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Secant-like methods for solving nonlinear integral equations of the Hammerstein type

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Abstract

We consider a one-parametric family of secant-type iterations for solving nonlinear equations in Banach spaces. We establish a semilocal convergence result for these iterations by means of a technique based on a new system of recurrence relations. This result is then applied to obtain existence and uniqueness results for nonlinear integral equations of the Hammerstein type. We also present a numerical example where the solution of a particular Hammerstein integral equation is approximated by different secant-type methods. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we study the solution of nonlinear integral equations of the Hammerstein type [4,6,7]:

$$x(s) = h(s) + \int_0^1 R(s,t)\,p(t,x(t))\,\mathrm{d}t, \tag{1}$$

where $x, h \in C[0,1]$, $R(s,t)$ is the kernel of a linear integral operator in $C[0,1]$ and $p(t,u)$ is a continuous function for $0 \leqslant t \leqslant 1$, $u \in U \subseteq \mathbb{R}$.

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In the function space $C[0,1]$ we introduce the operator F:

$$[F(x)](s) = x(s) - h(s) - \int_0^1 R(s,t)\,p(t,x(t))\,\mathrm{d}t \tag{2}$$

and rewrite (1) as a functional equation

$$F(x) = 0. \tag{3}$$

A linear and bounded operator from X to Y, denoted $[x,y;F]$, which satisfies the condition

$$[x,y;F](x-y) = F(x) - F(y)$$

is called a first order divided difference of F at the points x and y, see [9]. This condition does not uniquely determine the divided difference, with the exception of the case when X is one-dimensional. For the existence of divided differences in linear spaces, see [2].

The classical secant method is an efficient algorithm for solving nonlinear operator equation (3) [1,8]. As suggested by the secant iteration formula

$$x_{n+1} = x_n - [x_{n-1},x_n;F]^{-1}F(x_n), \quad x_{-1},x_0 \quad \text{pre-chosen},$$

there are two main elements for applying this method, the smoothness properties of the operator F, and the use of the first order divided difference of the operator F instead of the first derivative of F.

It is well known that for smooth equations, the classical secant method is superlineal convergent with Q-order at $(1+\sqrt{5})/2$ (see [9]).

Some Newton-like methods can be considered as generalized secant methods since they use only operator values. Considering methods based only on operator values, the following iteration is given:

$$\begin{aligned} & x_{-1},x_0 \quad \text{pre-chosen},\\ & y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad \lambda \in [0,1], \\ & x_{n+1} = x_n - [y_n,x_n;F]^{-1}F(x_n). \end{aligned} \tag{4}$$

In the real case, it is clear that the closer x_n and y_n are together, the higher the speed of the convergence. Moreover, observe that (4) is reduced to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$, since $y_n = x_n$ and $[y_n,x_n;F] = F'(x_n)$ (see [9]).

The use of the secant method is interesting, since the calculation of the first derivative F' is not required and the convergence of the method of successive substitutions is improved, although it is slower than Newton's method. For this, we consider iteration (4), whose speed of convergence is closed to that of Newton's iteration, when λ is near 1 (the Newton process), as we can see in the numerical example (Section 4), where the speed of convergence of (4) with $\lambda = 1 - 10^{-5}$ is similar to that of Newton's method but without evaluating the first derivative of the operator F.

In this paper, we analyse the convergence of (4) to a solution of Eq. (3), where F is a nonlinear operator defined on an open convex subset of a Banach space X with values in another Banach space Y. We use a technique consisting of a new system of recurrence relations.

We then consider the application of (4) to nonlinear integral equations of the Hammerstein type. We provide a semilocal convergence result for (4) which can be used to prove the existence of a unique solution of (1).

Next, we use a discretization scheme and approximate, by (4), the solution of a particular Hammerstein equation that is a version of the so-called H-equation which arises in the theory of radiative transfer (see [3]). We will see that iterations of (4) converge faster for increasing values of the parameter $\lambda \in [0,1]$.

