + \sqrt{L^2 + 8L_0 L}\right) \\ &\text{and} \\ L_3 &= \frac{1}{8}\left(4L_0 + \sqrt{L_0 L} + \sqrt{L^2 + 8L_0 L}\right). \end{aligned} \tag{1.13}$$

Moreover, notice that

$$h \leq \frac{1}{2} \Rightarrow h_1 \leq \frac{1}{2} \Rightarrow h_2 \leq \frac{1}{2} \Rightarrow h_3 \leq \frac{1}{2} \tag{1.14}$$

but not necessarily vice versa unless if $L_0 = L$ and

$$\begin{aligned} &\frac{h_1}{h} \to \frac{1}{2}, \qquad \frac{h_2}{h} \to \frac{1}{4}, \qquad \frac{h_2}{h_1} \to \frac{1}{2}, \qquad \frac{h_3}{h} \to 0, \\ &\frac{h_3}{h_1} \to 0 \quad \text{and} \quad \frac{h_3}{h_2} \to 0 \quad \text{as } \frac{L_0}{L} \to 0. \end{aligned} \tag{1.15}$$

Hence, the convergence domain for the Newton method (1.2) has been extended under the same computational cost, since in practice the computation of L requires the computation of L_0. Moreover, the error estimates on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^\star\|$ are more precise and the information on the location of the solution at least as precise.

In the case of the two-step Newton method (1.4) the sufficient convergence condition using only (1.5) is given by [6,15,18]

$$h_4 = L_4 \nu \leq \frac{1}{2}, \tag{1.16}$$

where

$$L_4 = \frac{4 + \sqrt{21}}{4} L. \tag{1.17}$$

In the present paper using (1.5) and (1.8) we show that (1.12) can be used as the sufficient convergence condition for the two-step Newton method (1.3). Moreover, we show that the sufficient convergence condition for (1.4) is given by

$$h_5 = L_5 \nu \leq \frac{1}{2}, \tag{1.18}$$

where

$$L_5 = \frac{1}{4}\left(3L_0 + L + \sqrt{(3L_0 + L)^2 + L(4L_0 + L)}\right). \tag{1.19}$$

Notice that

$$h_4 \leq \frac{1}{2} \Rightarrow h_5 \leq \frac{1}{2} \tag{1.20}$$

but not necessarily vice versa unless if $L_0 = L$ and

$$\frac{h_5}{h_4} \to \frac{1 + \sqrt{2}}{4 + \sqrt{21}} < 1 \quad \text{as } \frac{L_0}{L} \to 0. \tag{1.21}$$

Condition (1.18) can be weakened even further (see Lemma 3.3).

In the local convergence case using the Lipschitz condition

$$\|F'(x^\star)^{-1}(F'(x)-F'(y))\| \le l\|x-y\| \quad \text{for each } x \text{ and } y \in \mathcal{D} \text{ and some } l>0 \tag{1.22}$$

the convergence radius used in the literature (see Rheinboldt [20] and Traub [26]) for both the Newton method (1.2) and the two-step Newton method (1.3) is given by

$$R_0 = \frac{2}{3l}. \tag{1.23}$$

Here, we use the center-Lipschitz condition

$$\|F'(x^\star)^{-1}(F'(x)-F'(x^\star))\| \le l\|x-x^\star\| \quad \text{for each } x \in \mathcal{D} \text{ and some } l_0>0, \tag{1.24}$$

to show that the convergence radius for both the Newton method (1.2) and the two-step Newton method (1.3) is given by

$$R_0 = \frac{2}{2l_0+l}. \tag{1.25}$$

Note that again

$$l_0 \le l \tag{1.26}$$

holds in general and $\frac{l}{l_0}$ can be arbitrarily large [2,3,6]. We also have that

$$R_0 \le R \tag{1.27}$$

and

$$\frac{R}{R_0} \to 3 \quad \text{as } \frac{l_0}{l} \to 0. \tag{1.28}$$

The radius of convergence R was found by us in [2,3,6] only for the Newton method. Here, we also have this result for the two-step Newton method (1.3). Moreover, in view of (1.22) there exists $l_1>0$ such that

$$\|F'(x^\star)^{-1}(F'(x)-F'(x_0))\| \le l_1\|x-x_0\| \quad \text{for all } x \in \mathcal{D}. \tag{1.29}$$

Note that

$$l_1 \le l \tag{1.30}$$

holds and $\frac{l}{l_1}$ can be arbitrarily large. Although the convergence radius R does not change, the error bounds are more precise when using (1.29). Finally, the corresponding results for the two-step Newton method (1.4) are presented with

$$R = \frac{2}{2l_0+5l}. \tag{1.31}$$

Many high convergence order iterative methods can be written as two-step methods [1,3,6,7,14–18,26,27]. Therefore, the technique of recurrent functions or the technique of simplified majorizing sequences given in this study can be used to study other high convergence order iterative methods. As an example, we suggest the Chebyshev method or the method of tangent parabolas, defined by

$$x_{n+1} = x_n - (I-M_n)F'(x_n)^{-1}F(x_n) \quad \text{for each } n=0,1,2,\ldots, \tag{1.32}$$

where x_0 is an initial point and

$$M_n = \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n) \quad \text{for each } n=0,1,2,\ldots.$$

Here, $F''(x)$ denotes the second Fréchet-derivative of operator F [3,6,19,26]. The Chebyshev method can be written as a two-step method of the form

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - \frac{1}{2}F'(x_n)^{-1}F''(x_n)(y_n - x_n)^2 \quad \text{for each } n = 0, 1, 2, \ldots. \end{aligned} \tag{1.33}$$

The paper is organized as follows. The convergence results of the majorizing sequences for two-step Newton methods (1.3) and (1.4) are given in Sections 2 and 3 respectively. The semilocal and local convergence analysis of two-step Newton methods (1.3) and (1.4) is presented in Sections 4 and 5, respectively. Finally, numerical examples are given in Section 6.

2. Majorizing sequences for the two-step Newton method (1.3)

We present sufficient convergence conditions and bounds on the limit points of majorizing sequences for the two-step method (1.3).