Finally, we comment that the major difference between the present analysis and that of consisting of interpolation is the fact that for polynomial interpolation, uniform convergence of the interpolants cannot be guaranteed for every continuous function. Hence, the present analysis is based on semilocal convergence property. Indeed this is due to the fact that polynomial interpolation based on equally spaced grid points are not well behaved compared to, for example, Legendre–Gauss–Lobatto or Chebyshev grid points (see [4,5]).

2. Convergence analysis: an existence and uniqueness result for any operator

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear operator in an open convex subset $\Omega_0 \subseteq \Omega$. Let us assume that there exists a first order divided difference of F at every point $x, y \in \Omega_0$. Suppose that:

(A1) $\|x_0 - x_{-1}\| = \alpha$,
(A2) there exists $L_0^{-1} = [y_0, x_0; F]^{-1}$ such that $\|L_0^{-1}\| \leqslant \beta$,
(A3) $\|L_0^{-1}F(x_0)\| \leqslant \eta$,
(A4) $\|[x,y;F] - [u,v;F]\| \leqslant k(\|x-u\| + \|y-v\|)$, $k \geqslant 0$, $x,y,u,v \in \Omega_0$, $x \neq y$, $u \neq v$.

Under these conditions, we establish a system of recurrence relations from which the convergence of (4) is proved later. Let us denote

$$a_{-1} = \frac{\eta}{\alpha + \eta}, \qquad b_{-1} = \frac{k\beta\alpha^2}{\alpha + \eta}$$

and define the real sequences

$$a_n = f(a_{n-1})g(a_{n-1})b_{n-1}, \quad b_n = f(a_{n-1})^2 a_{n-1} b_{n-1}, \quad n \geqslant 0, \tag{5}$$

where

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = (1-\lambda) + (1+\lambda) f(x) x.$$

Note that f and g are increasing in $\mathbb{R} - \{1\}$, and that $f(x) > 1$ in $(0,1)$.

From the initial hypotheses, it follows that x_1 is well defined, since L_0^{-1} exists, and

$$\begin{aligned} &\|x_1 - x_0\| = \|L_0^{-1}F(x_0)\| \leqslant \eta = f(a_{-1})a_{-1}\|x_0 - x_{-1}\|, \\ &k\|L_0^{-1}\|\|x_0 - x_{-1}\| \leqslant k\beta\alpha = f(a_{-1})b_{-1}. \end{aligned} \tag{6}$$

Next, we prove the following recurrence relations for $n \geqslant 1$ by mathematical induction on n:

($\mathrm{i_n}$) there exists an $L_n^{-1} = [y_n, x_n; F]^{-1}$ such that $\|L_n^{-1}\| \leqslant f(a_{n-1})\|L_{n-1}^{-1}\|$,
($\mathrm{ii_n}$) $\|x_{n+1} - x_n\| \leqslant f(a_{n-1})a_{n-1}\|x_n - x_{n-1}\|$,
($\mathrm{iii_n}$) $k\|L_n^{-1}\|\|x_n - x_{n-1}\| \leqslant f(a_{n-1})b_{n-1}$.

Assuming $a_0 < 1$ and $x_1 \in \Omega_0$, from (5) and (6) we have

$$\|I - L_0^{-1}L_1\| \leqslant \|L_0^{-1}\|\|L_0 - L_1\| \leqslant k\|L_0^{-1}\|(\|y_1 - y_0\| + \|x_1 - x_0\|)$$

$$\leqslant k\|L_0^{-1}\|[(1-\lambda) + (1+\lambda)f(a_{-1})a_{-1}]\|x_0 - x_{-1}\| \leqslant a_0 < 1$$

and by the Banach lemma, L_1^{-1} exists and

$$\|L_1^{-1}\| \leqslant f(a_0)\|L_0^{-1}\|.$$

Hence (i_1) holds.

Using the following Taylor formula:

$$F(x_1) = (F'(x_0) - L_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\,\mathrm{d}t,$$

and taking into account (A_4) and $[x,x;F] = F'(x)$ we now obtain

$$\|F(x_1)\| \leqslant k[(1-\lambda)\|x_0 - x_{-1}\| + \|x_1 - x_0\|]\|x_1 - x_0\|$$

$$\leqslant kg(a_{-1})\|x_0 - x_{-1}\|\|x_1 - x_0\|.$$

As L_1^{-1} exists and x_2 is well defined, it follows that

$$\|x_2 - x_1\| \leqslant f(a_0)\|L_0^{-1}\|\|F(x_1)\| \leqslant f(a_0)a_0\|x_1 - x_0\|$$

and (ii_1) is true.

Note that

$$k\|L_1^{-1}\|\|x_1 - x_0\| \leqslant kf(a_0)\|L_0^{-1}\|\|x_1 - x_0\| \leqslant f(a_0)b_0$$

as a consequence of (6) and (i_1).

Finally, assuming $a_n < 1$ and $x_n \in \Omega_0$, for $n \geqslant 1$, items (i_{n+1})–(iii_{n+1}) follow analogously and the induction is complete.

To study the convergence of the sequence $\{x_n\}$, we analyse the sequences $\{a_n\}$ and $\{b_n\}$ given by (5). It is sufficient to see that $\{x_n\}$ is a Cauchy sequence and $a_n < 1$ for all $n \geqslant 0$.

Firstly, if we denote by $\{\alpha_n\}$ the Fibonacci sequence

$$\alpha_1 = \alpha_2 = 1 \quad \text{and} \quad \alpha_{n+2} = \alpha_{n+1} + \alpha_n, \quad n \geqslant 1 \tag{7}$$

and

$$s_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad n \geqslant 1, \tag{8}$$

the following properties can be proved, again by induction:

(P1) $$\alpha_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] > \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}, \quad n \geqslant 1,$$

(P2) $$s_n = \alpha_{n+2} - 1 \quad \text{and} \quad \beta_n = s_1 + s_2 + \cdots + s_n = \alpha_{n+4} - (n+3), \quad n \geqslant 1.$$

Secondly, some properties for the sequence $\{a_n\}$ and $\{b_n\}$ are proved in the following lemma.

Lemma 2.1. *Let* $\{a_n\}$ and $\{b_n\}$ *be the sequences defined in* (5) *and* $\lambda \in [0,1]$ *as a fixed element. If* $a_{-1} < (3-\sqrt{5})/2$ *and* $b_{-1} < a_{-1}(1-a_{-1})^2/(1+\lambda(2a_{-1}-1))$, *then*

(a) $\{a_n\}$ *and* $\{b_n\}$ *are decreasing*,
(b) $\gamma = b_0/b_{-1} \in (0,1)$ *and* $a_0/(1-a_0) < \gamma$,
(c) $a_n < \gamma^{\alpha_n} a_{n-1}$ *and* $b_n < \gamma^{\alpha_{n+1}} b_{n-1}$, *for* $n \geqslant 1$,
(d) $a_n < \gamma^{s_n} a_0$ *for* $n \geqslant 1$.

Proof. (a) The proof is by induction. From the hypotheses, it follows that $a_0 < a_{-1}$ and $b_0 < b_{-1}$. If it is true that $a_{j-1} > a_j$ and $b_{j-1} > b_j$ for $j = 0, 1, \ldots, n$, then

$$a_{n+1} < f(a_{n-1})g(a_{n-1})b_{n-1} = a_n \quad \text{and} \quad b_{n+1} < f(a_{n-1})^2 a_{n-1} b_{n-1} = b_n,$$

since f and g are increasing.
(b) This is immediately obvious from the hypotheses.
(c) This is again proved by induction. From $a_0 < a_{-1}$ and $b_0 = \gamma b_{-1}$ we deduce

$$a_1 < f(a_{-1})g(a_{-1})\gamma b_{-1} = \gamma a_0 \quad \text{and} \quad b_1 < f(a_{-1})^2 a_{-1} \gamma b_{-1} = \gamma b_0.$$

If it is supposed that $b_j < \gamma^{\alpha_{j+1}} b_{j-1}$ and $a_j < \gamma^{\alpha_j} a_{j-1}$ for $j = 1, 2, \ldots, n$, then

$$a_{n+1} < f(a_{n-1})g(a_{n-1})\gamma^{\alpha_{n+1}} b_{n-1} = \gamma^{\alpha_{n+1}} a_n$$

and

$$b_{n+1} < f(a_{n-1})^2 (\gamma^{\alpha_n} a_{n-1})\gamma^{\alpha_{n+1}} b_{n-1} < \gamma^{\alpha_{n+1}+\alpha_n} b_n = \gamma^{\alpha_{n+2}} b_n.$$

(d) This is a consequence of (c). □

The following semilocal convergence theorem shows that the sequence $\{x_n\}$ generated by (4) converges to a solution x^* of equation (3).