Lemma 2.1. *Let $L_0 > 0$, $L \geq L_0$ and $\nu > 0$ be given parameters. Set*

$$\alpha = \frac{2L}{L + \sqrt{L^2 + 8L_0L}}. \tag{2.1}$$

Suppose that

$$h_1 = L_1\nu \leq \frac{1}{2}, \tag{2.2}$$

where

$$L_1 = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0L}). \tag{2.3}$$

Then, scalar sequence $\{t_n\}$ given by

$$\begin{cases} t_0 = 0, \qquad s_0 = \nu, \\ t_{n+1} = s_n + \dfrac{L(s_n - t_n)^2}{2(1 - L_0s_n)} \\ s_{n+1} = t_{n+1} + \dfrac{L(t_{n+1} - s_n)^2}{2(1 - L_0t_{n+1})} \quad \text{for each } n = 0, 1, 2, \ldots \end{cases} \tag{2.4}$$

is well defined, increasing, bounded from above by

$$t^{**} = \frac{\nu}{1 - \alpha} \tag{2.5}$$

and converges to its unique least upper bound t^* which satisfies

$$\nu \leq t^* \leq t^{**}. \tag{2.6}$$

Moreover, the following estimates hold

$$t_{n+1} - s_n \leq \alpha(s_n - t_n) \leq \alpha^{2n+1}\nu, \tag{2.7}$$

$$s_n - t_n \leq \alpha(t_n - s_{n-1}) \leq \alpha^{2n}\nu \tag{2.8}$$

$$t^* - s_n \leq \frac{\alpha^{2n}\nu}{1 - \alpha} \tag{2.9}$$

and

$$t^* - t_n \leq \frac{\alpha^{2n}\nu}{1 - \alpha} + \alpha^{2n}\nu. \tag{2.10}$$

Proof. We first notice that $\alpha \in [\frac{1}{2}, 1)$ by (2.1). We shall show using mathematical induction that

$$\frac{L(s_k - t_k)}{2(1 - L_0 s_k)} \leq \alpha \tag{2.11}$$

and

$$\frac{L(t_{k+1} - s_k)}{2(1 - L_0 t_{k+1})} \leq \alpha. \tag{2.12}$$

If $k = 0$ in (2.11) we must have that

$$\frac{L(s_0 - t_0)}{2(1 - L_0 s_0)} \leq \alpha \quad \text{or} \quad \frac{L\nu}{2(1 - L_0 \nu)} \leq \alpha. \tag{2.13}$$

Using the value of α in (2.13) we can show instead that

$$\left(\frac{L}{2} + \frac{2LL_0}{L + \sqrt{L + \sqrt{L^2 + 8L_0 L}}}\right)\nu \leq \frac{2L}{L + \sqrt{L + \sqrt{L^2 + 8L_0 L}}}$$

which is (2.2). If $k = 0$ in (2.12) we must have

$$\frac{L(t_1 - s_0)}{2(1 - L_0 t_1)} \leq \alpha \quad \text{or} \quad (L^2 - 4L_0^2\alpha + 2L_0 L\alpha)\nu^2 + 8L_0\alpha\nu - 4\alpha \leq 0. \tag{2.14}$$

Case 1. $L^2 - 4L_0^2\alpha + 2L_0 L\alpha \geq 0$.
Then, (2.14) is satisfied provided that

$$\nu \leq \frac{-8L_0\alpha + \sqrt{(8L_0\alpha)^2 + 16\alpha(L^2 - 4L_0^2\alpha + 2L_0 L\alpha)}}{2(L^2 - 4L_0^2\alpha + 2L_0 L\alpha)} \tag{2.15}$$

or

$$\frac{2L_0\alpha + \sqrt{\alpha L^2 + 2L_0 L\alpha^2}}{2\alpha}\nu \leq 1. \tag{2.16}$$

In view of (2.2) and (2.16) we must show

$$\frac{2L_0\alpha + \sqrt{\alpha L^2 + 2L_0 L\alpha^2}}{2\alpha} \leq \frac{1}{4}(L + 4L_0 + \sqrt{L^2 + 8L_0 L})$$

or

$$2\sqrt{\alpha L^2 + 2L_0 L\alpha^2} \leq \alpha L + \alpha\sqrt{L^2 + 8L_0 L}$$

or

$$\alpha \geq \frac{2L}{L + \sqrt{L^2 + 8L_0 L}},$$

which is true as equality by (2.1).

Case 2. $L^2 - 4L_0^2\alpha + 2L_0L\alpha < 0$.

Then, again we must show that (2.15) is satisfied, which was shown in Case 1.

Case 3. $L^2 - 4L_0^2\alpha + 2L_0L\alpha = 0$.

Inequality (2.14) reduces to $2L_0\nu \leq 1$ which is true by (2.2). Hence, estimates (2.11) and (2.12) hold for $k = 0$. Let us assume they hold for $k \leq n$. Then, using (2.4), (2.11) and (2.12) we have in turn that

$$t_{k+1} - s_k = \frac{L(s_k - t_k)}{2(1 - L_0 s_k)}(s_k - t_k) \leq \alpha(s_k - t_k)$$
$$s_{k+1} - t_{k+1} = \frac{L(t_{k+1} - s_k)}{2(1 - L_0 t_{k+1})}(t_{k+1} - s_k) \leq \alpha(t_{k+1} - s_k)$$

leading to

$$t_{k+1} - s_k \leq \alpha(\alpha^2)^k \nu, \tag{2.17}$$
$$s_{k+1} - t_{k+1} \leq (\alpha^2)^{k+1}\nu, \tag{2.18}$$
$$\begin{aligned} t_{k+1} &\leq s_k + \alpha(\alpha^2)^k\nu \leq t_k + \alpha^{2k}\nu + \alpha\alpha^{2k}\nu \\ &\leq t_{k-1} + \alpha^{2(k-1)}\nu + \alpha\alpha^{2(k-1)}\nu + \alpha^{2k}\nu + \alpha\alpha^{2k}\nu \\ &\leq \cdots \leq t_0 + [\alpha^{2\cdot 0} + \cdots + \alpha^{2k}]\nu + \alpha[\alpha^{2\cdot 0} + \cdots + \alpha^{2k}]\nu \\ &= (1+\alpha)\frac{1 - \alpha^{2(k+1)}}{1 - \alpha^2}\nu < t^{**} \end{aligned} \tag{2.19}$$

and

$$s_{k+1} \leq (1+\alpha)\frac{1 - \alpha^{2(k+1)}}{1 - \alpha^2}\nu + \alpha^{2(k+1)}\nu. \tag{2.20}$$

In view of (2.11) and (2.17)–(2.19) we must show

$$\begin{aligned} &\frac{L}{2}(s_{k+1} - t_{k+1}) + L_0\alpha s_{k+1} - \alpha \leq 0 \\ &\text{or} \\ &\frac{L}{2}\alpha^{2(k+1)}\nu + L_0\alpha[(1+\alpha)\frac{1 - \alpha^{2(k+1)}}{1 - \alpha^2} + \alpha^{2(k+1)}]\nu - \alpha \leq 0. \end{aligned} \tag{2.21}$$

Estimate (2.21) motivates us to define recurrent functions f_k on $[0, \alpha^2]$ by

$$f_k(t) = \frac{L}{2}t^{k+1}\nu + L_0\sqrt{t}\left[(1 + \sqrt{t})\frac{1 - t^{k+1}}{1 - t} + t^{k+1}\right]\nu - \sqrt{t}. \tag{2.22}$$