We denote $\overline{B(x,r)} = \{y \in X; \|y-x\| \leqslant r\}$ and $B(x,r) = \{y \in X; \|y-x\| < r\}$.

Theorem 2.2. *Let* $x_{-1}, x_0 \in \Omega_0$ *and* $\lambda \in [0,1]$. *Let us suppose that* (A1)–(A4) *and the hypotheses of Lemma* 2.1 *are satisfied. If* $\overline{B(x_0,r_0)} \subseteq \Omega_0$, *where* $r_0 = \eta(1-a_0)(1-2a_0)$, *then the sequence* $\{x_n\}$ *generated by* (4) *is well defined and converges to a solution* x^* *of* (3) *with R-order of convergence of at least* $(1+\sqrt{5})/2$. *Moreover, it is proved that* x_n, $x^* \in \overline{B(x_0,r_0)}$ *and* x^* *is unique in* $B(x_0,\tau) \cap \Omega_0$, *where* $\tau = 1/(\beta k) - r_0 - (1-\lambda)\alpha$. *Furthermore, for all* $n \geqslant 0$,

$$\|x^* - x_n\| < \frac{\Delta^n}{1-\Delta}\eta\gamma^{\beta_{n-1}}, \tag{9}$$

where $\gamma = b_0/b_{-1}$, $\Delta = a_0/(1-a_0)$, $\beta_{-1} = 0 = \beta_0$ *and* $\beta_n = s_1 + s_2 + \cdots + s_n$, $n \geqslant 1$.

Proof. Since $a_0 < 1$, $a_n < 1$ is true for all $n \geqslant 1$. We will now determine that $\{x_n\}$ is a Cauchy sequence and $x_n \in \overline{B(x_0,r_0)}$, for $n \geqslant 1$. For $m \geqslant 1$ we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leqslant f(a_{n+m-2})a_{n+m-2} \cdots f(a_{n+1})a_{n+1}\ f(a_n)a_n\|x_{n+1} - x_n\| \\ &\quad + f(a_{n+m-3})a_{n+m-3} \cdots f(a_{n+1})a_{n+1} f(a_n)a_n\|x_{n+1} - x_n\|\end{aligned}$$

$$+\cdots+f(a_n)a_n\|x_{n+1}-x_n\|+\|x_{n+1}-x_n\|$$

$$=\left[\prod_{j=n}^{n+m-2} f(a_j)a_j+\prod_{j=n}^{n+m-3} f(a_j)a_j+\cdots+f(a_n)a_n+1\right]\|x_{n+1}-x_n\|. \tag{10}$$

As $\{a_j\}$ is decreasing, f is increasing and Lemma 2.1 applies, it follows for $n\geqslant 2$ that

$$\|x_{n+m}-x_n\|<\prod_{j=1}^{n-1}\gamma^{s_j}\Delta^n[\Delta^{m-1}+\Delta^{m-2}+\cdots+1]\|x_1-x_0\|,$$

where $\Delta<1$, since $a_0<\frac{1}{2}$.

Then

$$\|x_{n+m}-x_n\|<(\gamma^{s_1+s_2+\cdots+s_{n-1}})\frac{\Delta^n(1-\Delta^m)}{1-\Delta}\|x_1-x_0\|. \tag{11}$$

From (10), if $n=1$, we obtain

$$\|x_{m+1}-x_1\|<\frac{\Delta(1-\Delta^m)}{1-\Delta}\|x_1-x_0\| \tag{12}$$

and if $n=0$,

$$\|x_m-x_0\|<\frac{1-\Delta^m}{1-\Delta}\|x_1-x_0\|<\frac{\eta}{1-\Delta}=r_0. \tag{13}$$

Therefore, $x_n\in\overline{B(x_0,r_0)}$ for all $n\geqslant 1$, (4) is well defined and $\{x_n\}$ is a Cauchy sequence.

Consequently, $\lim_{n\to\infty}x_n=x^*\in\overline{B(x_0,r_0)}$. The inequality

$$\|F(x_n)\|\leqslant k((1-\lambda)\|x_{n-1}-x_{n-2}\|+\|x_n-x_{n-1}\|)\|x_n-x_{n-1}\|$$

follows from the equation

$$F(x_n)=(F'(x_{n-1})-L_{n-1})(x_n-x_{n-1})+\int_0^1(F'(x_{n-1}+t(x_n-x_{n-1}))-F'(x_{n-1}))(x_n-x_{n-1})\,\mathrm{d}t.$$

Hence, $\lim_{n\to\infty}\|F(x_n)\|=0$ and, since F is continuous at x^*,

$$\lim_{n\to\infty}\|F(x_n)\|=\|F(x^*)\|=0.$$

By letting $m\to\infty$ in (11)–(13), we obtain (9).

If z^* were another solution of (3) in $B(x_0,\tau)\cap\Omega_0$, and we consider

$$F(z^*)-F(x^*)=\int_{x^*}^{z^*}F'(x)\,\mathrm{d}x=\int_0^1 F'(x^*+t(z^*-x^*))(z^*-x^*)\,\mathrm{d}t=0$$

and the operator $P=\int_0^1 F'(x^*+t(z^*-x^*))\,\mathrm{d}t$, then

$$\|L_0^{-1}P-I\|\leqslant\|L_0^{-1}\|\,\|P-L_0\|\leqslant\|L_0^{-1}\|\int_0^1\|F'(x^*+t\,(z^*-x^*))-L_0\|\,\mathrm{d}t$$

$$=\|L_0^{-1}\|\int_0^1\|F'(x^*+t(z^*-x^*))-F'(x_0)+F'(x_0)-L_0\|\,\mathrm{d}t$$

$$\leqslant \beta\left(\int_0^1 2k\|x^* + t(z^*-x^*) - x_0\|\,\mathrm{d}t + \|F'(x_0) - L_0\|\right)$$

$$\leqslant \beta\left(\int_0^1 2k((1-t)\|x^*-x_0\| + t\|z^*-x_0\|)\,\mathrm{d}t + k(1-\lambda)\alpha\right)$$

$$= \beta k((1-\lambda)\alpha + \|x^*-x_0\| + \|z^*-x_0\|) < \beta k((1-\lambda)\alpha + r_0 + \tau) = 1,$$

which implies that the operator P is inversible, and consequently $z^* = x^*$.

Finally, properties (P1) and (P2) imply

$$\beta_{n-1} = \alpha_{n+3} - (n+2) > \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - (n+2)$$

and

$$\gamma^{\beta_{n-1}} < \frac{(\gamma^{(3+\sqrt{5})/(2\sqrt{5})})^{((1+\sqrt{5})/2)^n}}{\gamma^{n+2}} < \frac{\gamma^{((1+\sqrt{5})/2)^n}}{\gamma^{n+2}}.$$

In consequence, from (9),

$$\|x^* - x_n\| < \left(\frac{\Delta}{\gamma}\right)^n \frac{\eta}{\gamma^2(1-\Delta)}\gamma^{((1+\sqrt{5})/2)^n} < \frac{\eta}{\gamma^2(1-\Delta)}\gamma^{((1+\sqrt{5})/2)^n},$$

since $\Delta/\gamma < 1$ and the R-order of convergence is at least $(1+\sqrt{5})/2$, which concludes the proof. □