We need a relationship between two consecutive functions f_k. Using (2.22) we get that

$$f_{k+1}(t) = f_k(t) + \left[\frac{L}{2}t + L_0\sqrt{t}t + L_0\sqrt{t}(1 + \sqrt{t})\right](t - 1)t^k\nu \leq f_k(t), \tag{2.23}$$

since $\alpha \in [0, 1)$ and the quantity in the bracket for $t = \alpha^2$ is non-negative. Then, in view of (2.21)–(2.23) we must show that

$$\begin{aligned} &f_0(\alpha^2) \leq 0 \\ &\text{or} \\ &\left[\frac{L}{2}\alpha + L_0(1 + \alpha + \alpha^2)\right]\nu \leq 1. \end{aligned} \tag{2.24}$$

We have that α is the unique positive root of equation

$$2L_0t^2 + Lt - L = 0. \tag{2.25}$$

It follows from (2.24) and (2.25) that we must show

$$\frac{1}{2}(L + 2L_0 + 2L_0\alpha)\nu \leq 1 \tag{2.26}$$

or in view of (2.2)

$$\frac{1}{2}(L + 2L_0 + 2L_0\alpha) \leq \frac{1}{4}(L + 4L_0 + \sqrt{L^2 + 8L_0L})$$

or

$$\alpha \leq \frac{2L}{L + \sqrt{L^2 + 8L_0L}},$$

which is true as equality. The induction for (2.11) is completed. Estimate (2.12) is satisfied, if

$$\begin{aligned} &\frac{L}{2}(t_{k+1} - s_k) + \alpha L_0 t_{k+1} - \alpha \leq 0 \\ &\text{or} \\ &\frac{L}{2}\alpha\alpha^{2k}\nu + \alpha L_0(1+\alpha)\frac{1-\alpha^{2(k+1)}}{1-\alpha^2}\nu - \alpha \leq 0. \end{aligned} \tag{2.27}$$

Estimate (2.27) motivates us to define recurrent functions g_k on $[0, \alpha^2]$ by

$$g_k(t) = \frac{L}{2}\sqrt{t}t^{k+1}\nu + \sqrt{t}L_0(1+\sqrt{t})\frac{1-t^{k+1}}{1-t}\nu - \sqrt{t}. \tag{2.28}$$

We must have that

$$g_{k+1}(t) = g_k(t) + \left(\frac{L}{2}\sqrt{t}t + L_0\sqrt{t}(1+\sqrt{t})\right)(t-1)t^k\nu \leq g_k(t) \quad \text{for all } t \in [0, \alpha^2], \tag{2.29}$$

since $\alpha \in [0, 1)$. Hence, we have that $g_{k+1}(\alpha^2) \leq g_k(\alpha^2) \leq \cdots \leq g_1(\alpha^2)$ in view of (2.28), estimate (2.27) holds, if

$$g_1(\alpha^2) \leq 0 \tag{2.30}$$

or

$$\frac{1}{2}(L\alpha^2 + 2L_0(1+\alpha)(1+\alpha^2))\nu \leq 1. \tag{2.31}$$

We have by (2.25) that

$$\begin{aligned} L\alpha^2 + 2L_0(1+\alpha)(1+\alpha^2) &= \frac{L(L-L\alpha)}{2L_0} + 2L_0(1+\alpha)\left(1 + \frac{L-L\alpha}{2L_0}\right) \\ &= \frac{L^2 - L^2\alpha + 2L_0(1+\alpha)[2L_0 + L - L\alpha]}{2L_0} \\ &= \frac{L^2 - L^2\alpha + 4L_0^2 + 4L_0^2\alpha + 2L_0L + 2L_0L\alpha - 2L_0L\alpha - 2L_0L\alpha^2}{2L_0} \\ &= \frac{L(L - L\alpha - 2L_0\alpha^2) + 4L_0^2 + 4L_0^2\alpha + 2L_0L}{2L_0} \\ &= \frac{2L_0(L + 2L_0 + 2\alpha L_0)}{2L_0} = L + 2L_0 + 2\alpha L_0. \end{aligned}$$

So, we must have

$$\left(\frac{L}{2}+(1+\alpha)L_0\right)\nu \le 1. \tag{2.32}$$

Then, in view of (2.2) it suffices to show that

$$\frac{L}{2}+(1+\alpha)L_0 \le \frac{1}{4}\left(L+4L_0+\sqrt{L^2+8L_0L}\right)$$

or

$$\alpha \le \frac{-L+\sqrt{L^2+8L_0L}}{4L_0}=\frac{2L}{L+\sqrt{L^2+8L_0L}}$$

which is true as equality. The induction for (2.12) is complete. Hence, sequence $\{t_n\}$ is increasing, bounded from above by t^{**} given by (2.5) and as such it converges to its unique least upper bound t^* which satisfies (2.6). Moreover, using (2.17) and (2.18) we have that

$$t_{k+m}-s_k = t_{k+m}-s_{k+m}+s_{k+m}-s_k$$

and

$$\begin{aligned} s_{k+m}-s_k &= (s_{k+m}-s_{k+m-1})+(s_{k+m-1}-s_k)\le\cdots \\ &\le \alpha\alpha^{2(k+m-1)}\nu+\alpha^{2(k+m-1)}\nu+\alpha\alpha^{2(k+m-2)}\nu+\alpha^{2(k+m-2)}\nu+\cdots+\alpha\alpha^{2k}\nu+\alpha^{2k}\nu \end{aligned}$$

so

$$\begin{aligned} t_{k+m}-s_k &\le \alpha^{2(m+k)}\nu+\alpha\alpha^{2k}\nu(1+\cdots+\alpha^{2(m-1)})+\alpha^{2k}\nu(1+\cdots+\alpha^{2(m-1)}) \\ &= \alpha^{2k}(1+\cdots+\alpha^{2m})\nu+\alpha\alpha^{2k}(1+\cdots+\alpha^{2(m-1)})\nu. \end{aligned} \tag{2.33}$$

By letting $m\to\infty$ in (2.33) we obtain (2.9). Furthermore, we have that

$$\begin{aligned} s_{k+m}-t_k &\le s_{m+k}-s_{m+k-1}+s_{m+k-1}-t_k \\ &\le \alpha\alpha^{2(m+k-1)}\nu+\alpha^{2(k+m-1)}\nu+\cdots+s_k-t_k \\ &\le \alpha\alpha^{2k}(1+\cdots+\alpha^{2(m-1)})\nu+\alpha^{2k}(1+\cdots+\alpha^{2(m-1)})\nu+\alpha^{2k}\nu. \end{aligned} \tag{2.34}$$

By letting $m\to\infty$ in (2.34) we obtain (2.10). That completes the proof of the lemma. ⊠

Remark 2.1. Let us define sequence $\{\bar{t}_n\}$ by

$$\begin{cases} \bar{t}_0=0, \quad \bar{s}_0=\nu, \quad \bar{t}_1=\bar{s}_0+\dfrac{L_0(\bar{s}_0-\bar{t}_0)^2}{2(1-L_0\bar{s}_0)} \\ \bar{t}_{n+1}=\bar{s}_n+\dfrac{L(\bar{s}_n-\bar{t}_n)^2}{2(1-L_0\bar{s}_n)} \\ \bar{s}_{n+1}=\bar{t}_{n+1}+\dfrac{L(\bar{t}_{n+1}-\bar{s}_n)^2}{2(1-L_0\bar{t}_{n+1})} \quad \text{for each } n=0,1,2,\ldots. \end{cases} \tag{2.35}$$

Clearly, $\{\bar{t}_n\}$ converges under (2.2) and is tighter than $\{t_n\}$. Indeed, a simple inductive argument shows that

$$\bar{t}_n \le t_n \tag{2.36}$$

$$\bar{s}_n \le s_n \tag{2.37}$$

$$\bar{t}_{n+1}-\bar{s}_n \le t_{n+1}-s_n \tag{2.38}$$

$$\bar{s}_{n+1}-\bar{t}_{n+1} \le s_{n+1}-t_{n+1} \tag{2.39}$$

and

$$\bar{t}^* = \lim_{n\to\infty} \bar{t}_n \le t^*. \tag{2.40}$$

Moreover, a strict inequality holds in (2.36)–(2.39) if $L_0 < L$ for $n \ge 1$. Note also that sequence $\{\bar{t}_n\}$ may converge under weaker hypothesis than (2.2) (see [8] and the Lemmas that follow).

Next, we present a different technique for studying sequence $\{t_n\}$. This technique is easier but it provides a less precise upper bound on t^* and t^{**}. We will first simplify sequence $\{t_n\}$. Let $L = bL_0$ for some $b \ge 1$, $r_n = L_0 t_n$ and $q_n = L_0 s_n$. Then, we have that sequence $\{r_n\}$ is given by

$$\begin{cases} r_0 = 0, \qquad q_0 = L_0 \nu, \\ r_{n+1} = q_n + \dfrac{b(q_n - r_n)^2}{2(1-q_n)} \\ q_{n+1} = r_{n+1} + \dfrac{b(r_{n+1} - q_n)^2}{2(1-r_{n+1})}. \end{cases}$$

Then, set $p_n = 1 - r_n$, $m_n = 1 - q_n$ to obtain sequence $\{p_n\}$ given by

$$\begin{cases} p_0 = 1, \qquad m_0 = 1 - L_0 \nu, \\ p_{n+1} = m_n - \dfrac{b(m_n - p_n)^2}{2m_n} \\ m_{n+1} = p_{n+1} - \dfrac{b(p_{n+1} - m_n)^2}{2p_{n+1}}. \end{cases}$$

Finally, set $\beta_n = 1 - \frac{p_n}{m_{n-1}}$ and $\alpha_n = 1 - \frac{m_n}{p_n}$ to obtain the sequence $\{\beta_n\}$ defined by

$$\alpha_{n+1} = \frac{b}{2}\left(\frac{\beta_{n+1}}{1-\beta_{n+1}}\right)^2 \tag{2.41}$$

$$\beta_{n+1} = \frac{b}{2}\left(\frac{\alpha_n}{1-\alpha_n}\right)^2. \tag{2.42}$$

We also have by substituting and eliminating β_{n+1} that

$$\alpha_{n+1} = \frac{b^3}{8} \frac{\alpha_n^4}{(1-\alpha_n)^4 \left[1 - \frac{b}{2}\frac{\alpha_n^2}{(1-\alpha_n)^2}\right]^2}.$$

Moreover, it follows from (2.41) and (2.42) that the convergence of the sequences $\{\alpha_n\}$, $\{\beta_n\}$ (i.e. $\{r_n\}$, $\{q_n\}$) is related to the equation

$$x = \frac{b}{2}\frac{x^2}{(1-x)^2}$$

which has zeros

$$x = 0, \qquad x = \frac{4L_0}{L + L_0 + \sqrt{L^2 + 8L_0 L}} \quad \text{and} \quad x = \frac{L + L_0 + \sqrt{L^2 + 8L_0 L}}{4L_0}.$$

Hence, we arrived at the following.

Lemma 2.2. *Suppose that (2.2) holds. Then, sequence $\{t_n\}$ is increasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound t^*, which satisfies*

$$\nu \le t^* \le \frac{1}{L_0}.$$

The following is an obvious and useful extension of Lemma 2.1.

Lemma 2.3. *Suppose that there exists $N = 0, 1, 2, \ldots$ such that*

$$t_0 < s_0 < t_1 < s_1 < \cdots < s_N < t_{N+1} < \frac{1}{L_0} \quad \text{and} \quad h^N = L_2(s_N - t_N) \le \frac{1}{2}, \tag{2.43}$$

where L_2 is given in (2.3). Then, scalar sequence $\{t_n\}$ given in (2.4) is well defined, increasing, bounded from above by

$$t_N^{**} = \frac{s_N - t_N}{1 - \alpha}$$

and converges to its unique least upper bound t_N^* which satisfies

$$\nu \le t_N^* \le t_N^{**}.$$

Moreover, estimates (2.7)–(2.10) hold with $s_N - t_N$ replacing n for $n \ge N$. Notice that if $N = 0$, we obtain (1.11) and for $N = 1$ we obtain (1.12) [8].

3. Majorizing sequences for the two-step Newton method (1.4)

In this section we present majorizing sequences for the two-step method (1.4) along the lines of Section 2.

Lemma 3.1. *Let $L_0 > 0$, $L \ge L_0$ and $\nu > 0$ be given parameters. Set*

$$\alpha = \frac{L}{2L_0 + L}. \tag{3.1}$$

Suppose that

$$h_5 = L_5 \nu \le \frac{1}{2}, \tag{3.2}$$

where

$$L_5 = \frac{1}{4}\left(L + 3L_0 + \sqrt{(L + 3L_0)^2 + L(L + 4L_0)}\right). \tag{3.3}$$

Then, scalar sequence $\{t_n\}$ given by

$$\begin{cases} t_0 = 0, \qquad s_0 = \nu, \\ t_{n+1} = s_n + \dfrac{L(s_n - t_n)^2}{2(1 - L_0 t_n)} \\ s_{n+1} = t_{n+1} + \dfrac{L[(t_{n+1} - s_n) + 2(s_n - t_n)]}{2(1 - L_0 t_{n+1})}(t_{n+1} - s_n) \quad \text{for each } n = 0, 1, 2, \ldots \end{cases} \tag{3.4}$$

is well defined, increasing, bounded from above by

$$t^{**} = \frac{\nu}{1 - \alpha} \tag{3.5}$$

and converges to its unique least upper bound t^* which satisfies

$$\nu \le t^* \le t^{**}. \tag{3.6}$$

Moreover, the following estimates hold

$$t_{n+1} - s_n \le \alpha(s_n - t_n) \le \alpha^{2n+1}\nu, \tag{3.7}$$

$$s_n - t_n \le \alpha(t_n - s_{n-1}) \le \alpha^{2n}\nu \tag{3.8}$$

$$t^* - s_n \le \frac{\alpha^{2n}\nu}{1-\alpha} \tag{3.9}$$

and

$$t^* - t_n \le \frac{\alpha^{2n}\nu}{1-\alpha} + \alpha^{2n}\nu. \tag{3.10}$$

Proof. We first notice that $\alpha \in [\frac{1}{3}, 1)$ by (3.1). As in Lemma 2.1 we shall show that

$$\frac{L(s_k - t_k)}{2(1 - L_0 s_k)} \le \alpha \tag{3.11}$$

and

$$\frac{L(t_{k+1} - s_k) + 2L(s_k - t_k)}{2(1 - L_0 t_{k+1})} \le \alpha. \tag{3.12}$$