3. Application to nonlinear integral equations of the Hammerstein type

We use Theorem 2.2 to obtain an existence and uniqueness result of the solutions of (1). To apply Theorem 2.2 we must calculate the values k, β and η from the starting values x_{-1} and x_0, that must be fixed. So, we assume that p has a partial derivative with respect to the second variable, $p_2'(t,u) = (\partial/\partial u)p(t,u)$, and p_2' satisfies a Lipschitz condition

$$\|p_2'(t,u_1) - p_2'(t,u_2)\| \leqslant L\|u_1 - u_2\|, \quad L \geqslant 0.$$

Since F is a Fréchet differentiable on Ω_0, differentiating (2) we have

$$[F'(x)z](s) = z(s) - \int_0^1 R(s,t)p_2'(t,x(t))z(t)\,\mathrm{d}t.$$

Hence

$$[F'(x) - F'(y)](z)(s) = \int_0^1 R(s,t)(p_2'(t,y(t)) - p_2'(t,x(t))z(t)\,\mathrm{d}t.$$

Consequently,

$$\|F'(x) - F'(y)\| \leqslant L\|v\|\,\|x-y\|,$$

where $v(\rho) = \int_0^1 R(\rho,\sigma)\,\mathrm{d}\sigma$, and it follows, see [9], that F has a Lipschitz continuous divided difference in Ω_0:

$$[x,y;F] = \int_0^1 F'(y+\tau(x-y))\,\mathrm{d}\tau, \quad x,y \in \Omega_0.$$

Thus

$$\|[x,y;F]-[u,v;F]\| \leqslant k(\|x-u\|+\|y-v\|),$$

where $k = L\|v\|/2$.

In addition, it is clear that

$$[x,y;F]z(s) = z(s) - \int_0^1 R(s,t)\psi(t,x(t),y(t))z(t)\,\mathrm{d}t,$$

where $\psi(t,x(t),y(t)) = \int_0^1 p_2'(t,(y+\tau(x-y))(t))\,\mathrm{d}\tau$.

So, having fixed x_{-1} and x_0, we consider $y_0 = \lambda x_0 + (1-\lambda)x_{-1}$ where $\lambda \in [0,1]$ as a fixed element and $[y_0,x_0;F]z(s) = (I-T)z(s)$, where

$$Tz(s) = \int_0^1 R(s,t)\psi(t,y_0(t),x_0(t))z(t)\,\mathrm{d}t.$$

Hence

$$|T(z)(s)| = \left|\int_0^1 R(s,t)\psi(t,y_0(t),x_0(t))z(t)\,\mathrm{d}t\right| \leqslant M\|v\|\,\|z\|,$$

where $\|\cdot\|$ represents the max-norm in $C[0,1]$ and $M = \max_{t\in[0,1]}|\psi(t,y_0(t),x_0(t))|$.

If $M\|v\| < 1$, by the Banach lemma we have that L_0^{-1} exists and

$$\|L_0^{-1}\| \leqslant \frac{1}{1-M\|v\|}.$$

Since $\|F(x_0)\| \leqslant \|x_0-h\| + \|v\|\,\|p\|$, it follows that

$$\|L_0^{-1}F(x_0)\| \leqslant \frac{\|x_0-h\|+\|v\|\,\|p\|}{1-M\|v\|}.$$

Once the equation is known, the kernel R and the functions p and h are determined. Then the value $F(x_0)$ will be more explicit.

We can now give a result on existence and uniqueness of the solution of (1).

Theorem 3.1. *Following the previous notation, we consider* (2) *where* F: $C[0,1]\to C[0,1]$. *Let* $x_{-1},x_0 \in C[0,1]$ *and* $\lambda \in [0,1]$. *We assume* (A1)–(A4), $M\|v\| < 1$ *and the hypotheses of Lemma* 2.1, *where*

$$\beta = \frac{1}{1-M\|v\|}, \quad \eta = \frac{\|x_0-h\|+\|v\|\,\|p\|}{1-M\|v\|} \quad \text{and} \quad k = \frac{L\|v\|}{2}.$$

Then there is at least one solution of (1) *in* $\overline{B(x_0,r_0)}$, *where* $r_0 = \eta(1-a_0)/(1-2a_0)$, *and it is unique in* $B(x_0,1/(\beta k)-r_0-(1-\lambda)\alpha)$.