If $k = 0$, (3.11) is satisfied, if

$$\frac{1}{4}(2L_0 + L)\nu \le \frac{1}{2}$$

which is true, since $\frac{2L_0+L}{4} \le L_2$. For $k = 0$, (3.12) becomes

$$\frac{\frac{L^2\nu^2}{2} + 2L\nu}{2\left(1 - L_0\left(\nu + \frac{L\nu^2}{2}\right)\right)} \le \frac{L}{2L_0 + L}$$

or

$$L(4L_0 + L)\nu^2 + 4(3L_0 + L)\nu - 4 \le 0 \tag{3.13}$$

which is true by (3.2). Hence, estimates (3.11) and (3.12) hold for $k = 0$. Then, assume they hold for all $k \le n$. As in Lemma 2.1, we have that

$$t_{k+1} - s_k \le \alpha^{2k+1}\nu, \tag{3.14}$$

$$s_{k+1} - t_{k+1} \le (\alpha^2)^{k+1}\nu, \tag{3.15}$$

$$t_{k+1} = (1+\alpha)\frac{1 - \alpha^{2(k+1)}}{1 - \alpha^2}\nu < t^{**} \tag{3.16}$$

and

$$s_{k+1} \le (1+\alpha)\frac{1 - \alpha^{2(k+1)}}{1 - \alpha^2}\nu + \alpha^{2(k+1)}\nu. \tag{3.17}$$

In view of (3.14)–(3.16), estimate (3.11) is satisfied if

$$\frac{L}{2}\alpha^{2n}\nu + L_0\alpha(1+\alpha)\frac{1 - \alpha^{2n}}{1 - \alpha^2}\nu - \alpha \le 0. \tag{3.18}$$

Estimate (3.18) motivates us to define recurrent functions f_k on $[0, \alpha^2]$ by

$$f_k(t) = \frac{L}{2}t^k\nu + L_0\sqrt{t}(1+\sqrt{t})\frac{1 - t^k}{1 - t}\nu - \sqrt{t}. \tag{3.19}$$

Then, we have that

$$f_{k+1}(t) = f_k(t) + \frac{1}{2}g(t)(t-1)t^{k-1}\nu \le f_k(t), \tag{3.20}$$

by the choice of α, where,

$$g(t) = 2L_0\sqrt{t}(1+\sqrt{t}) + Lt.$$

In view of (3.20) we have that for $t = \alpha^2$

$$f_{k+1}(\alpha^2) \le f_k(\alpha^2). \tag{3.21}$$

Hence, it follows from (3.21) that (3.18) is satisfied, if

$$f_1(\alpha^2) \le 0 \tag{3.22}$$

or

$$\frac{2L_0 + L}{2}\nu \le 1. \tag{3.23}$$

But (3.23) is true by (3.2). Similarly, (3.12) is satisfied if

$$\frac{L}{2}\alpha^{2k+1}\nu + L\alpha^{2k}\nu + \alpha L_0(1+\alpha)\frac{1-\alpha^{2(k+1)}}{1-\alpha^2}\nu - \alpha \le 0 \tag{3.24}$$

leading to the introduction of functions f_k^1 on $[0, \alpha^2]$ by

$$f_k^1(t) = \frac{L}{2}\sqrt{t}t^k\nu + Lt^k\nu + \sqrt{t}L_0(1+\sqrt{t})\frac{1-t^{k+1}}{1-t}\nu - \sqrt{t}. \tag{3.25}$$

Then, we have that

$$f_{k+1}^1(t) = f_k^1(t) + g^1(t)(t-1)t^k\nu \le f_k^1(t), \tag{3.26}$$

where

$$g^1(t) = \frac{L}{2}\sqrt{t} + L_0\sqrt{t}(1+\sqrt{t}) + L. \tag{3.27}$$

Hence, it follows from (3.26) that (3.24) is satisfied if $f_0^1(\alpha^2) \le 0$, (since $f_k^1(\alpha^2) \le f_{k-1}(\alpha^2) \le \cdots \le f_0^1(\alpha^2)$), which reduces to showing (3.13). The rest of the proof is identical to the proof of Lemma 2.1. The proof of Lemma 3.1 is complete. ⊠

Remark 3.1. Let us define sequence $\{\bar{t}_n\}$ by

$$\begin{cases} \bar{t}_0 = 0, \quad \bar{s}_0 = \nu, \quad \bar{t}_1 = \bar{s}_0 + \dfrac{L_0(\bar{s}_0 - \bar{t}_0)^2}{2(1-L_0\bar{t}_0)} \\ \bar{s}_1 = \bar{t}_1 + \dfrac{L(\bar{t}_1 - \bar{s}_0)^2 + 2L_0(\bar{s}_0 - \bar{t}_0)(\bar{t}_1 - \bar{s}_0)}{2(1-L_0\bar{t}_1)} \\ \bar{t}_{n+1} = \bar{s}_n + \dfrac{L(\bar{s}_n - \bar{t}_n)^2}{2(1-L_0\bar{t}_n)} \\ \bar{s}_{n+1} = \bar{t}_{n+1} + \dfrac{L[(\bar{t}_{n+1} - \bar{s}_n) + 2(\bar{s}_n - \bar{t}_n)](\bar{t}_{n+1} - \bar{s}_n)}{2(1-L_0\bar{t}_{n+1})} \quad \text{for each } n = 0, 1, 2, \ldots. \end{cases} \tag{3.28}$$

Then, sequence $\{\bar{t}_n\}$ is at least as tight as majorizing sequence $\{t_n\}$.

Using the sequence of modifications of sequence $\{t_n\}$ following Remark 2.1 we obtain in turn that

$$\begin{aligned}
r_0 &= 0, \qquad q_0 = L_0\nu,\\
r_{n+1} &= q_n + \frac{b(q_n-r_n)^2}{2(1-q_n)}\\
q_{n+1} &= r_{n+1} + \frac{b[(r_{n+1}-q_n)+2(q_n-r_n)](r_{n+1}-q_n)}{2(1-q_{n+1})}.\\
p_{n+1} &= m_n - \frac{b(m_n-p_n)^2}{2m_n}\\
m_{n+1} &= p_{n+1} - \frac{b[(p_{n+1}-m_n)-2b(p_n-m_n)](p_{n+1}-m_n)}{2m_{n+1}}.\\
\alpha_{n+1} &= \frac{b\beta_{n+1}(1-\alpha_n)(1-\alpha_{n+1})+2b\alpha_n\beta_{n+1}}{2(1-\beta_{n+1})(1-\alpha_n)(1-\beta_{n+1})}\\
\beta_{n+1} &= \frac{b}{2}\left(\frac{\alpha_n}{1-\alpha_n}\right)^2.
\end{aligned}$$

Hence, we arrive at the following.

Lemma 3.2. *Suppose that (3.2) holds. Then, the sequence $\{t_n\}$ is increasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound which satisfies*

$$\nu \le t^* \le \frac{1}{L_0}.$$

We also get the following.

Lemma 3.3. *Suppose that there exists $N = 0, 1, 2, \ldots$ such that*

$$\begin{aligned}
&t_0 < s_0 < t_1 < s_1 < \cdots < s_N < t_{N+1} < \frac{1}{L_0}\\
&\textit{and}\\
&h^N = L_5(s_N - t_N) \le \frac{1}{2}
\end{aligned} \tag{3.29}$$

where L_5 is given in (3.3). Then, the conclusions of Lemma 2.3 hold but with sequence $\{t_n\}$ given by (3.4).

4. Convergence of the two-step Newton method (1.3)

We present the semilocal convergence of two-step method (1.3) followed by the local convergence. From now on $U(\omega,\rho)$ and $\bar{U}(\omega,\rho)$ stand, respectively, for the open and the closed ball in X with center ω and radius $\rho > 0$.

First, for the semilocal convergence, we use (1.3) to obtain the identities

$$x_{n+1} - y_n = [-F'(y_n)^{-1}F'(x_0)]\left[F'(x_0)^{-1}\int_0^1 [F'(x_n+t(y_n-x_n)) - F'(x_n)](y_n-x_n)dt\right], \tag{4.1}$$

$$\begin{aligned}
y_{n+1} - x_{n+1} \;=\; & [-F'(x_{n+1})^{-1}F'(x_0)]\left[F'(x_0)^{-1}\int_0^1 [F'(y_n+t(x_{n+1}-y_n))\right.\\
&\left. - F'(y_n)](x_{n+1}-y_n)dt\right].
\end{aligned} \tag{4.2}$$

Moreover, if $F(x^*) = F(y^*) = 0$, we have that

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^*+t(y^*-x^*))(y^*-x^*)dt. \tag{4.3}$$

Then, using (4.1)–(4.3), it is standard to show (cf [2,3,6–8,18]) the following.

Theorem 4.1. *Let $F : \mathcal{D} \subset X \to Y$ be Fréchet differentiable. Suppose that there exists $x_0 \in \mathcal{D}$ and parameters $L_0 > 0, L \geq L_0, \nu \geq 0$ such that for each $x, y \in \mathcal{D}$,*

$$F'(x_0)^{-1} \in Ł(Y, X),$$
$$\|F'(x_0)^{-1}F(x_0)\| \leq \nu,$$
$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq L_0\|x - x_0\|,$$
$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq L\|x - y\|.$$

Moreover, suppose that hypotheses of Lemma 2.1, Lemma 2.2 or Lemma 2.3 hold and

$$\bar{U}(x_0, t^*) \subseteq \mathcal{D},$$

where t^* is given in Lemma 2.1. Then, the sequence $\{x_n\}$ generated by two-step method (1.3) is well defined, remains in $\bar{U}(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \bar{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - y_n\| \leq \bar{t}_{n+1} - \bar{s}_n,$$
$$\|y_n - x_n\| \leq \bar{s}_n - \bar{t}_n,$$
$$\|x_n - x^*\| \leq t^* - \bar{t}_n$$

and

$$\|y_n - y^*\| \leq t^* - \bar{s}_n$$

where the sequence $\{\bar{t}_n\}$ is given in (2.35). Furthermore, if there exists $r \geq t^*$ such that

$$\bar{U}(x_0, r) \subseteq \mathcal{D}$$

and

$$L_0(t^* + r) < 2,$$

then, the limit point x^* is the unique solution of equation $F(x) = 0$ in $\bar{U}(x_0, r)$.

Remark 4.1. (a) The limit point t^* can be replaced by $\frac{1}{L_0}$ or t^{**} (given in closed form in (2.5)) in Theorem 4.1.

(b) As already noted in the introduction the earlier results in the literature [9–27] use $L_0 = L$ in their theorems which clearly reduce to Theorem 4.1 (if $L = L_0$). The advantages of our approach have already been stated in the introduction.

Second, for the local convergence we obtain the identities

$$y_n - x^* = [-F'(x_n)^{-1}F'(x^*)]\left[F'(x^*)^{-1}\int_0^1 [F'(x^* + t(x_n - x^*)) - F'(x_n)](x_n - x^*)dt\right] \tag{4.4}$$

and

$$x_{n+1} - x^* = [-F'(y_n)^{-1}F'(x^*)]\left[F'(x^*)^{-1}\int_0^1 [F'(x^* + t(y_n - x^*)) - F'(y_n)](y_n - x^*)dt\right] \tag{4.5}$$

we can arrive at [2,3,6,8] the following.

Theorem 4.2. *Let $F : \mathcal{D} \subset X \to Y$ be Fréchet differentiable. Suppose that there exist $x^* \in \mathcal{D}$ and parameters $l_0 > 0, l_1 > 0, l > 0$ such that for each $x, y \in \mathcal{D}$,*

$$F(x^*) = 0,$$
$$F'(x^*)^{-1} \in Ł(Y, X),$$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq l_0\|x - x^*\|,$$
$$\|F'(x^*)^{-1}[F'(x) - F'(x_0)]\| \leq l_1\|x - x_0\|,$$
$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq l\|x - y\|$$

and

$$\bar{U}(x^*, R) \subseteq \mathcal{D},$$

where

$$R = \frac{2}{2l_0 + l}.$$

Then, the sequence $\{x_n\}$ *generated by two-step method* (1.3) *is well defined for each* $n = 0, 1, 2, \ldots$ *and converges to* $x^* \in \bar{U}(x_0, R)$ *provided that* $x_0 \in U(x^*, R)$*. Moreover the following estimates hold for each* $n = 0, 1, 2, \ldots$

$$\|y_n - x^*\| \leq \frac{\bar{l}\|x_n - x^*\|^2}{2(1 - l_0\|x_n - x^*\|)}$$

and

$$\|x_{n+1} - x^*\| \leq \frac{l\|y_n - x^*\|^2}{2(1 - l_0\|y_n - x^*\|)}$$

where

$$\bar{l} = \begin{cases} l_1 & \text{if } n = 0 \\ l & \text{if } n => 0. \end{cases}$$

Remark 4.2. If $l_1 = l = l_0$ the result reduces to [20,26] in the case of the Newton method. The radius is then given by $R_0 = \frac{2}{3l}$.