Table 1

j	$\bar{x}_j$	j	$\bar{x}_j$	j	$\bar{x}_j$
0	1.0000000000000000	7	0.8643075347402679	14	0.8141388910219971
1	0.9566527838959406	8	0.8555128396460574	15	0.8095584770363422
2	0.9319739617269809	9	0.8483515334503204	16	0.8121077060370004
3	0.9129724633617881	10	0.8416103236537321	17	0.8083909132037373
4	0.8977438552742314	11	0.8304716661779638	18	0.8049765496475915
5	0.8848953323716495	12	0.8245463960276492	19	0.8017901827237186
6	0.8738883606190559	13	0.8191213272527618	20	0.7988090566367779

Table 2
$x_{-1}(t_j) = 1.35,\ x_0(t_j) = 1\ (j = 0, 1, \ldots, 20)$

n	$\lambda = 0$	$\lambda = 0.6$	$\lambda = 0.85$	$\lambda = 1 - 10^{-5}$
1	1.14120×10^{-2}	8.05623×10^{-3}	6.21652×10^{-3}	4.93910×10^{-3}
2	3.24264×10^{-4}	1.08921×10^{-4}	3.66835×10^{-5}	3.99254×10^{-6}
3	6.08624×10^{-7}	5.91357×10^{-8}	5.82438×10^{-9}	2.61646×10^{-12}
4	3.21652×10^{-11}	4.19664×10^{-13}	5.10703×10^{-15}	0.0

4. Numerical example

Next, we solve a particular equation of type (1), which is quoted in [10]:

$$x(s) = 1 - \frac{\mu}{2} \int_0^1 \frac{s}{t+s} \frac{1}{x(t)} \, \mathrm{d}t, \quad s \in [0,1], \ \ \mu \in [0,1] \text{ fixed}, \tag{14}$$

in $\Omega_0 = \{x \in C[0,1]\colon x \text{ is positive}\}$.

Now, Eq. (14) is discretized to replace it with a finite dimensional problem. For the direct numerical solution of (14) we choose $\mu = \frac{1}{2}$ and introduce the points $t_j = j/m$ $(j = 0, 1, \ldots, m)$, where m is an integer according to the precision required. The composite trapezoidal rule with mesh size $1/m$ is used. A scheme is then designed for the determination of numbers $x(t_j)$. So

$$0 = x(t_j) - 1 + \frac{1}{4m} \left[\frac{1}{2} \frac{t_j}{t_j + t_0} \frac{1}{x(t_0)} + \sum_{k=1}^{m-1} \frac{t_j}{t_j + t_k} \frac{1}{x(t_k)} + \frac{1}{2} \frac{t_j}{t_j + t_m} \frac{1}{x(t_m)} \right].$$

For the solution of (1), a resonable choice for the starting point x_0, in the application of the Secant method ((4) for $\lambda = 0$), is the function $h(s)$. Therefore, for (14), we take $x_0(t_j) = 1$ $(j = 1, 2, \ldots, m)$. The choice of $x_{-1}(t_j) = 1.35$ $(j = 1, 2, \ldots, m)$ is motivated for the conditions imposed in Lemma 2.1 for a_{-1}, which depends on $\|x_0 - x_{-1}\| = \alpha$. In Table 1 the approximation of the solution $\bar{x}$ is shown, using 16 significant decimal places and $m = 20$, when the secant method is applied to the previous scheme.

Table 2 contains the errors $\|x_n - \bar{x}\|_\infty$ for the iterates x_n generated by (4) for different values of the parameter λ. Observe, for a precision of 16 significant decimal places, that the solution $\bar{x}$ is obtained in four steps $(n = 4)$ when iteration (4), with $\lambda = 1 - 10^{-5}$, is applied.

Table 3
Error for Newton's method using 16 significant decimal places

n	$\|\bar{x} - z_n\|_\infty$
1	4.93901×10^{-3}
2	3.99072×10^{-6}
3	2.58171×10^{-12}
4	0.0

As we can see in Table 2, the higher λ is, the faster iteration (4) converges.

Finally, note that (4), where λ is near one, gives similar approximations to those obtained by Newton's sequence $\{z_n\}$, without using F' (see Table 3).

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