If $l_1 = l$ the result reduces to [2,3,6] in the case of the Newton method. The radius is again given by R. However, if $l_1 < l$, then the error bounds are finer (see $\bar{l}$ and $\|y_0 - x^*\|$).

5. Convergence of the two-step Newton method (1.4)

As in Section 4, we obtain the following identities for the semilocal convergence, but using (4.1), (4.3) and

$$y_{n+1} - x_{n+1} = [-F'(x_{n+1})^{-1}F'(x_0)]\left[F'(x_0)^{-1}\int_0^1 [F'(y_n + t(x_{n+1} - y_n)) - F'(y_n)](x_{n+1} - y_n)dt\right]. \tag{5.1}$$

Then, again we arrive at the following.

Theorem 5.1. *Let* $F : \mathcal{D} \subset X \to Y$ *be Fréchet differentiable. Suppose that there exist* $x_0 \in \mathcal{D}$ *and parameters* $L_0 > 0, L \geq L_0, \nu \geq 0$ *such that for each* $x, y \in \mathcal{D}$,

$$F'(x_0)^{-1} \in Ł(Y, X),$$
$$\|F'(x_0)^{-1}F(x_0)\| \leq \nu,$$
$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq L_0\|x - x_0\|,$$
$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq L\|x - y\|.$$

Moreover, suppose that hypotheses of Lemma 3.1, *Lemma* 3.2 *and Lemma* 3.3 *hold and*

$$\bar{U}(x_0, t^*) \subseteq \mathcal{D},$$

where t^* *is given in* (3.5)*. Then, the sequence* $\{x_n\}$ *generated by two-step method* (1.4) *is well defined, remains in* $\bar{U}(x_0, t^*)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x^* \in \bar{U}(x_0, t^*)$ *of equation*

$F(x) = 0$. Furthermore, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|x_{n+1} - y_n\| \le \bar{t}_{n+1} - \bar{s}_n,$$
$$\|y_n - x_n\| \le \bar{s}_n - \bar{t}_n,$$
$$\|x_n - x^*\| \le t^* - \bar{t}_n$$

and

$$\|y_n - y^*\| \le t^* - \bar{s}_n$$

where the sequence $\{\bar{t}_n\}$ is given in (3.28). If there exists $r \ge t^*$ such that

$$\bar{U}(x_0, r) \subseteq \mathcal{D}$$

and

$$L_0(t^* + r) < 2,$$

then, the limit point x^* is the unique solution of equation $F(x) = 0$ in $\bar{U}(x_0, r)$.

Remark 5.1. These remarks as similar to the Remarks in 4.2 are omitted.

The identities for the local convergence case using (1.4) are (4.4) and

$$x_{n+1} - x^* = [-F'(x_n)^{-1}F'(x^*)]\left[F'(x^*)^{-1}\int_0^1 [F'(x^* + t(y_n - x^*)) - F'(y_n)](y_n - x^*)dt + (F'(y_n) - F'(x_n))(y_n - x^*)\right]$$

to obtain the following.

Theorem 5.2. *Let $F : \mathcal{D} \subset X \to Y$ be Fréchet differentiable. Suppose that there exist $x^* \in \mathcal{D}$ and parameters $l_0 > 0, l_1 > 0, l > 0$ such that for each $x, y \in \mathcal{D}$,*

$$F(x^*) = 0,$$
$$F'(x^*)^{-1} \in Ł(Y, X),$$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le l_0\|x - x^*\|,$$
$$\|F'(x^*)^{-1}[F'(x) - F'(x_0)]\| \le l_1\|x - x_0\|,$$
$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \le l\|x - y\|$$

and

$$\bar{U}(x^*, R) \subseteq \mathcal{D},$$

where

$$R = \frac{2}{2l_0 + 5l}.$$

Then, the sequence $\{x_n\}$ generated by two-step method (1.4) is well defined for each $n = 0, 1, 2, \ldots$ and converges to $x^* \in \bar{U}(x_0, R)$ provided that $x_0 \in U(x^*, R)$. Moreover the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|y_n - x^*\| \le \frac{\bar{l}\|x_n - x^*\|^2}{2(1 - l_0\|x_n - x^*\|)}$$

and

$$\|x_{n+1} - x^*\| \le \frac{l[\|y_n - x^*\| + 2\|y_n - x_n\|]\|y_n - x^*\|}{2(1 - l_0\|y_n - x^*\|)}$$

where $\bar{l}$ is given in Theorem 4.2.

Remark 5.2. We are not aware of any results in the literature involving the local convergence of the two-step Newton method (1.4). But if there is, see Remark 4.2.

6. Numerical examples

In the semilocal convergence the old convergence conditions are not satisfied but the new conditions are satisfied. Moreover in the local convergence case our convergence ball is larger than the older ones. We present six numerical examples. The first four involve the semilocal convergence and the last two the local convergence.

Example 1: *Semilocal convergence for the two-step Newton method* (1.3). In the following example, we consider the real function

$$x^3 - 0.49 = 0. \tag{6.1}$$

We take the starting point $x_0 = 1$ and we consider the domain $\Omega = B(x_0, 0.5)$. In this case, we obtain

$$\nu = 0.17, \tag{6.2}$$

$$L = 3 \tag{6.3}$$

and

$$L_0 = 2.5. \tag{6.4}$$

Notice that Kantorovich hypothesis $L\nu \leq 0.5$ is not satisfied, but condition (2.2) in Lemma 2.1 is satisfied since

$$L_1 = 2.66333\cdots$$

and

$$h_1 = L_1\nu = 0.452766\cdots \leq 0.5.$$

So, the two-step Newton method starting from $x_0 \in B(x_0, 0.5)$ converges to the solution of (6.1) from Theorem 4.1.

Example 2: *Semilocal convergence for the two-step Newton method* (1.3). Let $X = Y = \mathcal{C}[0, 1]$, the space of continuous functions defined in $[0, 1]$ equipped with the max-norm. Let $\Omega = \{x \in \mathcal{C}[0, 1]; \|x\| \leq R\}$, such that $R > 1$ and F defined on Ω and given by

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 G(s, t)x(t)^3\, dt, \quad x \in C[0, 1], s \in [0, 1],$$

where $f \in \mathcal{C}[0, 1]$ is a given function, λ is a real constant and the kernel G is the Green function

$$G(s, t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

In this case, for each $x \in \Omega$, $F'(x)$ is a linear operator defined on Ω by the following expression:

$$[F'(x)(v)](s) = v(s) - 3\lambda \int_0^1 G(s, t)x(t)^2 v(t)\, dt, \quad v \in C[0, 1], s \in [0, 1].$$

If we choose $x_0(s) = f(s) = 1$, it follows $\|I - F'(x_0)\| \leq 3|\lambda|/8$. Thus, if $|\lambda| < 8/3$, $F'(x_0)^{-1}$ is defined and

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\lambda|}.$$

Moreover,

$$\|F(x_0)\| \leq \frac{|\lambda|}{8},$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\lambda|}{8 - 3|\lambda|}.$$

On the other hand, for $x, y \in \Omega$ we have

$$[(F'(x) - F'(y))v](s) = 3\lambda \int_0^1 G(s,t)(x(t)^2 - y^2(t))v(t)\,dt.$$

Consequently,

$$\|F'(x) - F'(y)\| \le \|x-y\| \frac{3|\lambda|(\|x\| + \|y\|)}{8} \le \|x-y\| \frac{6R|\lambda|}{8},$$
$$\|F'(x) - F'(1)\| \le \|x-1\| \frac{1 + 3|\lambda|(\|x\| + 1)}{8} \le \|x-1\| \frac{1 + 3(1+R)|\lambda|}{8}.$$

Choosing $\lambda = 1$ and $R = 2.6$, we have

$$\nu = \frac{1}{5},$$
$$L = 3.12$$

and

$$L_0 = 2.16.$$

Hence, condition (1.7), $2L\nu = 1.248 \le 1$ is not satisfied, but condition (2.2) $L_1\nu = 0.970685 \le 1$ is satisfied. We can ensure the convergence of $\{x_n\}$ by Theorem 4.1.

Example 3: *Semilocal convergence for the two-step Newton method* (1.4). Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, equipped with the max-norm. Consider the following nonlinear boundary value problem

$$\begin{cases} u'' = -u^3 - \gamma\, u^2 \\ u(0) = 0, \qquad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)\,(u^3(t) + \gamma\, u^2(t))\,dt \tag{6.5}$$

where, $\mathcal{Q}$ is the Green function:

$$\mathcal{Q}(s,t) = \begin{cases} t\,(1-s), & t \le s \\ s\,(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |\mathcal{Q}(s,t)|\,dt = \frac{1}{8}.$$

Then problem (6.5) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)\,(x^3(t) + \gamma\, x^2(t))\,dt.$$

Set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R_0)$. It is easy to verify that $U(u_0, R_0) \subset U(0, R_0 + 1)$ since $\|u_0\| = 1$. If $2\,\gamma < 5$, the operator F' satisfies conditions of Theorem 5.1 with

$$\nu = \frac{1+\gamma}{5 - 2\,\gamma}, \qquad L = \frac{\gamma + 6\,R_0 + 3}{4(5 - 2\,\gamma)}, \qquad L_0 = \frac{2\,\gamma + 3\,R_0 + 6}{8(5 - 2\,\gamma)}.$$

Note that $L_0 < L$. Choosing $R_0 = 1$ and $\gamma = 0.6$, condition (1.16) $\frac{4+\sqrt{21}}{4} L\nu = 0.570587\cdots \le 0.5$ is not satisfied, but condition (3.2) is satisfied as

$$\frac{1}{4}\left(3L_0 + L + \sqrt{(3L_0 + L)^2 + L(4L_0 + L)}\right)\nu = 0.381116\cdots \le 0.5.$$

So, we can ensure the convergence of $\{x_n\}$ by Theorem 5.1.

Example 4: *Semilocal convergence for the two-step Newton method* (1.4). Let $\mathcal{X} = [-1, 1]$, $\mathcal{Y} = \mathbb{R}$, $x_0 = 0$ and $F : \mathcal{X} \to \mathcal{Y}$ be the polynomial:

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{9}.$$

In this case, since $\|F'(0)^{-1}F(0)\| \leq 0.13333\cdots = \nu$, $L = \frac{22}{10}$ and $L_0 = \frac{13}{10}$, condition (1.16) $\frac{4+\sqrt{21}}{4} L\nu = 0.629389\cdots \leq 0.5$ is not satisfied, but condition (3.2) $\frac{1}{4}\left(L+3L_0+\sqrt{(L+3L_0)^2+L(L+4L_0)}\right)\nu = 0.447123\cdots \leq 0.5$, is satisfied. Hence, by Theorem 5.1, the sequence $\{x_n\}$ generated by the two step Newton method (1.4), is well defined and converges to a solution x^* of $F(x) = 0$.

Example 5: *Local convergence for both two step Newton methods*. Let $X = Y = \mathbb{R}^3$, $D = U(0, 1)$, $x^* = (0, 0, 0)$ and define the function F on D by

$$F(x, y, z) = (e^x - 1, y^2 + y, z). \tag{6.6}$$

We have that for $u = (x, y, z)$

$$F'(u) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & 2y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.7}$$

Using the norm of the maximum of the rows and (6.6)–(6.7) we see that since $F'(x^*) = \operatorname{diag}\{1, 1, 1\}$, we can define parameters for the Newton method by

$$l = l_1 = e, \tag{6.8}$$

and

$$l_0 = 2. \tag{6.9}$$

Then the two-step Newton method (1.3) starting from $x_0 \in B(x^*, R^*)$ converges to a solution of (6.6). Note that this radius is greater than the Rheinboldt or Traub one [26] given by $R^*_{TR} = \frac{2}{3e} < \frac{2}{4+e} = R^*$. Moreover, hypotheses of Theorems 5.3 hold. Note that again $l_0 < l$. Then, the two-step Newton method (1.4) starting from $x_0 \in B(x^*, R)$, where $R = \frac{2}{2l_0+5l} = \frac{2}{4+5e}$, converges to x^*.

Example 6: *Local convergence for both two step Newton methods*. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, the space of continuous functions defined on $[0, 1]$, equipped with the max norm and $\mathcal{D} = \overline{U}(0, 1)$. Define the function F on $\mathcal{D}$ given by

$$F(h)(x) = h(x) - 5\int_0^1 x\,\theta\, h(\theta)^3\, d\theta. \tag{6.10}$$

Then, we have:

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\,\theta\, h(\theta)^2\, u(\theta)\, d\theta \quad \text{for all } u \in \mathcal{D}.$$

Using (6.10), hypotheses of Theorem 4.2 hold for $x^\star(x) = 0$ $(x \in [0, 1])$, $l = l_1 = 15$ and $l_0 = 7.5$.

Then the two-step Newton method (1.3) starting from $x_0 \in B(x^*, R^*)$ converges to a solution of (6.6). Note that the radius R^* is bigger than Rheinboldt or Traub one [26] given by $R^*_{TR} = \frac{2}{45} < \frac{1}{15} = R^*$. Moreover, hypotheses of Theorem 5.2 hold for the same value of the constants. Note that again $l_0 < l$. Then, the two-step Newton method (1.4) starting from $x_0 \in B(x^*, R)$, where $R = \frac{2}{2l_0+5l} = \frac{1}{45}$, converges to x^*.

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